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QUADRATIC APPROXIMATION OF GENERALIZED TRIBONACCI SEQUENCES

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Abstract

In this paper, we give quadratic approximation of generalized Tribonacci sequence $\{V_n\}_{n\geq 0}$ defined by $V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}$ $(n \geq 3)$ and use this result to give the matrix form of the n-th power of a companion matrix of $\{V_n\}_{n\geq 0}$. Then we re-prove the cubic identity or Cassini-type formula for $\{V_n\}_{n\geq 0}$ and the Binet's formula of the generalized Tribonacci quaternions.

Keywords: Binet's formula, companion matrix, generalized Tribonacci sequence, Narayana number, Padovan number, quadratic approximation, Tribonacci number.

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1. Introduction

Let $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ be a companion matrix of the Tribonacci sequence $\{T_n\}_{n\geq 0}$ defined by the third-order linear recurrence relation

$$(1) \qquad T_0 = T_1 = 0,\ T_2 = 1,\ T_n = T_{n-1} + T_{n-2} + T_{n-3}\ (n \geq 3).$$

Then, by an inductive argument, the n-th power Q^n has the matrix form

$$(2) \qquad Q^n = \begin{bmatrix} T_{n+2} & T_{n+1} + T_n & T_{n+1} \\ T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \end{bmatrix}\ (n \geq 2).$$

For further properties of Tribonacci numbers, we refer to [3, 5, 6].

The property $\det(Q^n) = (\det(Q))^n = 1$ and equation (2) provides an alternate proof of the Cassini-type (or cubic-type) formula for $\{T_n\}_{n\geq 0}$:

$$T_n^3 + T_{n-1}^2 T_{n+2} + T_{n-2} T_{n+1}^2 - 2T_{n-1} T_n T_{n+1} - T_{n-2} T_n T_{n+2} = 1. \tag{3}$$

Now, let's think of the other access method in order to give the matrix form equation (2) of Q^n. This method gives the motivation of our research. That is, our research is based on the following observation. It is well known [11] that the usual Tribonacci numbers can be expressed using Binet's formula

$$T_n = \frac{\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)} + \frac{\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}. \tag{4}$$

where α, ω_1 and ω_2 are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$. Furthermore, $\alpha = \frac{1}{3} + A_T + B_T$, $\omega_1 = \frac{1}{3} + \epsilon A_T + \epsilon^2 B_T$ and $\omega_2 = \frac{1}{3} + \epsilon^2 A_T + \epsilon B_T$, where

$$A_T = \sqrt[3]{\frac{19}{27} + \sqrt{\frac{11}{27}}},\ B_T = \sqrt[3]{\frac{19}{27} - \sqrt{\frac{11}{27}}},$$

and $\epsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$.

From the Binet's formula equation (4), using the classic identities $\alpha + \omega_1 + \omega_2 = 1$, $\alpha\omega_1 + \alpha\omega_2 + \omega_1\omega_2 = -1$, we have for any integer $n \geq 2$:

$$\begin{aligned} &\alpha T_n + (1+\omega_1\omega_2)T_{n-1} + T_{n-2} \\ &\quad = \frac{\alpha^{n-2}(\alpha^3 + (1+\omega_1\omega_2)\alpha + 1)}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{\omega_1^{n-2}(\alpha\omega_1^2 + (1+\omega_1\omega_2)\omega_1 + 1)}{(\alpha-\omega_1)(\omega_1-\omega_2)} \\ &\quad + \frac{\omega_2^{n-2}(\alpha\omega_2^2 + (1+\omega_1\omega_2)\omega_2 + 1)}{(\alpha-\omega_2)(\omega_1-\omega_2)} = \alpha^{n-1}. \end{aligned}$$

Then, we obtain

$$\alpha T_n + (1+\omega_1\omega_2)T_{n-1} + T_{n-2} = \alpha^{n-1}\ (n \geq 2). \tag{5}$$

Multipling equation (5) by α, using $\alpha\omega_1\omega_2 = 1$, if we then interchange the role of α, and ω_1 and α, and ω_2, we obtain the quadratic approximation of $\{T_n\}_{n\geq 0}$

$$\text{Quadratic app. of } \{T_n\} : \begin{cases} \alpha^n = T_n\alpha^2 + (T_{n-1}+T_{n-2})\alpha + T_{n-1}, \\ \omega_1^n = T_n\omega_1^2 + (T_{n-1}+T_{n-2})\omega_1 + T_{n-1}, \\ \omega_2^n = T_n\omega_2^2 + (T_{n-1}+T_{n-2})\omega_2 + T_{n-1}, \end{cases} \tag{6}$$

where α, ω_1 and ω_2 are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$.

In equation (5), if we replace α with the companion matrix Q and change T_{n-1} into the matrix $T_{n-1}I_3$, where I_3 is the 3×3 identity matrix, then we obtain the matrix form equation (2) of Q^n

$$Q^n = T_nQ^2 + (T_{n-1} + T_{n-2})Q + T_{n-1}I_3 \left(= \begin{bmatrix} T_{n+2} & T_{n+1}+T_n & T_{n+1} \\ T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1}+T_{n-2} & T_{n-1} \end{bmatrix} \right).$$

The Tribonacci sequence has been generalized in many ways, for example, by changing the recurrence relation while preserving the initial terms, by altering the initial terms but maintaining the recurrence relation, by combining of these two techniques, and so on (for more details see [4, 8, 10, 12]).

In this paper, we consider one type of generalized Tribonacci sequences. In fact, the sequence $\{V_n\}_{n\geq 0}$ defined by Shannon and Horadam [10] depending on six positive integer parameters V_0, V_1, V_2, r, s and t used in the third-order linear recurrence relation:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3} \quad (n \geq 3). \tag{7}$$

In this paper, as mentioned above, we provide quadratic approximation of $\{V_n\}_{n\geq 0}$ and use this result to give the matrix form of the n-th power of a companion matrix of $\{V_n\}_{n\geq 0}$. Then, we re-prove the Cassini-type formula for the sequence $\{V_n\}_{n\geq 0}$ and Binet's formula of the generalized Tribonacci quaternions.

2. QUADRATIC APPROXIMATION OF THE GENERALIZED TRIBONACCI SEQUENCES $\{V_n\}$

We consider the generalized Tribonacci sequence $\{V_n(V_0, V_1, V_2; r, s, t)\}$, or briefly $\{V_n\}$, defined as in (7), where V_0, V_1, V_2 are arbitrary integers and r, s, t, are real numbers. This sequence has been studied by Shannon and Horadam [10], Yalavigi [13] and Pethe [8]. If we set $r = s = t = 1$ and $V_0 = V_1 = 0$, $V_2 = 1$, then $\{V_n\}$ is the well-known Tribonacci sequence, and if $r = s = t = 1$ and $V_0 = 3$, $V_1 = 1$, $V_2 = 3$, then $\{V_n\}$ is the Tribonacci-Lucas investigated by Elia in [2].

As the elements of this Tribonacci-type number sequence provide third order iterative relation, its characteristic equation is $x^3 - rx^2 - sx - t = 0$, whose roots are $\alpha = \alpha(r, s, t) = \frac{r}{3} + A_V + B_V$, $\omega_1 = \frac{r}{3} + \epsilon A_V + \epsilon^2 B_V$ and $\omega_2 = \frac{r}{3} + \epsilon^2 A_V + \epsilon B_V$, where

$$A_V = \sqrt[3]{\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}},\ B_V = \sqrt[3]{\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}}, \tag{8}$$

with $\Delta = \Delta(r, s, t) = \frac{r^3t}{27} - \frac{r^2s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}$ and $\epsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ (see [1]).

In this paper, $\Delta(r,s,t)>0$, then the cubic equation $x^3-rx^2-sx-t=0$ has one real and two nonreal solutions, the latter being conjugate complex. Thus, the Binet formula for the generalized Tribonacci numbers can be expressed as:

$$V_n=\frac{P\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)}-\frac{Q\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)}+\frac{R\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}, \tag{9}$$

where $P=V_2-(\omega_1+\omega_2)V_1+\omega_1\omega_2V_0$, $Q=V_2-(\alpha+\omega_2)V_1+\alpha\omega_2V_0$ and $R=V_2-(\alpha+\omega_1)V_1+\alpha\omega_1V_0$.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Padovan, Narayana and third-order Jacobsthal. For example, $\{V_n(0,0,1;1,1,1)\}_{n\geq 0}$, $\{V_n(0,1,0;0,1,1)\}_{n\geq 0}$, are Tribonacci and Padovan sequences, respectively. In particular, the Binet formula for the generalized Tribonacci sequence $\{U_n\}_{n\geq 0}=\{V_n(0,0,1;r,s,t)\}_{n\geq 0}$ is expressed as follows.

Lemma 1. *The Binet formula for the generalized Tribonacci sequence* $\{U_n\}_{n\geq 0}$ *is*

$$U_n=\frac{\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)}-\frac{\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)}+\frac{\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}, \tag{10}$$

where α, ω_1 *and* ω_2 *are the roots of the cubic equation* $x^3-rx^2-sx-t=0$.

Proof. The validity of this formula can be confirmed using the recurrence relation. ■

In [10], using an inductive argument, authors give the matrix form of the n-th power of a companion matrix $M=\begin{bmatrix} r & s & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$ of $\{V_n\}_{n\geq 0}$

$$\begin{bmatrix} V_{n+2}\\ V_{n+1}\\ V_n\end{bmatrix}=\begin{bmatrix} r & s & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}^n\begin{bmatrix} V_2\\ V_1\\ V_0\end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} r & s & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}^n=\begin{bmatrix} U_{n+2} & sU_{n+1}+tU_n & tU_{n+1}\\ U_{n+1} & sU_n+tU_{n-1} & tU_n\\ U_n & sU_{n-1}+tU_{n-2} & tU_{n-1}\end{bmatrix}\ (n\geq 2), \tag{12}$$

where U_n is defined by equation (10).

And then give the Cassini-type identity for $\{U_n\}$ by taking the determinant of both sides of the matrix form equation (12)

$$U_n^3+U_{n-1}^2U_{n+2}+U_{n-2}U_{n+1}^2-2U_{n-1}U_nU_{n+1}-U_{n-2}U_nU_{n+2}=t^{n-2}, \tag{13}$$

for $n \geq 2$.

More generally,

$$\begin{bmatrix} V_{n+4} & V_{n+3}+V_{n+2} & V_{n+3} \\ V_{n+3} & V_{n+2}+V_{n+1} & V_{n+2} \\ V_{n+2} & V_{n+1}+V_n & V_{n+1} \end{bmatrix} = M^n \begin{bmatrix} V_4 & V_3+V_2 & V_3 \\ V_3 & V_2+V_1 & V_2 \\ V_2 & V_1+V_0 & V_1 \end{bmatrix}, \tag{14}$$

for $n \geq 0$. Or equivalently, we can write the Cassini-type identity for $\{V_n\}$ by taking the determinant of both sides of the matrix form equation (14)

$$V_{n+2}^3 + V_{n+1}^2 V_{n+4} + V_n V_{n+3}^2 - V_{n+2}(2V_{n+1}V_{n+3} + V_n V_{n+4}) = t^n g_V(0), \tag{15}$$

where $g_V(0) = V_2^3 + V_1^2 V_4 + V_0 V_3^2 - V_2(2V_1V_3 + V_0V_4)$.

Now, we give the quadratic approximation of $\{V_n\}$ and then use this result to obtain the matrix form equation (14).

Theorem 2. *Let $\{V_n\}_{n\geq 0}$, α, ω_1 and ω_2 be as above. Then, we have for all integer $n \geq 0$*

(16)

$$\textit{Quadratic app. of } \{V_n\}: \begin{cases} P\alpha^{n+2} = \alpha^2 V_{n+2} + \alpha(sV_{n+1} + tV_n) + tV_{n+1}, \\ Q\omega_1^{n+2} = \omega_1^2 V_{n+2} + \omega_1(sV_{n+1} + tV_n) + tV_{n+1}, \\ R\omega_2^{n+2} = \omega_2^2 V_{n+2} + \omega_2(sV_{n+1} + tV_n) + tV_{n+1}, \end{cases}$$

where $P = V_2 - (\omega_1 + \omega_2)V_1 + \omega_1\omega_2 V_0$, $Q = V_2 - (\alpha + \omega_2)V_1 + \alpha\omega_2 V_0$ and $R = V_2 - (\alpha + \omega_1)V_1 + \alpha\omega_1 V_0$.

Proof. Using the Binet's formula equation (9), we have

$$\begin{aligned} &\alpha V_{n+2} + (s + \omega_1\omega_2)V_{n+1} + tV_n \\ &\quad = \frac{P\alpha^n(\alpha^3 + (s+\omega_1\omega_2)\alpha + t)}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{Q\omega_1^n(\alpha\omega_1^2 + (s+\omega_1\omega_2)\omega_1 + t)}{(\alpha-\omega_1)(\omega_1-\omega_2)} \\ &\quad + \frac{R\omega_2^n(\alpha\omega_2^2 + (s+\omega_1\omega_2)\omega_2 + t)}{(\alpha-\omega_2)(\omega_1-\omega_2)} = (V_2 - (\omega_1+\omega_2)V_1 + \omega_1\omega_2 V_0)\alpha^{n+1}, \end{aligned}$$

the latter given that $\alpha\omega_1^2 + (s+\omega_1\omega_2)\omega_1 + t = 0$ and $\alpha\omega_2^2 + (s+\omega_1\omega_2)\omega_2 + t = 0$. Then, we get

$$\alpha V_{n+2} + (s + \omega_1\omega_2)V_{n+1} + tV_n = (V_2 - (\omega_1+\omega_2)V_1 + \omega_1\omega_2 V_0)\alpha^{n+1}. \tag{17}$$

Multiplying equation (17) by α and using $\alpha\omega_1\omega_2 = t$, we have

$$\begin{aligned} P\alpha^{n+2} &= \alpha^2 V_{n+2} + \alpha(s+\omega_1\omega_2)V_{n+1} + \alpha t V_n \\ &= \alpha^2 V_{n+2} + \alpha(sV_{n+1} + tV_n) + tV_{n+1}, \end{aligned}$$

where $P = V_2 - (\omega_1 + \omega_2)V_1 + \omega_1\omega_2 V_0$. If we change α, ω_1 and ω_2 role above process, we obtain the desired result equation (16). ∎

Now, we can re-prove equations equation (14) and equation (15) by using the above quadratic approximation of $\{V_n\}$ in equation (16).

Corollary 3. Let $M = \begin{bmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ be a companion matrix of $\{V_n\}$. Then the matrix form of the n-th power M^n is given by equation (14) and the Cassini-type formula for $\{V_n\}$ is given by equation (15).

Proof. In equation (16), if we change α, ω_1 and ω_2 into the matrix M and change tV_{n+1} into the matrix $tV_{n+1}I_3$, then we have

$$M^n(V_2M^2+(sV_1+tV_0)M+tV_1I_3) = V_{n+2}M^2+(sV_{n+1}+tV_n)M+tV_{n+1}I_3. \tag{18}$$

In fact, equation (18) holds for the following reason: Since

$$M\begin{bmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{bmatrix} = \begin{bmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{bmatrix} \text{ and } M^n\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix} = \begin{bmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{bmatrix},$$

we have

$$\begin{aligned}
&M^n(V_2M^2+(sV_1+tV_0)M+tV_1I_3)\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix} \\
&\quad= V_2M^{n+2}\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix}+(sV_1+tV_0)M^{n+1}\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix}+tV_1M^n\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix} \\
&\quad= V_2\begin{bmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \end{bmatrix}+(sV_1+tV_0)\begin{bmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \end{bmatrix}+tV_1\begin{bmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{bmatrix} \\
&\quad= \begin{bmatrix} V_2V_{n+4}+(sV_1+tV_0)V_{n+3}+tV_1V_{n+2} \\ V_2V_{n+3}+(sV_1+tV_0)V_{n+2}+tV_1V_{n+1} \\ V_2V_{n+2}+(sV_1+tV_0)V_{n+1}+tV_1V_n \end{bmatrix}.
\end{aligned}$$

Using $V_{n+3} = rV_{n+2}+sV_{n+1}+tV_n$ and $V_{n+4} = (r^2+s)V_{n+2}+(rs+t)V_{n+1}+rtV_n$, we have

$$M^n(V_2M^2 + (sV_1 + tV_0)M + tV_1I_3)\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix}$$

$$= \begin{bmatrix} V_4V_{n+2} + (sV_{n+1} + tV_n)V_3 + tV_{n+1}V_2 \\ V_3V_{n+2} + (sV_{n+1} + tV_n)V_2 + tV_{n+1}V_1 \\ V_2V_{n+2} + (sV_{n+1} + tV_n)V_1 + tV_{n+1}V_0 \end{bmatrix}$$

$$= (V_{n+2}M^2 + (sV_{n+1} + tV_n)M + tV_{n+1}I_3)\begin{bmatrix} V_2 \\ V_1 \\ V_0 \end{bmatrix}.$$

Thus from equation (18) we have

$$M^n(V_2M^2 + (sV_1{+}tV_0)M + tV_1I_3) = M^n\begin{bmatrix} V_4 & sV_3 + tV_2 & tV_3 \\ V_3 & sV_2 + tV_1 & tV_2 \\ V_2 & sV_1 + tV_0 & tV_1 \end{bmatrix}$$

$$= M^n\begin{bmatrix} V_4 & V_3 + V_2 & V_3 \\ V_3 & V_2 + V_1 & V_2 \\ V_2 & V_1 + V_0 & V_1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & s-t & t \end{bmatrix}$$

and

$$V_{n+2}M^2 + (sV_{n+1} + tV_n)M + tV_{n+1}I_3$$

$$= \begin{bmatrix} V_{n+4} & sV_{n+3} + tV_{n+2} & tV_{n+3} \\ V_{n+3} & sV_{n+2} + tV_{n+1} & tV_{n+2} \\ V_{n+2} & sV_{n+1} + tV_n & tV_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} V_{n+4} & V_{n+3} + V_{n+2} & V_{n+3} \\ V_{n+3} & V_{n+2} + V_{n+1} & V_{n+2} \\ V_{n+2} & V_{n+1} + V_n & V_{n+1} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & s-t & t \end{bmatrix}.$$

Since the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & s-t & t \end{bmatrix}$ is invertible, we obtain the desired result equation (14) and by taking the determinant of both sides of the matrix form equation (14) we obtain the desired result equation (15). The proof is completed. ■

Remark 4. For some positive integer k, if $r = k$, $s = 0$ and $t = 1$, then $\{U_n\}$ is the k-Narayana sequence $\{b_{k,n}\}$ investigated by Ramírez and Sirvent in [9]. The k-Narayana numbers can be expressed using Binet's formula

$$b_{k,n} = \frac{\alpha_k^n}{(\alpha_k - \omega_{k,1})(\alpha_k - \omega_{k,2})} - \frac{\omega_{k,1}^n}{(\alpha_k - \omega_{k,1})(\omega_{k,1} - \omega_{k,2})} + \frac{\omega_{k,2}^n}{(\alpha_k - \omega_{k,2})(\omega_{k,1} - \omega_{k,2})}. \tag{19}$$

where α_k, $\omega_{k,1}$ and $\omega_{k,2}$ are the roots of the cubic equation $x^3 - kx^2 - 1 = 0$. Furthermore, using the notation of equation (8), the roots given in [9] can be written as $\alpha_k = \frac{k}{3} + A_k + B_k$, $\omega_{k,1} = \frac{k}{3} + \epsilon A_k + \epsilon^2 B_k$ and $\omega_{k,2} = \frac{k}{3} + \epsilon^2 A_k + \epsilon B_k$, where

$$A_k = \sqrt[3]{\frac{k^3}{27} + \frac{1}{2} + \sqrt{\frac{k^3}{27} + \frac{1}{4}}},\ B_k = \sqrt[3]{\frac{k^3}{27} + \frac{1}{2} - \sqrt{\frac{k^3}{27} + \frac{1}{4}}},$$

and $\epsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ is a cubic root of unity.

From the Binet's formula equation (19), using the identities $\alpha_k + \omega_{k,1} + \omega_{k,2} = k$, $\alpha_k\omega_{k,1} + \alpha_k\omega_{k,2} + \omega_{k,1}\omega_{k,2} = 0$, we have for any integer $n \geq 2$:

$$\begin{aligned}\alpha_k b_{k,n+2} &+ (\omega_{k,1}\omega_{k,2}) b_{k,n+1} + b_{k,n} \\ &= \frac{\alpha_k^n(\alpha_k^3 + (\omega_{k,1}\omega_{k,2})\alpha_k + 1)}{(\alpha_k - \omega_{k,1})(\alpha_k - \omega_{k,2})} - \frac{\omega_{k,1}^n(\alpha_k\omega_{k,1}^2 + (\omega_{k,1}\omega_{k,2})\omega_{k,1} + 1)}{(\alpha_k - \omega_{k,1})(\omega_{k,1} - \omega_{k,2})} \\ &+ \frac{\omega_{k,2}^n(\alpha_k\omega_{k,2}^2 + (\omega_{k,1}\omega_{k,2})\omega_{k,2} + 1)}{(\alpha_k - \omega_{k,2})(\omega_{k,1} - \omega_{k,2})} = \alpha_k^{n+1},\end{aligned}$$

the latter given that $\alpha_k\omega_{k,1}^2 + (\omega_{k,1}\omega_{k,2})\omega_{k,1} + 1 = 0$ and $\alpha_k\omega_{k,2}^2 + (\omega_{k,1}\omega_{k,2})\omega_{k,2} + 1 = 0$. Then, we get

$$\alpha_k b_{k,n+2} + (\omega_{k,1}\omega_{k,2}) b_{k,n+1} + b_{k,n} = \alpha_k^{n+1}. \tag{20}$$

Multipling equation (20) by α_k, using $\alpha_k\omega_{k,1}\omega_{k,2} = 1$, and if we change α_k, $\omega_{k,1}$ and $\omega_{k,2}$ in the role above process, we obtain the quadratic approximation of $\{b_{k,n}\}_{n\geq 0}$

$$\text{Quadratic app. of } \{b_{k,n}\} : \begin{cases} \alpha_k^n = b_{k,n}\alpha_k^2 + b_{k,n-2}\alpha_k + b_{k,n-1}, \\ \omega_{k,1}^n = b_{k,n}\omega_{k,1}^2 + b_{k,n-2}\omega_{k,1} + b_{k,n-1}, \\ \omega_{k,2}^n = b_{k,n}\omega_{k,2}^2 + b_{k,n-2}\omega_{k,2} + b_{k,n-1}, \end{cases} \tag{21}$$

where α_k, $\omega_{k,1}$ and $\omega_{k,2}$ are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$.

In equation (20), if we replace α_k with the companion matrix Q_k and change T_{n-1} into the matrix $T_{n-1}I_3$, where I_3 is the 3×3 identity matrix, then we obtain the matrix form of Q_k^n:

$$Q_k^n = b_{k,n}Q_k^2 + b_{k,n-2}Q_k + b_{k,n-1}I_3 \left(= \begin{bmatrix} b_{k,n+2} & b_{k,n} & b_{k,n+1} \\ b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ b_{k,n} & b_{k,n-2} & b_{k,n-1} \end{bmatrix} \right),$$

where $Q_k = \begin{bmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

The next corollary gives an alternative proof of the Binet's formula for the generalized Tribonacci quaternions (see [1, Theorem 2.1]).

Corollary 5. For any integer $n \geq 0$, the n-th generalized Tribonacci quaternion is

$$Q_{V,n} = \frac{P\underline{\alpha}\alpha^n}{(\alpha-\omega_1)(\alpha-\omega_2)} - \frac{Q\underline{\omega_1}\omega_1^n}{(\alpha-\omega_1)(\omega_1-\omega_2)} + \frac{R\underline{\omega_2}\omega_2^n}{(\alpha-\omega_2)(\omega_1-\omega_2)}, \tag{22}$$

where P, Q and R as in equation (9), $\underline{\alpha} = 1+\alpha\mathbf{i}+\alpha^2\mathbf{j}+\alpha^3\mathbf{k}$, $\underline{\omega_1} = 1+\omega_1\mathbf{i}+\omega_1^2\mathbf{j}+\mathbf{k}$ and $\underline{\omega_2} = 1+\omega_2\mathbf{i}+\omega_2^2\mathbf{j}+\mathbf{k}$. If $V_0 = V_1 = 0$ and $V_2 = 1$, we get the classic Tribonacci quaternion.

Proof. For the equation (16), we have

$$\begin{aligned}
&\alpha^2 Q_{V,n+2} + \alpha\left(sQ_{V,n+1} + tQ_{V,n}\right) + tQ_{V,n+1} \\
&= \alpha^2\left(V_{n+2} + V_{n+3}\mathbf{i} + V_{n+4}\mathbf{j} + V_{n+5}\mathbf{k}\right) \\
&+ \alpha\left(sV_{n+1} + tV_n + (sV_{n+2} + tV_{n+1})\mathbf{i} + (sV_{n+3} + tV_{n+2})\mathbf{j} + (sV_{n+4} + tV_{n+3})\mathbf{k}\right) \\
&+ t\left(V_{n+1} + V_{n+2}\mathbf{i} + V_{n+3}\mathbf{j} + V_{n+4}\mathbf{k}\right) \\
&= \alpha^2 V_{n+2} + \alpha\left(sV_{n+1} + tV_n\right) + tV_{n+1} + \left(\alpha^2 V_{n+3} + \alpha\left(sV_{n+2} + tV_{n+1}\right) + tV_{n+2}\right)\mathbf{i} \\
&+ \left(\alpha^2 V_{n+4} + \alpha\left(sV_{n+3} + tV_{n+2}\right) + tV_{n+3}\right)\mathbf{j} \\
&+ \left(\alpha^2 V_{n+5} + \alpha\left(sV_{n+4} + tV_{n+3}\right) + tV_{n+4}\right)\mathbf{k}.
\end{aligned}$$

From the identity $P\alpha^{n+2} = \alpha^2 V_{n+2} + \alpha(sV_{n+1} + tV_n) + tV_{n+1}$ in Theorem 2, we obtain

$$\alpha^2 Q_{V,n+2} + \alpha\left(sQ_{V,n+1} + tQ_{V,n}\right) + tQ_{V,n+1} = P\underline{\alpha}\alpha^{n+2}. \tag{23}$$

Similarly, we have

$$\omega_1^2 Q_{V,n+2} + \omega_1\left(sQ_{V,n+1} + tQ_{V,n}\right) + tQ_{V,n+1} = Q\underline{\omega_1}\omega_1^{n+2}, \tag{24}$$

$$\omega_2^2 Q_{V,n+2} + \omega_2\left(sQ_{V,n+1} + tQ_{V,n}\right) + tQ_{V,n+1} = R\underline{\omega_2}\omega_2^{n+2}. \tag{25}$$

Subtracting equation (24) from equation (23) gives

$$(\alpha+\omega_1)Q_{V,n+2} + (sQ_{V,n+1} + tQ_{V,n}) = \frac{P\underline{\alpha}\alpha^{n+2} - Q\underline{\omega_1}\omega_1^{n+2}}{\alpha-\omega_1}. \tag{26}$$

Similarly, subtracting equation (25) from equation (23) gives

$$(\alpha+\omega_2)Q_{V,n+2} + (sQ_{V,n+1} + tQ_{V,n}) = \frac{P\underline{\alpha}\alpha^{n+2} - R\underline{\omega_2}\omega_2^{n+2}}{\alpha-\omega_2}. \tag{27}$$

Finally, subtracting equation (27) from equation (26), we obtain

$$\begin{aligned} Q_{V,n+2} &= \frac{1}{\omega_1-\omega_2}\left(\frac{P\underline{\alpha}\alpha^{n+2}-Q\underline{\omega_1}\omega_1^{n+2}}{\alpha-\omega_1}-\frac{P\underline{\alpha}\alpha^{n+2}-R\underline{\omega_2}\omega_2^{n+2}}{\alpha-\omega_2}\right) \\ &= \frac{P\underline{\alpha}\alpha^{n+2}}{(\alpha-\omega_1)(\alpha-\omega_2)}-\frac{Q\underline{\omega_1}\omega_1^{n+2}}{(\alpha-\omega_1)(\omega_1-\omega_2)}+\frac{R\underline{\omega_2}\omega_2^{n+2}}{(\alpha-\omega_2)(\omega_1-\omega_2)}. \end{aligned}$$

So, the corollary is proved. ■

3. Conclusions

Sequences of integer numbers have been studied over several years, with emphasis on studies of the well known Fibonacci sequence (and then the Lucas sequence) that is related to the golden ratio. In this paper, we also contribute for the study of Tribonacci sequence giving some identities which some of them involve generalized Tribonacci numbers. In the future, we intend to discuss the invertibility of these type matrices associated with these sequence (or quadratic approximation of generalized Tribonacci numbers with negative subscripts) using the identities given by Kuhapatanakul and Sukruan in [7].

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