Discussiones Mathematicae
General Algebra and Applications 38 (2018) 5–18
doi:10.7151/dmgaa.1286

WEAK RELATIVE COMPLEMENTS IN ALMOST DISTRIBUTIVE LATTICES

Ramesh Sirisetti

Department of Mathematics
GIT, GITAM University
Visakhapatnam- 530 045, India

e-mail: ramesh.sirisetti@gmail.com

and

G. Jogarao

Department of Mathematics
AIET Bhogapuram- 531 162, India

e-mail: jogarao.gunda@gmail.com

Abstract

In this paper, the concept of relative complementation in almost distributive lattice is generalized. We obtain several properties on the sets of weak relative complement elements. We prove a sufficient condition for a weakly relatively complemented almost distributive lattice with dense elements to become a generalized stone almost distributive lattice.

Keywords: dense elements, relative complements, weak relative complementation, almost distributive lattice, generalized stone almost distributive lattice.

2010 Mathematics Subject Classification: 06D75, 06B10.

1. Introduction

The class of distributive lattices plays a key role in the theory of lattice (Boolean algebras). Many authors generalized the concept of distributive lattice in different aspects, one of them, Swamy and Rao [8] introduced the concept of **A**lmost **D**istributive **L**attice (ADL) as a common abstraction of ring theoretic and lattice theoretic generalization of a Boolean algebra which satisfies almost all conditions of a distributive lattice $(L, \wedge, \vee, 0)$ except the commutativity of $\wedge$, $\vee$ and the right distributivity of $\vee$ over $\wedge$. In fact, each one of these three conditions are equivalent to each other. The authors also introduced relatively complemented ADLs and studied broadly. In [7], Ramesh and Rao introduced weakly relatively complemented ADLs and obtained some equivalent conditions for an ADL to become weakly relatively complemented. For an ADL L with dense elements, the authors introduced the set $B_D(L) = \{a \in L/$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is a dense element$\}$ and proved that $B_D(L)$ is always a weakly relatively complemented ADL.

In this paper, we introduce weak relative complements in ADLs and obtain several properties on them. We present a class of weakly relatively complemented subADLs in an ADL. We characterize weakly relatively complemented ADLs in terms of annihilator ideals. We derive a sufficient condition for an ADL to become a weakly relatively complemented ADL. Also, we obtain some necessary and sufficient conditions for a weakly relatively complemented ADL to become a Boolean algebra. Finally, we obtain a sufficient condition for a weakly relatively complemented ADL with dense elements to become a generalized stone ADL.

2. Preliminaries

At first, we remind that the notion of almost distributive lattice and necessary properties.

Definition [8]**.** An algebra $(L, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is said to be an *almost distributive lattice* (abbreviated: ADL), if it satisfies the following

(i) $0 \wedge a = 0$

(ii) $a \vee 0 = a$

(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(vi) $(a \vee b) \wedge b = b$

for all $a, b, c \in L$.

Definition [8]**.** Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define

$$x \wedge y = \begin{cases} x_0 & \text{if } x = x_0 \\ y & \text{if } x \neq x_0 \end{cases} \qquad\qquad x \vee y = \begin{cases} y & \text{if } x = x_0 \\ x & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \wedge, \vee, x_0)$ is an ADL with x_0 as its zero element. This ADL, which is not a lattice, is called a *discrete ADL*.

Throughout this paper L stands for an ADL $(L, \wedge, \vee, 0)$ unless otherwise mentioned. For any $a, b \in L$, we say that a *is less than or equal to* b and write $a \leq b$ if $a \wedge b = a$ or, equivalently $a \vee b = b$. It is easy to observe that $\leq$ is a partial ordering on L. An element $a \in L$ is said to be the *greatest element*, if $x \leqslant a$ for all $x \in L$.

Lemma 1 [8]**.** *For any $a, b, c \in L$, we have the following:*

(i) $a \wedge 0 = 0$ *and* $0 \vee a = a$

(ii) $a \wedge a = a \vee a = a$

(iii) $a \vee (b \vee a) = a \vee b$

(iv) $\wedge$ *is associative*

(v) $a \wedge b \wedge c = b \wedge a \wedge c$

(vi) $a \wedge b = 0 \iff b \wedge a = 0$

(vii) $a \wedge b \leq b$ *and* $a \leq a \vee b$

(viii) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(ix) $a \vee b = b \vee a \Longleftrightarrow a \wedge b = b \wedge a$.

Lemma 2 [8]**.** *The following are equivalent in L:*

(i) $(a \wedge b) \vee a = a$, *for all* $a, b \in L$

(ii) $a \wedge b = b \wedge a$, *for all* $a, b \in L$

(iii) $a \vee b = b \vee a$, *for all* $a, b \in L$

(iv) $(L, \wedge, \vee)$ *is a distributive lattice.*

A non-empty subset I (resp., F) of L is said to be an *ideal* (resp., *filter*), if for any $a, b \in I$ (resp., F) and $x \in L$, $a \vee b$, $a \wedge x \in I$ (resp., $a \wedge b, x \vee a \in F$). For any non-empty subset S of L, $(S] = \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_1, s_2, \ldots, s_n \in S,\, x \in L$ and n is a positive integer$\}$ is the smallest ideal containing S. In particular, for any $a \in L$, $(a] = \{a \wedge x \mid x \in L\}$ is the principal ideal generated by a. For $a, b \in L$, the greatest lower bound of $(a]$ and $(b]$ is $(a \wedge b]$ and the least upper bound of $(a]$ and $(b]$ is $(a \vee b]$. For any non-empty subset A of L, the set $A^* = \{x \in L \mid a \wedge x = 0$, for all $a \in A\}$ is called the *annihilator* of A in L. It becomes an ideal in L. In particular, for any $a \in L$, $\{a\}^* = (a)^*$, where $(a) = (a]$ is the principal ideal generated by a.

Lemma 3 [3, 4]**.** *For any $a, b, c \in L$, we have the following:*

(i) $a \leq b \Longrightarrow (b)^* \subseteq (a)^*$

(ii) $(a)^{***} = (a)^*$

(iii) $(a \vee b)^* = (a)^* \cap (b)^*$

(iv) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$

(v) $(a)^* \subseteq (b)^* \Longleftrightarrow (b)^{**} \subseteq (a)^{**}$

(vi) $a \in (a)^{**}$

(vii) $(a \vee b)^* = (b \vee a)^*$

(viii) $(a \wedge b)^* = (b \wedge a)^*$

(ix) $(a)^* = (b)^* \Longleftrightarrow (a)^{**} = (b)^{**}$.

An element d in L is said to be *dense*, if $(d)^* = \{0\}$. Let us denote by D the set of dense elements in L. Then D is a filter (provided $D \neq \emptyset$). Moreover, if $d \in D$, then $d \vee x$, $x \vee d \in D$ for all $x \in L$. An element $m \in L$ is said to be *maximal*, if $m \wedge x = x$ for all $x \in L$. It is easy to observe that every maximal element is dense.

Definition [8]**.** Given a, b in L, an element x of L is said to be a *relative complement of a with respect to b*, if $a \wedge x = 0$ and $a \vee x = a \vee b$.

Definition [7]**.** L is said to be *weakly relatively complemented*, if for any $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.

Theorem 4 [7]**.** *If every non-zero element is dense in L, then L is weakly relatively complemented.*

3. Weak relative complements

In this section, we introduce weak relative complements in ADLs. We present a class of weakly relatively complemented subADLs in an ADL. We obtain several properties on the sets of weak relative complements.

Definition. Given a, b in L, an element x of L is said to be a *weak relative complement of a with respect to b*, if $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.

Example 5. Every relative complement element is a weak relative complement.

Remark 6. But a weak relative complement element need not be relative complement. For, see the following:

Example 7 [7]**.** Let $X_2 = \{0, a\}$ and $X_3 = \{0, b_1, b_2\}$ be two discrete ADLs. Then $X_2 \times X_3 = \{(0,0), (0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. Take $L = \{0, c_1, c_2, c_3, m_1, m_2\}$, where $0 = (0,0), c_1 = (0,b_1), c_2 = (0,b_2), c_3 = (a,0), m_1 = (a,b_1), m_2 = (a,b_2)$. Define $\wedge$, $\vee$ on L as follows:

$\wedge$	0	c_1	c_2	c_3	m_1	m_2
0	0	0	0	0	0	0
c_1	0	c_1	c_2	0	c_1	c_2
c_2	0	c_1	c_2	0	c_1	c_2
c_3	0	0	0	c_3	c_3	c_3
m_1	0	c_1	c_2	c_3	m_1	m_2
m_2	0	c_1	c_2	c_3	m_1	m_2

$\vee$	0	c_1	c_2	c_3	m_1	m_2
0	0	c_1	c_2	c_3	m_1	m_2
c_1	c_1	c_1	c_1	m_1	m_1	m_1
c_2	c_2	c_2	c_2	m_2	m_2	m_2
c_3	c_3	m_1	m_2	c_3	m_1	m_2
m_1	m_1	m_1	m_1	m_1	m_1	m_1
m_2	m_2	m_2	m_2	m_2	m_2	m_2

Then $(L, \wedge, \vee, 0)$ is an ADL but not a lattice. For any $a, b \in L$, define x by

$$x = \begin{cases} b, & \text{if } a \wedge b = 0 \\ 0, & \text{if } a \text{ is dense or } a \vee b = a \\ c_3, & \text{otherwise.} \end{cases}$$

Then $x \in L$, $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. Therefore x is a weak relative complement of a with respect to b. Now, for $c_3, c_1 \in L$, c_2 is a weak relative complement element of c_3 with respect to c_1. But c_2 is not a relative complement element of c_3 with respect to c_1 (because $c_3 \wedge c_2 = 0$ and $c_3 \vee c_2 = m_2 \neq m_1 = c_3 \vee c_1$).

Given $a, b \in L$, the relative complement of a with respect to b exists and is unique [8], but a weak relative complement of a with respect to b exists and need not be unique. See Example 7, for $c_3, c_1 \in L$, there exist $x_1 = c_1, x_2 = c_2 \in L$ such that $c_3 \wedge x_1 = 0$, $(c_3 \vee x_1)^* = (c_3 \vee c_1)^*$ and $c_3 \wedge x_2 = 0$, $(c_3 \vee x_2)^* = (c_3 \vee c_1)^*$. So that $x_1 = c_1 \neq c_2 = x_2$. Therefore x_1 and x_2 are two different weak relative complements of c_3 with respect to c_1.

Given a, b in L, the set of weak relative complements of a with respect to b is denoted by $\langle a, b \rangle$. For any $a \in L$, $\langle a, 0 \rangle$ and $\langle 0, a \rangle$ are non-empty.

The following theorem has a straightforward proof from the definition of a weakly relatively complemented ADL.

Theorem 8. *L is weakly relatively complemented if and only if $\langle a, b \rangle$ is non-empty, for all $a, b \in L$.*

L is a *disjunctive ADL* [2], if for any $x, y \in L$, $x \neq y$ implies $(x)^* \neq (y)^*$.

Theorem 9 [7]. *If L is disjunctive, then every weak relative complement element is relative complement.*

Now, we have the following theorem.

Theorem 10. *Let $a, b \in L$ such that $\langle a, b\rangle \neq \phi$. Then $\langle a, b\rangle$ is closed under $\wedge$ and $\vee$. Moreover $\langle a, b\rangle \cup \{0\}$ is a weakly relatively complemented subADL of L.*

Proof. Let $x, y \in \langle a, b\rangle$. Then $a \wedge x = 0 = a \wedge y$ and $(a \vee x)^* = (a \vee b)^* = (a \vee y)^*$. Therefore $a \wedge (x \wedge y) = (a \wedge x) \wedge y = 0$ (by Lemma 1(iv)) and $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = 0$. Now,

$$\begin{aligned}[a \vee (x \wedge y)]^{**} &= [(a \vee x) \wedge (a \vee y)]^{**} \quad \text{(by Def. 1.1(L5)[8])}\\ &= (a \vee x)^{**} \cap (a \vee y)^{**} \quad \text{(by Lemma 3(iv))}\\ &= (a \vee b)^{**}.\end{aligned}$$

Therefore $[a \vee (x \wedge y)]^* = (a \vee b)^*$. Hence $x \wedge y \in \langle a, b\rangle$. Similarly,

$$\begin{aligned}[a \vee (x \vee y)]^{**} &= [(a \vee (x \vee y))^*]^* \\ &= [(a)^* \cap (x \vee y)^*]^* && \text{(by Lemma 3(iii))}\\ &= [(a)^* \cap ((x)^* \cap (y)^*)]^* && \text{(by Lemma 3(iii))}\\ &= [((a)^* \cap (x)^*) \cap ((a)^* \cap (y)^*)]^* \\ &= [(a \vee x)^* \cap (a \vee y)^*]^* && \text{(by Lemma 3(iii))}\\ &= [(a \vee b)^* \cap (a \vee b)^*]^* \\ &= (a \vee b)^{**}.\end{aligned}$$

Therefore $[a \vee (x \vee y)]^* = (a \vee b)^*$. Hence $x \vee y \in \langle a, b\rangle$ and $\langle a, b\rangle$ is closed under $\wedge$ and $\vee$. Thus $\langle a, b\rangle \cup \{0\}$ is a subADL of L. Let $x \in \langle a, b\rangle$ and $x \neq 0$. Then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. For $t \in \langle a, b\rangle$,

$$\begin{aligned} t \in (x)^* &\Rightarrow t \wedge x = 0 \\ &\Rightarrow t \wedge x = 0 = t \wedge a && \text{(since } t \in \langle a, b\rangle)\\ &\Rightarrow t \in (a \vee x)^* = (a \vee b)^* && \text{(since } x \in \langle a, b\rangle)\\ &\Rightarrow t \in (a \vee t)^* && \text{(since } (a \vee t)^* = (a \vee b)^*)\\ &\Rightarrow t \wedge t = 0 = t.\end{aligned}$$

Therefore x is dense in $\langle a, b\rangle$. Hence every non-zero element in $\langle a, b\rangle \cup \{0\}$ is dense. By Theorem 4, $\langle a, b\rangle \cup \{0\}$ is a weakly relatively complemented subADL of L. ■

Lemma 11. *For any $a, b \in L$, we have the following:*

(i) $\langle a, a\rangle = \{0\}$

(ii) $\langle a, 0\rangle = \{0\}$

(iii) $\langle a \vee b, b\rangle = \{0\}$

(iv) $\langle a, a \wedge b\rangle = \{0\}$.

Proof. (i) and (ii) are trivial. (iii) Let $x \in < a \vee b, b >$. Then $(a \vee b) \wedge x = 0$ and $[(a \vee b) \vee x]^* = [(a \vee b) \vee b]^*$. Therefore $x \in (a \vee b)^*$ and $[(a \vee b) \vee x]^* = (a \vee b)^*$ as $(a \vee b) \vee b = a \vee b$ (by Lemma 1(vii)). So that $x \in [(a \vee b) \vee x]^*$. Hence $x \wedge x = 0 = x$.

(iv) Let $x \in < a, a \wedge b >$. Then $a \wedge x = 0$ and $(a \vee x)^* = [a \vee (a \wedge b)]^*$. Therefore $x \in (a)^*$ and $(a)^* = (a \vee x)^*$ as $a \vee (a \wedge b) = a$ (by Lemma 1(vii)). So that $x \in (a \vee x)^*$. Hence $x \wedge x = 0 = x$. ■

Lemma 12. *For any $a, b \in L$, we have the following:*

(i) *If $0 \in \langle a, b \rangle$, then $(a)^* \subseteq (b)^*$*
(ii) *If $a \in \langle a, b \rangle$, then $a = 0 = b$*
(iii) *If d is dense in L and $d \in \langle a, b \rangle$, then $a = 0$ and b is dense.*
(iv) *$b \in \langle a, b \rangle$ if and only if $a \wedge b = 0$.*

Proof. (i) If $0 \in \langle a, b \rangle$, then $a \wedge 0 = 0$ and $(a \vee 0)^* = (a \vee b)^*$. Therefore $(a)^* = (a)^* \cap (b)^*$ (by Lemma 3(iii)). Hence $(a)^* \subseteq (b)^*$.

(ii) If $a \in \langle a, b \rangle$, then $a \wedge a = 0$ and $(a \vee a)^* = (a \vee b)^*$. Therefore $a = 0$ and $(0)^* = (b)^*$. Hence $b = 0$.

(iii) If $d \in \langle a, b \rangle$, then $a \wedge d = 0$ and $(a \vee d)^* = (a \vee b)^*$. Therefore $a = 0$ and $(a \vee b)^* = (a \vee d)^* = \{0\}$ (since d is dense). So that $(b)^* = \{0\}$. Hence $a = 0$ and b is dense.

(iv) If $b \in \langle a, b \rangle$, then $a \wedge b = 0$. The other direction is trivial. ■

Given an ADL L with dense elements, define $B_D(L) = \{a \in L \mid$ there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b$ is a dense element$\}$ [7]. It is always a weakly relatively complemented ADL (by Theorem 4.2 [7]).

Lemma 13. *For any $a, b \in L$, we have the following:*

(i) *If a is dense, then $\langle a, b \rangle = \{0\}$*
(ii) *If b is dense, then $\langle a, b \rangle \subseteq B_D(L)$*
(iii) *If b is maximal, then $\langle a, b \rangle \subseteq B_D(L)$*
(iv) *If b is dense, then $\langle 0, b \rangle = D$.*

Proof. (i) If a is dense, then $a \wedge 0 = 0$ and $(a \vee 0)^* = (a \vee b)^* = \{0\}$, for any $b \in L$(since $a, a \vee b$ are dense). Therefore $0 \in \langle a, b \rangle$. So that $\langle a, b \rangle \neq \phi$. Let $x \in \langle a, b \rangle$. Then $a \wedge x = 0$. Therefore $x = 0$ (since a is dense). Hence $\langle a, b \rangle = \{0\}$.

(ii) Let $x \in \langle a, b \rangle$. Then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. Since b is dense, $(a \vee x)^* = (a \vee b)^* = \{0\}$ (since $a \vee b$ is dense) . Therefore $a \vee x$ is dense. Hence $x \in B_D(L)$. Thus $\langle a, b \rangle \subseteq B_D(L)$.

(iii) Since every maximal element is dense, $\langle a, b \rangle \subseteq B_D(L)$.

(iv) Suppose b is dense. Now,

$$\begin{aligned}\langle 0,b\rangle &= \{x\in L \mid 0\wedge x=0 \text{ and } (0\vee x)^*=(0\vee b)^*\}\\ &= \{x\in L \mid (x)^*=(b)^*\}\\ &= \{x\in L \mid (x)^*=\{0\}\} \qquad \text{(since } b \text{ is dense)}\\ &= \{x\in L \mid x\in D\}\\ &= D.\end{aligned}$$

Therefore $\langle 0,b\rangle = D$. ■

Theorem 14. *If L is weakly relatively complemented and $a,b\in L$, then, $(b)^*\subseteq (x)^*$, for all $x\in\langle a,b\rangle$.*

Proof. Let $a,b\in L$. Then there exists $x\in L$ such that $a\wedge x=0$ and $(a\vee x)^*=(a\vee b)^*$. Now, for this x,

$$\begin{aligned}(x)^{**} &= [x\wedge(x\vee a)]^{**} && \text{(by Lemma 1.4(2)[8])}\\ &= (x)^{**}\cap(x\vee a)^{**} && \text{(by Lemma 3(iv))}\\ &= (x)^{**}\cap(a\vee b)^{**} && \text{(since } (a\vee x)^{**}=(a\vee b)^{**})\\ &= [x\wedge(a\vee b)]^{**} && \text{(by Lemma 3(iv))}\\ &= [(x\wedge a)\vee(x\wedge b)]^{**} && \text{(by Definition 1.1(L4)[8])}\\ &= (x\wedge b)^{**} && \text{(since } x\wedge a=0 \text{ and by Lemma 1(vi)).}\end{aligned}$$

Therefore $(x)^{**}=(x)^{**}\cap(b)^{**}$. So that $(x)^{**}\subseteq(b)^{**}$ and hence $(b)^*\subseteq(x)^*$ (by Lemma 3(ii)(v)). ■

By Theorem 8 and 14, we have the following:

Corollary 15. *Let $a,b\in L$ such that $\langle a,b\rangle\neq\phi$. Then $(b)^*\subseteq(x)^*$, for all $x\in\langle a,b\rangle$.*

Theorem 16. *For any $a,b,c\in L$, we have the following:*

(i) $\langle a,c\rangle=\langle a\wedge c,c\rangle$

(ii) $\langle a,c\rangle\cap\langle b,c\rangle\subseteq\langle a\vee b,c\rangle$.

Proof. (i) Let $x\in\langle a,c\rangle$. Then $a\wedge x=0$ and $(a\vee x)^*=(a\vee c)^*$. Therefore $a\wedge c\wedge x=0$. Now,

$$\begin{aligned}
[(a \wedge c) \vee x]^{**} &= [x \vee (a \wedge c)]^{**} && \text{(by Lemma 3(vii))} \\
&= [(x \vee a) \wedge (x \vee c)]^{**} \\
&= (x \vee a)^{**} \cap (x \vee c)^{**} && \text{(by Lemma 3(iv))} \\
&= (a \vee c)^{**} \cap [(x)^{*} \cap (c)^{*}]^{*} && \text{(since } (a \vee x)^{**} = (a \vee c)^{**}) \\
&= (a \vee c)^{**} \cap (c)^{**} && \text{(since } (c)^{*} \subseteq (x)^{*} \text{ and by Corollary 15)} \\
&= ((a \vee c) \wedge c)^{**} \\
&= (c)^{**}.
\end{aligned}$$

Therefore $[(a \wedge c) \vee x]^{*} = (c)^{*}$. Hence $x \in \langle a \wedge c, c \rangle$. On the other hand, let $x \in \langle a \wedge c, c \rangle$, then $a \wedge c \wedge x = 0$ and $((a \wedge c) \vee x)^{*} = ((a \wedge c) \vee c)^{*} = (c)^{*}$ (by Lemma 1.4(1) [8]). Therefore $a \wedge x \in (c)^{*}$. So that $a \wedge x \wedge ((a \wedge c) \vee x) = 0$. Hence $a \wedge x = 0$. For $t \in L$,

$$\begin{aligned}
t \in (a \vee x)^{*} &\Rightarrow t \wedge a = 0 = t \wedge x \\
&\Rightarrow t \wedge a \wedge c = 0 = t \wedge x \\
&\Rightarrow t \in ((a \wedge c) \vee x)^{*} \\
&\Rightarrow t \wedge c = 0 && \text{(since } ((a \wedge c) \vee x)^{*} = (c)^{*}) \\
&\Rightarrow t \in (a \vee c)^{*}.
\end{aligned}$$

Therefore $(a \vee x)^{*} \subseteq (a \vee c)^{*}$. Similarly we can prove $(a \vee c)^{*} \subseteq (a \vee x)^{*}$. So that $(a \vee c)^{*} = (a \vee x)^{*}$. Hence $x \in \langle a, c \rangle$. Thus $\langle a, c \rangle = \langle a \wedge c, c \rangle$.

(ii) Let $x \in \langle a, c \rangle \cap \langle b, c \rangle$. Then $a \wedge x = 0 = b \wedge x$ and $(a \vee x)^{*} = (a \vee c)^{*}$, $(b \vee x)^{*} = (b \vee c)^{*}$. Therefore $(a \vee b) \wedge x = 0$. For $t \in L$,

$$\begin{aligned}
t \in (a \vee b \vee x)^{*} &\Rightarrow t \wedge a = t \wedge b = t \wedge x = 0 \\
&\Rightarrow t \in (a \vee x)^{*} = (a \vee c)^{*} \\
&\Rightarrow t \wedge c = 0 \\
&\Rightarrow t \in (a \vee b \vee c)^{*}.
\end{aligned}$$

Therefore $(a \vee b \vee x)^{*} \subseteq (a \vee b \vee c)^{*}$. Similarly $(a \vee b \vee c)^{*} \subseteq (a \vee b \vee x)^{*}$. So that $(a \vee b \vee x)^{*} = (a \vee b \vee c)^{*}$ and $(a \vee b) \wedge x = 0$. Hence $x \in \langle a \vee b, c \rangle$. Thus $\langle a, c \rangle \cap \langle b, c \rangle \subseteq \langle a \vee b, c \rangle$. ∎

Theorem 17. *For any $a, b, c \in L$, we have the following:*

(i) $\langle a \wedge b, c \rangle = \langle b \wedge a, c \rangle$

(ii) $\langle a \vee b, c \rangle = \langle b \vee a, c \rangle$.

Proof. Let $a, b, c \in L$. Then, for any $x \in L$,

$$\begin{aligned}
(a \wedge b) \wedge x = 0 &\iff (b \wedge a) \wedge x = 0 && \text{(by Lemma 1(v))} \\
[(a \wedge b) \vee x]^* &\iff (a \wedge b)^* \cap (x)^* && \text{(by Lemma 3(iii))} \\
&\iff (b \wedge a)^* \cap (x)^* && \text{(by Lemma 3(viii))} \\
&\iff [(b \wedge a) \vee x]^*.
\end{aligned}$$

Therefore $\langle a \wedge b, c\rangle = \langle b \wedge a, c\rangle$. Similarly we can prove $\langle a \vee b, c\rangle = \langle b \vee a, c\rangle$. ∎

Lemma 18. *If L is weakly relatively complemented and $a, b, c \in L$, then, $\langle a, b\rangle = \langle a, c\rangle \iff (a \vee b)^* = (a \vee c)^*$.*

Proof. Suppose that $\langle a, b\rangle = \langle a, c\rangle$. Let $x \in \langle a, b\rangle$. Then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^* = (a \vee c)^*$. Conversely suppose that $(a \vee b)^* = (a \vee c)^*$. Let $x \in \langle a, b\rangle$. Then $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. Therefore $a \wedge x = 0$ and $(a \vee x)^* = (a \vee c)^*$. So that $x \in \langle a, c\rangle$. Hence $\langle a, b\rangle \subseteq \langle a, c\rangle$. Similarly we can prove $\langle a, c\rangle \subseteq \langle a, b\rangle$. Thus $\langle a, b\rangle = \langle a, c\rangle$. ∎

Theorem 19. *Let I be an ideal in a weakly relatively complemented ADL L. Then I^* is a weakly relatively complemented subADL of L.*

Proof. It is easy to prove that I^* is a subADL of L. Let $a, b \in I^* \subseteq L$. Then there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$. Since $a, b \in I^*$, we have $a \wedge y = 0 = b \wedge y$, for all $y \in I$. So that $y \wedge (a \vee b) = 0$, for all $y \in I$. That $y \in (a \vee b)^*$, for all $y \in I$. Therefore $I \subseteq (a \vee b)^* = (a \vee x)^*$. Hence $a \vee x \in (a \vee x)^{**} \subseteq I^*$ (by Lemma 3(v)(vi)). So $x = (a \vee x) \wedge x \in I^*$ (since I^* is an ideal). Therefore I^* is weakly relatively complemented. ∎

The above theorem characterizes a class of weakly relatively complemented subADLs in an ADL. It is not known if an arbitrary subADL of a weakly relatively complemented ADL is weakly relatively complemented.

Theorem 20. *If $(a)^*$ is a principal ideal for all $a \in L$, then L is weakly relatively complemented.*

Proof. Let $a, b \in L$. Then there exist $x, y \in L$ such that $(a)^* = (x]$ and $(b)^* = (y]$. For $t \in L$,

$$\begin{aligned}
t \in (a \vee x)^* &\Rightarrow t \in (a)^* \cap (x)^* && \text{(by Lemma 3(iii))} \\
&\Rightarrow t \in (x] \cap (x)^* = \{0\} && \text{(by Definition 3.1[3])} \\
&\Rightarrow t = 0.
\end{aligned}$$

Therefore $a \vee x$ is dense in L. Similarly we can prove $b \vee y$ is also dense in L.

Take $c = x \wedge b$. Then $a \wedge c = a \wedge x \wedge b = 0 \wedge b = 0$ (since $a \wedge x = 0$). Now, for $t \in L$,

$$\begin{aligned} t \in (a \vee c)^* &\Rightarrow t \wedge a = 0 = t \wedge c \\ &\Rightarrow t \wedge a \wedge b = 0 = t \wedge x \wedge b \\ &\Rightarrow t \wedge b \wedge (a \vee x) = 0 \\ &\Rightarrow t \wedge b = 0 && \text{(since } a \vee x \text{ is dense)} \\ &\Rightarrow t \wedge (a \vee b) = 0 \\ &\Rightarrow t \in (a \vee b)^*. \end{aligned}$$

Therefore $(a \vee c)^* \subseteq (a \vee b)^*$. Now,

$$\begin{aligned} (a \vee c) \wedge (a \vee b) &= a \vee (c \wedge b) && \text{(by Definition 1.1(}L5\text{) [8])} \\ &= a \vee (x \wedge b \wedge b) \\ &= a \vee (x \wedge b) && \text{(by Lemma 1(ii))} \\ &= a \vee c. \end{aligned}$$

Therefore $(a \vee c) \leq (a \vee b)$. So that $(a \vee b)^* \subseteq (a \vee c)^*$(by Lemma 3(i)). Hence $(a \vee b)^* = (a \vee c)^*$. Thus L is weakly relatively complemented. ∎

4. SOME RESULTS ON WEAKLY RELATIVELY COMPLEMENTED ADLS WITH DENSE ELEMENTS

In this section, we prove necessary and sufficient conditions for a weakly relatively complemented ADL with dense elements to become a Boolean algebra. We derive a necessary condition for a generalized stone ADL with dense elements. Finally, we prove a sufficient condition for a weakly relatively complemented ADL with dense elements to become a generalized Stone ADL.

Theorem 21. *If L is weakly relatively complemented with dense elements, then the following are equivalent:*

(i) *L has exactly one dense element*
(ii) *L is a Boolean algebra*
(iii) *L is disjunctive.*

Proof. Let L be a weakly relatively complemented ADL.

(i) $\Rightarrow$ (ii) Assume that L has exactly one dense element, say d. For any $a \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee d)^* = \{0\}$ (since $a \vee d$ is dense). Then $a \vee x$ is dense and $a \vee x = d$. Therefore $a \leq a \vee x = d$, for any $a \in L$. Therefore d is the greatest element in L. Hence L is complemented and bounded. For any $a, b \in L$,

$$\begin{aligned} a \wedge b &= a \wedge b \wedge d && \text{(since } d \text{ is the greatest element)} \\ &= b \wedge a \wedge d && \text{(by Lemma 1(v))} \\ &= b \wedge a. \end{aligned}$$

Therefore L is a bounded distributive lattice with the least element 0 and the greatest element d (by Theorem 1.13 [8]). Hence L is Boolean algebra.

(ii) $\Rightarrow$ (iii) Assume that L is a Boolean algebra. Let $a, b \in L$. Suppose $(a)^* = (b)^*$. For this $a, b \in L$, there exist $x, y \in L$ such that $a \wedge x = 0 = b \wedge y$ and $a \vee x = 1 = b \vee y$, where 1 is the greatest element in L. Therefore $x \in (a)^*$ and $y \in (b)^*$. So that $x \in (b)^*$ and $y \in (a)^*$ (since $(a)^* = (b)^*$). Hence $b \wedge x = 0 = a \wedge y$. Now,

$$\begin{aligned} a &= a \wedge 1 \\ &= a \wedge (b \vee y) \\ &= (a \wedge b) \vee (a \wedge y) && \text{(by Definition 1.1}(L4)\text{ [8])} \\ &= a \wedge b. && \text{(since } a \wedge y = 0\text{)} \end{aligned}$$

Therefore $a \leq b$. Similarly we can prove $b \leq a$. Hence $a = b$. Thus L is disjunctive.

(iii) $\Rightarrow$ (i) Assume that L is disjunctive. Suppose L has two dense elements, say a, b. Therefore $(a)^* = (b)^*$. Since L is disjunctive, $a = b$. Hence L has exactly one dense element. ■

L is a *generalized stone ADL* [4], if for any $a \in L, (a)^* \vee (a)^{**} = L$. Now, we have the following:

Theorem 22. *Every generalized stone ADL L with dense elements is weakly relatively complemented.*

Proof. Suppose that L is a generalized stone ADL. Let $x, y \in L$. Then $(x)^* \vee (x)^{**} = L = (y)^* \vee (y)^{**}$. Choose a dense element in L such that $d = a \vee b$ for some $a \in (x)^*$ and $b \in (x)^{**}$ (since L is a stone ADL (i.e., $L = (x)^* \vee (x)^{**}$)). For $s \in L$,

$$\begin{aligned} s \in (a \vee x)^* &\Rightarrow s \wedge a = 0 = s \wedge x \\ &\Rightarrow s \wedge a = 0 = s \wedge b && \text{(since } s \in (x)^* \text{ and } b \in (x)^{**}\text{)} \\ &\Rightarrow s \wedge (a \vee b) = 0 \\ &\Rightarrow s \wedge d = 0 \\ &\Rightarrow s = 0. && \text{(since } d \text{ is dense)} \end{aligned}$$

Therefore $a \vee x$ is dense. Take $t = a \wedge y$. Then $x \wedge t = x \wedge a \wedge y = 0$ (since

$a \wedge x = 0$ and by Lemma 1(vi)), and

$$\begin{aligned}
(x \vee t) \wedge (x \vee y) &= x \vee (t \wedge y) && \text{(by Definition 1.1}(L5)\text{ [8])} \\
&= x \vee (a \wedge y \wedge y) \\
&= x \vee (a \wedge y) && \text{(by Lemma 1(ii))} \\
&= x \vee t.
\end{aligned}$$

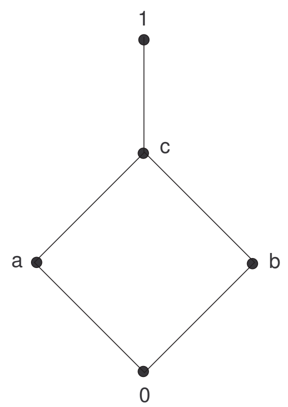
Therefore $(x \vee t) \leqslant x \vee y$. Hence $(x \vee y)^* \subseteq (x \vee t)^*$(by Lemma 3(i)). For $s \in L$,

$$\begin{aligned}
s \in (x \vee t)^* &\Rightarrow s \wedge x = 0 = s \wedge t \\
&\Rightarrow s \wedge x \wedge y = 0 = s \wedge a \wedge y \\
&\Rightarrow s \wedge y \wedge (x \vee a) = 0 && \text{(by Lemma 1(v))} \\
&\Rightarrow s \wedge y = 0 && \text{(since } x \vee a \text{ is dense)} \\
&\Rightarrow s \wedge (x \vee y) = 0 \\
&\Rightarrow s \in (x \vee y)^*.
\end{aligned}$$

Therefore $(x \vee t)^* \subseteq (x \vee y)^*$. Hence $(x \vee t)^* = (x \vee y)^*$. Thus L is weakly relatively complemented. ■

Remark 23. The converse of above statement need not be true. For, see the following:

Example 24. Let $L = \{0, a, b, c, 1\}$ be an ADL whose Hasse-diagram is



Then L is weakly relatively complemented. For $a, b \in L$, $(a)^* = \{0, b\}$ and $(a)^{**} = \{0, a\}$. Therefore $(a)^* \vee (a)^{**} = \{0, a, b, c\} \neq L$ and hence L is not a generalized stone ADL.

L is a *normal ADL* [5], if for any $x, y \in L$, $x \wedge y = 0$ implies $(x)^* \vee (y)^* = L$. Now, we have the following.

Theorem 25. *If L is weakly relatively complemented and normal with dense elements, then L is a generalized stone ADL.*

Proof. Let d be a dense element in L and $x \in L$. Then there exists $y \in L$ such that $x \wedge y = 0$ and $(x \vee y)^* = (x \vee d)^* = \{0\}$. Therefore $x \wedge y = 0$ and $x \vee y$ is dense. So that $y \in (x)^*$ also $(x)^{**} \subseteq (y)^*$ (by Lemma 3(v)). For $s, t \in L$,

$$\begin{aligned}
s \in (x)^* \text{ and } t \in (y)^* &\Rightarrow s \wedge x = 0 = t \wedge y \\
&\Rightarrow s \wedge x \wedge t = 0 = t \wedge y \wedge s \\
&\Rightarrow s \wedge t \wedge (x \vee y) = 0 && \text{(by Lemma 1(v)} \\
&\Rightarrow s \wedge t = 0 && \text{(since } x \vee y \text{ is dense)} \\
&\Rightarrow t \in (x)^{**} && \text{(since } s \in (x)^*) \\
&\Rightarrow (y)^* \subseteq (x)^{**}.
\end{aligned}$$

Therefore $(x)^{**} = (y)^*$. Since L is normal and $x \wedge y = 0$, $(x)^* \vee (y)^* = L$. Therefore $(x)^* \vee (x)^{**} = L$. Hence L is a generalized stone ADL. ■

Acknowledgement

The authors are very much thankful to the referee for valuable comments that greatly improved the manuscript.

References

[1] G. Birkhoff, Lattice theory (Amer. Math. Soc. Colloquium Pub., 1967).

[2] G.C. Rao and G. Nanaji Rao, *Dense elements in almost distributive lattices*, Southeast Asian Bull. Math. **27** (2004) 1081–1088.

[3] G.C. Rao and M. Sambasiva Rao, *Annihilator ideal in almost distributive lattices*, Int. Math. Forum **4** (2009) 733–746.

[4] G.C. Rao and M. Sambasiva Rao, *Annulets in almost distributive lattices*, European. J. Pure and Applied Math. **2** (2009) 58–72.

[5] G.C.Rao and S. Ravi Kumar, *Normal almost distributive lattices*, Southeast Asian Bull. Math. **32** (2008) 831–841.

[6] S. Burris and H.P. Sankappanavar, A course in universal algebra (Springer-Verlag, 1980).

[7] S. Ramesh and G. Jogarao, *Weakly relatively complemented almost distributive lattices*, Palestine J. Math. **6** (2017) 1–10.

[8] U.M. Swamy and G.C. Rao, *Almost distributive lattices*, J. Austral. Math. Soc. **31** (1981) 77–91.
doi:10.1017/S1446788700018498

Received 28 February 2017
Revised 27 June 2017
Accepted 6 November 2017