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THE HORADAM HYBRID NUMBERS

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Abstract

In this paper we introduce the Horadam hybrid numbers and give some their properties: Binet formula, character and generating function.

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1. Introduction

The hybrid numbers were introduced by Özdemir in [7] as a new generalization of complex, hyperbolic and dual numbers.

Let $\mathbb{K}$ be the set of hybrid numbers $\mathbf{Z}$ of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\epsilon + d\mathbf{h}, \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}$, ϵ, $\mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1,\ \epsilon^2 = 0,\ \mathbf{h}^2 = 1 \tag{2}$$

and

$$\mathbf{ih} = -\mathbf{hi} = \epsilon + \mathbf{i}. \tag{3}$$

If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\epsilon + d_1\mathbf{h}$, and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\epsilon + d_2\mathbf{h}$, are any two hybrid numbers then equality, addition, substraction and multiplication by scalar are defined.

Equality: $\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\epsilon + (d_1 + d_2)\mathbf{h}$,
substraction: $\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\epsilon + (d_1 - d_2)\mathbf{h}$,
multiplication by scalar $s \in \mathbb{R}$: $s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\epsilon + sd_1\mathbf{h}$.

Addition operation in the hybrid numbers is both commutative and associative. Zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{Z}$ is $-\mathbf{Z} = -a - b\mathbf{i} - c\epsilon - d\mathbf{h}$. This means that $(\mathbb{K}, +)$ is an Abelian group.

The hybrid numbers multiplication is defined using (2) and (3). Note that using the formulas (2) and (3) we can find the product of any two hybrid units. The following Table presents products of $\mathbf{i}$, ϵ, and $\mathbf{h}$

$\cdot$	$\mathbf{i}$	ϵ	$\mathbf{h}$
$\mathbf{i}$	-1	$1-\mathbf{h}$	$\epsilon+\mathbf{i}$
ϵ	$\mathbf{h}+1$	0	$-\epsilon$
$\mathbf{h}$	$-\epsilon-\mathbf{i}$	ϵ	1

Table 1.

Using the rules given in Table 1. the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions.

The conjugate of a hybrid number $\mathbf{Z}$ is defined by

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\epsilon + d\mathbf{h}} = a - b\mathbf{i} - c\epsilon - d\mathbf{h}. \tag{4}$$

The real number

$$C(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2 \tag{5}$$

is called the character of the hybrid number $\mathbf{Z}$.

We use the following notation for
scalar part: $S(\mathbf{Z}) = a \in \mathbb{R}$,
vector part: $V(\mathbf{Z}) = b\mathbf{i} + c\epsilon + d\mathbf{h} \in \mathbb{K}$.
For the basics on hybrid number theory and algebraic and geometric properties of hybrid numbers, see [7].

2. The Horadam numbers

Let p, q, n be integers. For $n \geq 0$ Horadam (see [4]) defined the numbers $W_n = W_n(W_0, W_1; p, q)$ by the recursive equation

$$W_n = p \cdot W_{n-1} - q \cdot W_{n-2}, \tag{6}$$

for $n \geq 2$ with fixed real numbers W_0, W_1.

For the historical reasons these numbers were later called the Horadam numbers.

Let α, β be the roots of the equation

$$x^2 - px + q = 0.$$

Then the Binet formula for the Horadam number has the form

$$W_n = A\alpha^n + B\beta^n, \tag{7}$$

where

$$\begin{cases} A = \dfrac{W_1 - W_0\beta}{\alpha - \beta} \\ \\ B = \dfrac{W_0\alpha - W_1}{\alpha - \beta} \end{cases} \tag{8}$$

see [4].

For special W_0, W_1, p, q the equation (6) defines the well-known numbers named as the numbers of the Fibonacci type. We list some of them

(a) $W_n(0,1;1,-1) = F_n$ – the Fibonacci numbers,

(b) $W_n(2,1;1,-1) = L_n$ – the Lucas numbers,

(c) $W_n(0,1;2,-1) = P_n$ – the Pell numbers,

(d) $W_n(2,2;2,-1) = Q_n$ – the Pell-Lucas numbers,

(e) $W_n(0,1;1,-2) = J_n$ – the Jacobsthal numbers,

(f) $W_n(2,1;1,-2) = j_n$ – the Jacobsthal-Lucas numbers.

The following Table presents the initial words of mentioned type of the Horadam numbers for $n = 0, 1, \ldots, 10$.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
L_n	2	1	3	4	7	11	18	29	47	76	123
P_n	0	1	2	5	12	29	70	169	408	985	2378
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726
J_n	0	1	1	3	5	11	21	43	85	171	341
j_n	2	1	5	7	17	31	65	127	257	511	1025

Table 2.

These numbers have many applications in distinct areas of mathematics. In [5] Horadam defined the complex Fibonacci numbers and the Fibonacci quaternions. In [3] Halici derived generating functions for the Fibonacci quaternions and Binet formulas for the Fibonacci and Lucas quaternions. In [6] Nurkan and Güven defined the dual Fibonacci quaternions and dual Lucas quaternions. In [2] Halici investigated the complex Fibonacci quaternions.

Interesting results concerning Pell quaternions, Pell-Lucas quaternions have been obtained quite recently and can be found in [1, 9]. Jacobsthal quaternions and Jacobsthal-Lucas quaternions were introduced in [8].

Motivated by their investigations and results in this paper we introduce and study the Horadam hybrid numbers and we give some properties of them.

3. The Horadam hybrid numbers

The nth Horadam hybrid number H_n is defined as

$$H_n = W_n + \mathbf{i}W_{n+1} + \epsilon W_{n+2} + \mathbf{h}W_{n+3}. \tag{9}$$

For special values of W_n we obtain the definitions of

(a) nth Fibonacci hybrid number FH_n

$$FH_n = F_n + \mathbf{i}F_{n+1} + \epsilon F_{n+2} + \mathbf{h}F_{n+3},$$

(b) nth Pell hybrid number PH_n

$$PH_n = P_n + \mathbf{i}P_{n+1} + \epsilon P_{n+2} + \mathbf{h}P_{n+3},$$

(c) nth Jacobsthal hybrid number JH_n

$$JH_n = J_n + \mathbf{i}J_{n+1} + \epsilon J_{n+2} + \mathbf{h}J_{n+3}.$$

In the same way we can define nth Lucas hybrid number LH_n, nth Pell-Lucas hybrid number QH_n and nth Jacobsthal-Lucas hybrid number jH_n substituting in place of coefficients of hybrid numbers Lucas numbers, Pell-Lucas numbers and Jacobsthal-Lucas numbers, respectively.

Using the above definitions we can write initial Fibonacci, Pell and Jacobsthal hybrid numbers, i.e.,

$$\begin{aligned}
&FH_0 = \mathbf{i} + \epsilon + 2\mathbf{h},\\
&FH_1 = 1 + \mathbf{i} + 2\epsilon + 3\mathbf{h},\\
&FH_2 = 1 + 2\mathbf{i} + 3\epsilon + 5\mathbf{h},\\
&\ldots
\end{aligned}$$

$$\begin{aligned}
PH_0 &= \mathbf{i} + 2\epsilon + 5\mathbf{h},\\
PH_1 &= 1 + 2\mathbf{i} + 5\epsilon + 12\mathbf{h},\\
PH_2 &= 2 + 5\mathbf{i} + 12\epsilon + 29\mathbf{h},\\
&\dots
\end{aligned}$$

$$\begin{aligned}
JH_0 &= \mathbf{i} + \epsilon + 3\mathbf{h},\\
JH_1 &= 1 + \mathbf{i} + 3\epsilon + 5\mathbf{h},\\
JH_2 &= 1 + 3\mathbf{i} + 5\epsilon + 11\mathbf{h},\\
&\dots
\end{aligned}$$

We will give the Binet formula for the Horadam hybrid numbers.

Theorem 1. *Let $n \geq 0$ be integer. Then*

$$H_n = A\alpha^n \left(1 + \mathbf{i}\alpha + \epsilon\alpha^2 + \mathbf{h}\alpha^3\right) + B\beta^n \left(1 + \mathbf{i}\beta + \epsilon\beta^2 + \mathbf{h}\beta^3\right), \tag{10}$$

where A, B are defined by (8).

Proof. Using the definition of Horadam hybrid number (9) and the Binet formula for the Horadam numbers (7) we have

$$\begin{aligned}
H_n &= (A\alpha^n + B\beta^n) + \mathbf{i}\left(A\alpha^{n+1} + B\beta^{n+1}\right)\\
&\quad + \epsilon\left(A\alpha^{n+2} + B\beta^{n+2}\right) + \mathbf{h}\left(A\alpha^{n+3} + B\beta^{n+3}\right)
\end{aligned}$$

and after calculations we obtain (10). ∎

Remark 2. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -1$ we have $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$ and the Binet formula for the Fibonacci hybrid number FH_n has the form

$$\begin{aligned}
FH_n &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \left(1 + \frac{1+\sqrt{5}}{2}\mathbf{i} + \frac{3+\sqrt{5}}{2}\epsilon + \frac{4+2\sqrt{5}}{2}\mathbf{h}\right)\\
&\quad - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \left(1 + \frac{1-\sqrt{5}}{2}\mathbf{i} + \frac{3-\sqrt{5}}{2}\epsilon + \frac{4-2\sqrt{5}}{2}\mathbf{h}\right).
\end{aligned}$$

Remark 3. For $W_0 = 0$, $W_1 = 1$, $p = 2$ and $q = -1$ we have $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$, $A = \frac{1}{2\sqrt{2}}$, $B = -\frac{1}{2\sqrt{2}}$ and the Binet formula for the Pell hybrid number PH_n has the form

$$\begin{aligned}
PH_n &= \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^n \left(1 + (1+\sqrt{2})\mathbf{i} + (3+2\sqrt{2})\epsilon + (7+5\sqrt{2})\mathbf{h}\right)\\
&\quad - \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\right)^n \left(1 + (1-\sqrt{2})\mathbf{i} + (3-2\sqrt{2})\epsilon + (7-5\sqrt{2})\mathbf{h}\right).
\end{aligned}$$

Remark 4. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -2$ we have $\alpha = 2$, $\beta = -1$, $A = \frac{1}{3}$, $B = -\frac{1}{3}$ and the Binet formula for the Jacobsthal hybrid number JH_n has the form

$$JH_n = \frac{1}{3}2^n\left(1 + 2\mathbf{i} + 4\epsilon + 8\mathbf{h}\right) - \frac{1}{3}(-1)^n\left(1 - \mathbf{i} + \epsilon - \mathbf{h}\right).$$

Using (5) we will calculate the character of the Horadam hybrid number.

Theorem 5. *Let $n \geq 0$ be integer. Then*

$$\begin{aligned} C(H_n) &= W_n^2\left(1 - p^2q^2\right) + W_nW_{n+1}\left(2q + 2p^3q - 2pq^2\right) \\ &\quad + W_{n+1}^2\left(1 - 2p - p^4 + 2p^2q - q^2\right). \end{aligned} \tag{11}$$

Proof. Let $W_{n+2} = pW_{n+1} - qW_n$, $W_{n+3} = pW_{n+2} - qW_{n+1} = (p^2 - q)W_{n+1} - pqW_n$ and $C(H_n) = W_n^2 + W_{n+1}^2 - 2W_{n+1}W_{n+2} - W_{n+3}^2$. Then $C(H_n) = W_n^2 + W_{n+1}^2 - 2W_{n+1}\left(pW_{n+1} - qW_n\right) - \left((p^2 - q)W_{n+1} - pqW_n\right)^2$ and by calculations the result follows. ∎

Using (7) one can prove

Remark 6. Let $n \geq 0$ be integer. Then

$$\begin{aligned} C(H_n) &= A^2\alpha^{2n}\left(1 + \alpha^2 - 2\alpha^3 - \alpha^6\right) + B^2\beta^{2n}\left(1 + \beta^2 - 2\beta^3 - \beta^6\right) \\ &\quad + 2AB\alpha^n\beta^n\left(1 + \alpha\beta - \alpha\beta^2 - \alpha^2\beta - \alpha^3\beta^3\right). \end{aligned} \tag{12}$$

Remark 7. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -1$ we have the character of the Fibonacci hybrid number FH_n

$$C(FH_n) = -5F_{n+1}^2 - 6F_nF_{n+1}.$$

Remark 8. For $W_0 = 0$, $W_1 = 1$, $p = 2$ and $q = -1$ we have the character of the Pell hybrid number PH_n

$$C(PH_n) = -3P_n^2 - 28P_{n+1}^2 - 22P_nP_{n+1}.$$

Remark 9. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -2$ we have the character of the Jacobsthal hybrid number JH_n

$$C(JH_n) = -3J_n^2 - 10J_{n+1}^2 - 16J_nJ_{n+1}.$$

Next we shall give the ordinary generating functions for the Horadam hybrid numbers.

Theorem 10. *The generating function for the Horadam hybrid number sequence* $\{H_n\}$ *is*

$$\sum_{n=0}^{\infty} H_n t^n = \frac{H_0 + t(H_1 - pH_0)}{1 - pt + qt^2}.$$

Proof. Assuming that the generating function of the Horadam hybrid number sequence $\{H_n\}$ has the form $G(t) = \sum_{n=0}^{\infty} H_n t^n$, we obtain that

$$\begin{aligned}(1 - pt + qt^2)G(t) &= (1 - pt + qt^2) \cdot (H_0 + H_1 t + H_2 t^2 + \cdots) \\ &= H_0 + H_1 t + H_2 t^2 + \cdots - pH_0 t - pH_1 t^2 - pH_2 t^3 \\ &\quad - \cdots + qH_0 t^2 + qH_1 t^3 + qH_2 t^4 + \cdots = H_0 + t(H_1 - pH_0),\end{aligned}$$

since $H_n = pH_{n-1} - qH_{n-2}$ and the coefficients of t^n for $n \geq 2$ are equal to zero. Moreover, $H_0 = W_0 + \mathbf{i}W_1 + \epsilon(pW_1 - qW_0) + \mathbf{h}(p^2 W_1 - pqW_0 - qW_1)$ and $H_1 - pH_0 = (W_1 - pW_0) - \mathbf{i}qW_0 - \epsilon qW_1 + \mathbf{h}(-pqW_1 + q^2 W_0)$. ■

Remark 11. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -1$ we have the generating function for the Fibonacci hybrid number sequence $\{FH_n\}$

$$\sum_{n=0}^{\infty} FH_n t^n = \frac{FH_0 + t(FH_1 - FH_0)}{1 - t - t^2} = \frac{(\mathbf{i} + \epsilon + 2\mathbf{h}) + t(1 + \epsilon + \mathbf{h})}{1 - t - t^2}.$$

Remark 12. For $W_0 = 0$, $W_1 = 1$, $p = 2$ and $q = -1$ we have the generating function for the Pell hybrid number sequence $\{PH_n\}$

$$\sum_{n=0}^{\infty} PH_n t^n = \frac{PH_0 + t(PH_1 - 2PH_0)}{1 - 2t - t^2} = \frac{(\mathbf{i} + 2\epsilon + 5\mathbf{h}) + t(1 + \epsilon + 2\mathbf{h})}{1 - 2t - t^2}.$$

Remark 13. For $W_0 = 0$, $W_1 = 1$, $p = 1$ and $q = -2$ we have the generating function for the Jacobsthal hybrid number sequence $\{JH_n\}$

$$\sum_{n=0}^{\infty} JH_n t^n = \frac{JH_0 + t(JH_1 - JH_0)}{1 - t - 2t^2} = \frac{(\mathbf{i} + \epsilon + 3\mathbf{h}) + t(1 + 2\epsilon + 2\mathbf{h})}{1 - t - 2t^2}.$$

4. Remarks

In this paper, the sequences of Horadam hybrid numbers, Fibonacci hybrid numbers, Pell hybrid numbers and Jacobsthal hybrid numbers were introduced. Some properties involving these sequences, including the Binet formulae and the ordinary generating functions were presented.

Horadam numbers are special case of numbers defined by linear recurrence relation and they can be defined for negative value of n. Horadam hybrid numbers also can be studied for Horadam numbers with integer domain. Then more properties of the Horadam hybrid numbers can be given.

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