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RELATION BETWEEN BE-ALGEBRAS AND g-HILBERT ALGEBRAS

Akbar Rezaei

Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, Iran

e-mail: rezaei@pnu.ac.ir

and

Arsham Borumand Saeid

Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman, Kerman, Iran

e-mail: arsham@uk.ac.ir

Abstract

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "true" and as a generalization of this was defined the notion of g-Hilbert algebra. In this paper, we investigate the relationship between g-Hilbert algebras, gi-algebras, implication gruopoid and BE-algebras. In fact, we show that every g-Hilbert algebra is a self distributive BE-algebras and conversely. We show cannot remove the condition self distributivity. Therefore we show that any self distributive commutative BE-algebras is a gi-algebra and any gi-algebra is strong and transitive if and only if it is a commutative BE-algebra. We prove that the MV-algebra is equivalent to the bounded commutative BE-algebra.

Keywords: (Heyting, implication, (g-)Hilbert) algebra, BE/CI-algebra, dual $(S/Q/BCK)$-algebra, gi-algebra, implication groupoid, pre-logic, MV-algebra.

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1. Introduction and Preliminaries

H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra [13]. Rezaei *et al.* got some results on BE-algebras and introduced the notion of commutative ideals in BE-algebras and proved several characterizations of such ideals [17, 18]. Walendziak investigated the relationship between BE-algebras, implication algebras, and J-algebras [21]. Moreover, he defined commutative BE-algebras and stated that these algebras are equivalent to the commutative dual BCK-algebras. The concept of Hilbert algebra was introduced by Henkin and Skolem for investigations in intuitionistic and other non-classical logics. The prepositions of Hilbert algebras in algebraic logic were displayed by Chajda *et al.* Jun and Abbott. Also, we proved that every Hilbert algebra is a self distributive BE-algebra and commutative self distributive BE-algebra is a Hilbert algebra and we showed that cannot remove the conditions of commutativity and self distributivity [19]. Meng introduced the notion of CI-algebras as a generalization of BE-algebras and showed that any commutative (self distributive) CI-algebra is a BE-algebra [16]. Then Borumand Saeid proved that CI-algebras are equivalent to dual Q-algebras [2]. Since Heyting algebras generalize the well-known idea of Boolean algebras and most simply defined as a certain type of lattice, we investigated the relationship between BE-algebras and Heyting algebras and showed that a Heyting algebra is equivalent to the bounded commutative self distributive BE-algebra. Furthermore, we showed that every dual S-algebra is a commutative BE-algebras but the converse may be not true [20]. The g-Hilbert algebras are in fact the pre-logics and several results and examples can be found in [6]. In fact they showed that Hilbert algebras rise as quotient algebras of pre-logics by a congruence induced by a natural quasiorder. Furthermore, Borzooei *et al.* obtained some properties of g-Hilbert algebra and showed that any branch in commutative g-Hilbert algebra is a Boolean algebra [3].

In this paper, we show that the every g-Hilbert algebra is a self distributive BE-algebra and conversely. Also, we state relationship between gi-algebras, implication groupoids, MV-algebras and BE-algebras. Also, we show that any self distributive commutative BE-algebra is a gi-algebra and a gi-algebra P is strong and transitive if and only if P is a commutative BE-algebra. In fact, it is known that many properties of among algebraic structures are similar to ones of BE/CI-algebras, which motivates us to explore the interrelations. The concepts and methods of their respective algebras can therefore be applied to study deeply BE/CI-algebras.

2. Relation between BE-algebras and g-Hilbert algebras

Definition 2.1 [16]**.** By a *CI-algebra* we shall mean an algebra $(X;*,1)$ of type $(2,0)$ satisfying the following axioms:

$(CI1)$ $x*x=1$,

$(CI2)$ $1*x=x$,

$(CI3)$ $x*(y*z)=y*(x*z)$,

for all $x,y,z\in X$.

We introduce a relation "$\leq$" on X by $x\leq y$ if and only if $x*y=1$. CI-algebra X is said to be a *BE-algebra* if (BE) $x*1=1$, for all $x\in X$ [13]. CI/BE-algebra X is said to be *self distributive* if $x*(y*z)=(x*y)*(x*z)$, for all $x,y,z\in X$ [13]. CI/BE-algebra X is said to be *commutative* if $x*(x*y)=y*(y*x)$, for all $x,y\in X$ [21, 18].

Proposition 2.2. *If X is a self distributive CI-algebra, then it is a BE-algebra.*

Proof. Let $x\in X$. Using $(CI1)$ and self distributivity we have

$$x*1=x*(1*1)=(x*1)*(x*1)=1.$$

∎

Definition 2.3 [21]**.** An algebra $(X;*,1)$ of type $(2,0)$ is called a dual BCK-algebra (or briefly, $DBCK$-algebra) if

- $(BE1)$ $x*x=1$ for all $x\in X$,
- $(BE2)$ $x*1=1$ for all $x\in X$,
- $(DBCK1)$ $x*y=y*x=1\Longrightarrow x=y$,
- $(DBCK2)$ $(x*y)*((y*z)*(x*z))=1$,
- $(DBCK3)$ $x*((x*y)*y)=1$,

for all $x,y,z\in X$.

Proposition 2.4 [21]**.** *Any $DBCK$-algebra is a BE-algebra.*

Proposition 2.5 [21]**.** *If X is a commutative BE-algebra, then*

$$x*y=1 \text{ and } y*x=1 \text{ imply } x=y, \text{ for all } x,y\in X.$$

Theorem 2.6 [21]**.** *If X is a commutative BE-algebra, then X is a $DBCK$-algebra.*

Corollary 2.7 [21]**.** *X is a commutative BE-algebra if and only if it is a commutative $DBCK$-algebra.*

Definition 2.8 [10]**.** A *Hilbert algebra* is an algebra $(H; \to, 1)$ of type $(2, 0)$ satisfying the following axioms:

$(H1)$ $x \to (y \to x) = 1$,

$(H2)$ $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,

$(H3)$ $x \to y = 1$ and $y \to x = 1$ imply $x = y$,

for all $x, y, z \in H$.

It was proved by A. Diego that the class of all Hilbert algebras forms a variety, i.e., it is determined by a set of identities [10]. The following result is also adopted from [12, 10].

Proposition 2.9. *If* $(H; \to, 1)$ *be a Hilbert algebra, then,*

(i) $x \to x = 1$,

(ii) $1 \to x = x$,

(iii) $x \to 1 = 1$,

(iv) $x \to (y \to z) = y \to (x \to z)$,

(v) $x \to (y \to z) = (x \to y) \to (x \to z)$,

for all $x, y, z \in H$.

It can be easily checked that the binary relation "$\leq$ " introduced in a Hilbert algebra $(H; \to, 1)$ by

$$x \leq y \text{ if and only if } x * y = 1$$

is a partial order on H with 1 as the greatest element.

Definition 2.10 [3]**.** A *generalized Hilbert algebra* (or briefly, *g-Hilbert algebra*) is an algebra $(X; \to, 1)$ of type $(2, 0)$ which satisfies the following axioms:

$(GH1)$ $1 \to x = x$,

$(GH2)$ $x \to x = 1$,

$(GH3)$ $z \to (y \to x) = y \to (z \to x)$,

$(GH4)$ $z \to (y \to x) = (z \to y) \to (z \to x)$,

for all $x, y, z \in X$.

Example 2.11 [3]**.** Let $(X; \leq, 1)$ be an unital poset and implication " $\to$ " on X is defined as follows:

$$x \to y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$$

Then $(X; \to, 1)$ is a g-Hilbert algebra.

Theorem 2.12 [3]**.** *Any Hilbert algebra is a g-Hilbert algebra.*

The following example show that the converse of Theorem 2.12 is not correct in general.

Example 2.13 [3]**.** Let $X = \{1, a, b\}$ be a set with the following table.

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then $(X; *, 1)$ is g-Hilbert algebra but it is not a Hilbert algebra, since $a * b = b * a = 1$ but $a \neq b$.

Theorem 2.14 [3]**.** *An algebra $(X; \to, 1)$ is an implication algebra if and only if it is a commutative g-Hilbert algebra.*

Theorem 2.15 [19]**.** *Any Hilbert algebra is a self distributive BE-algebra.*

Theorem 2.16 [19]**.** *Any commutative self distributive BE-algebra is a Hilbert algebra.*

Lemma 2.17. *Any g-Hilbert algebra is a (self distributive) BE-algebra.*

Proof. Let $(X; \to, 1)$ be a g-Hilbert algebra. Set "$* := \to$ ". Hence $(X; *, 1)$ is a BE-algebra. Because it is sufficient prove that $x * 1 = 1$, for all $x \in X$. Using $(GH4)$, we have

$$x * 1 = x * (1 * 1) = (x * 1) * (x * 1) = 1. \qquad \blacksquare$$

Lemma 2.18. *Any self distributive BE-algebra is a g-Hilbert algebra.*

The following example, we show that the condition self distributivity in Lemma 2.18, is necessary.

Example 2.19 [21]**.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and " $*$ " be the binary operation on $\mathbb{N}_0$ is defined as follows:

$$x * y = \begin{cases} 0, & \text{if } y \leq x, \\ y - x, & \text{if } x < y. \end{cases}$$

Then $(\mathbb{N}_0; *, 1)$ is a BE-algebra. But it is not self distributive. Also, it is not a g-Hilbert algebra, because

$$3 * (4 * 5) = 3 * (5 - 4) = 3 * 1 = 0 \neq (3 * 4) * (3 * 5) = 1 * 2 = 2 - 1 = 1.$$

Theorem 2.20. *A BE-algebra is a g-Hilbert algebra if and only if it is self distributive.*

3. Relation between BE-algebras and gi-algebras

Definition 3.1 [7]**.** An algebra $\mathcal{A} = (A; *, 1)$ of type $(2, 0)$ is called an *implication groupoid* if it satisfies the following axioms:

$(IG1)$ $x * x = 1$,

$(IG2)$ $1 * x = x$,

for all $x \in X$. If $\mathcal{A}$, moreover, satisfies also self distributive (i.e., for all $x, y, z \in X$,

$$x * (y * z) = (x * y) * (x * z),$$

we call it *distributive implication groupoid.*

The concept of *implication algebra* was introduced by J.C. Abbott to describe properties of logical connective *"implication"* in a classical logic [1].

Proposition 3.2. *Any CI/BE-algebra is an implication groupoid.*

Proof. The axioms $(IG1)$ and $(IG2)$ are $(CI1)$ and $(CI2)$. ∎

Proposition 3.3. *Any distributive implication groupoid with condition $(CI3)$ is a BE-algebra.*

Proof. Let $x \in X$. We have $x * 1 = x * (x * x) = (x * x) * (x * x) = 1$. Hence (BE) is valid. ∎

Definition 3.4 [1]**.** A groupoid $(X; *)$ is called an *implication algebra* if it satisfies the following axioms:

$(I1)$ $(x * y) * x = x$,

$(I2)$ $(x * y) * y = (y * x) * x$,

$(I3)$ $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$.

Chen and Oliveira proved that in any implication algebra $(X; *)$ the identity $x * x = y * y$ holds for all $x, y \in X$, i.e., $x * x$ is an algebraic constant which is denoted by 1 [9]. It is well-known that every implication algebra is also a Hilbert algebra (and hence an implication groupoid).

Theorem 3.5 [7]**.** *A distributive implication groupoid is an implication algebra if and only if it is commutative.*

Definition 3.6 [23]**.** An algebra with a *generalized implication* (or briefly, *gi-algebra*) is a poset $(P, \leq)$ with a binary operation " $*$ " if it satisfies the following axioms:

$(I1)$ $\quad y \leq x * y,$

$(I2)$ $\quad x \leq y * z \;\Rightarrow\; y \leq x * z,$

for all $x, y, z \in X$.

Proposition 3.7. *Any self distributive commutative BE-algebra is a gi-algebra.*

Proof. Let $(X; *, 1)$ be a BE-algebra. Define a binary operation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

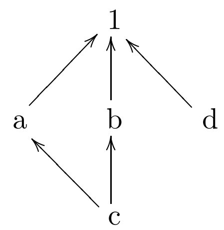
We note that "$\leq$" is reflexive by $(CI1)$. Since X is self distributive, we can see that the relation "$\leq$" is transitive. In fact, let $x \leq y$ and $y \leq z$. Then

$$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1.$$

Hence $x \leq z$. Now, since X is commutative, we can see that the relation is antisymmetric by Proposition 2.5. Hence "$\leq$" is a partial order on X and so $(X; \leq)$ is a poset. Also, since $y * (x * y) = 1$, we have $y \leq x * y$ and so $(I1)$. By using $(CI3)$ the condition $(I2)$ is valid. ■

The following example show that the converse of Proposition 3.7 is not true in general.

Example 3.8 [23]**.** Let $P = \{a, b, c, d, 1\}$ be a poset with the following diagram.



We define a binary operation " $*$ " on P by the following Cayley table.

$*$	1	a	b	c	d
1	1	a	b	c	1
a	1	1	b	b	1
b	1	a	1	a	1
c	1	1	1	1	1
d	1	a	b	c	1

Then $(P; *)$ is a gi-algebra but it is not a BE-algebra, since $1 * d = 1 \neq d$.

Definition 3.9 [23]**.** A *gi*-algebra P is said to be *strong* if:

$$(S) \quad x * y = 1 \text{ implies } x \leq y, \text{ for all } x, y \in P.$$

The strong *gi*-algebras are equivalent to the dual weak BCK-algebras introduced by Cirulis [5].

Definition 3.10 [23]**.** A *gi*-algebra P is said to be *transitive* if:

$$(T) \quad x * y \leq (z * x) * (z * y), \text{ for all } x, y, z \in P.$$

Proposition 3.11 [23]**.** *Let P be a gi-algebra. If P is transitive, then the condition $(CI3)$ holds.*

Theorem 3.12 [23]**.** *Let P be a gi-algebra. Then P is strong and transitive if and only if P is a $DBCK$-algebra.*

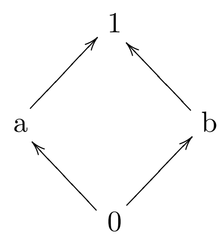
Corollary 3.13. *Let P be a gi-algebra. Then P is strong and transitive if and only if P is a commutative BE-algebra.*

Proof. It follows directly by Theorem 3.12 and Corollary 2.7. ■

The following example show that in Corollary 3.13 two conditions strongly and transitivity are necessary.

Example 3.14 [23]**.** (1) A *gi*-algebra from Example 3.8 is not strong, since $1 * d = 1 \nleq d$. Also, it is not a BE-algebra.

(2) Let $P = \{0, a, b, 1\}$ be a poset with the following diagram.



We define a binary operation "$*$" on P by the following Cayley table.

$*$	1	a	b	0
1	1	a	b	0
a	1	1	b	a
b	1	a	1	b
0	1	1	1	1

Then $(P;*)$ is a strong gi-algebra which is not transitive, since

$$0*a = 1 \not\leq a = (b*0)*(b*a).$$

Also, it is not a BE-algebra, since

$$a*(b*0) = a*b = b \neq b*(a*0) = b*a = a.$$

Definition 3.15 [6]**.** By a *pre-logic* it is meant a triplet $\mathcal{A} = (A;*,1)$ where A is a non-empty set, "$*$" is a binary operation on A and $1 \in A$ is a nullary operation such that the following hold:

$(P1)$ $x*x = 1$,

$(P2)$ $1*x = x$,

$(P3)$ $x*(y*z) = (x*y)*(x*z)$,

$(P4)$ $x*(y*z) = y*(x*z)$,

for all $x, y, z \in P$.

Lemma 3.16 [6]**.** *Let $\mathcal{A} = (A;*,1)$ be a pre-logic. Then*

(i) $x*1 = 1$,

(ii) $x*(y*x) = 1$,

for all $x, y \in X$.

Comparing Definition 3.15 with Proposition 2.9, we see that every Hilbert algebra is a pre-logic. Also, every self distributive CI/BE-algebra is equivalent to pre-logic.

Example 3.17 [6]**.** Let $A = \{1, a, b, c\}$ and the binary operation "$*$" is defined by the following table.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	1
c	1	a	1	1

Then $(A;*,1)$ is a pre-logic which is not a Hilbert algebra, since

$$b*c = c*b = 1, \text{ but } c \neq b.$$

The notion of MV-algebra, originally introduced by Chang, ia an attempt as developing a theory of algebraic systems that would correspond to the many valued propositional calculus: the axioms for this calculus are known as the Lukasiewicz axioms [8]. In this section, we state relation between the MV-algebra and the bounded commutative BE-algebra.

Definition 3.18 [4]**.** A BE-algebra $(X;*,1)$ is called bounded if there exists the smallest element 0 of X (i.e. $0*x=1$, for all $x\in X$).

The element 0 is the bottom element in X, since $0\leq x$, for all $x\in X$. Also, $x*0$ is called a *pseudocomplement* of x and we write $x^*=x*0$.

Example 3.19 [4]**.** (1) The interval $[0,1]$ of real numbers with the operation "$*$" defined by

$$x*y=\min\{1-x+y,1\},\ for\ all\ x,y\in X$$

is a bounded BE-algebra.

(2) Let $(X;*,1)$ be a BE-algebra, $0\notin X$ and $\bar{X}=X\cup\{0\}$. If we extensively define

$$0*x=0*0=1\ \ and\ \ x*0=0\ \ for\ all\ x\in X.$$

Then $(\bar{X};*,0,1)$ is a bounded BE-algebra with 0 as the smallest element.

Theorem 3.20 [14]**.** *A bounded commutative dual* BCK*-algebra* $(X,*,1,0)$ *is a* MV*-algebra* $(X,\oplus,\prime,0)$ *with the operations "*$\oplus$*" and "*$\prime$*" defined as following:*

$$x\oplus y=x'*y,\ \text{and}\ x'=x^*,$$

for all $x,y\in X$.

Theorem 3.21 [14]**.** *A* MV*-algebra* $(X,\oplus,\prime,0)$ *is a bounded commutative dual* BCK*-algebra* $(X,*,1,0)$ *with the operations "*$*$*" and the top element* 1 *defined as following:*

$$x*y=x'\oplus y,\ \text{and}\ 1=0',$$

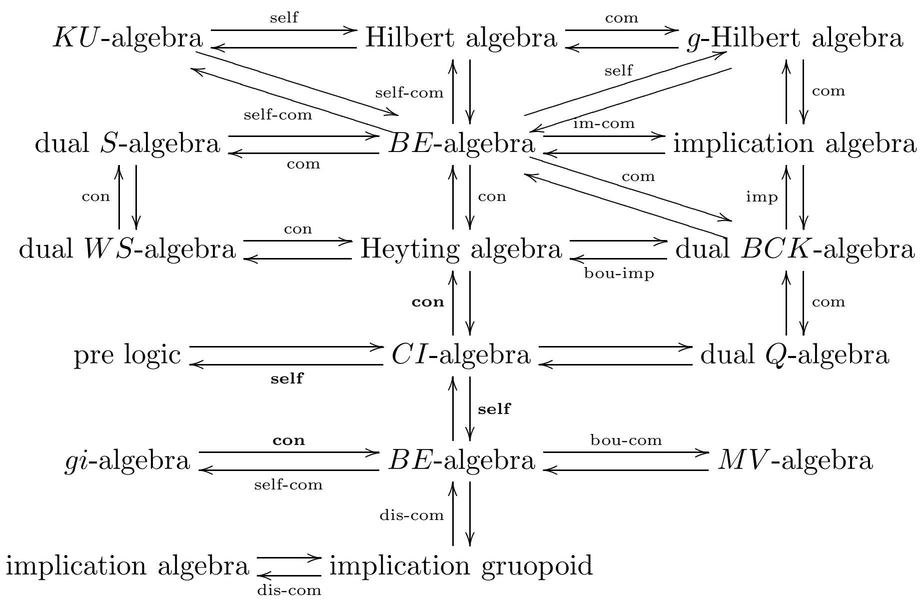
for all $x,y\in X$.

Corollary 3.22. *Any* MV*-algebra is equivalent to a bounded commutative BE-algebra.*

Proof. It follows directly by Theorems 3.20, 3.21 and Corollary 2.7. ■

4. Conclusion

Hilbert algebras represent the algebraic counterpart of the implicative fragment of intuitionistic propositional logic. In fact, Hilbert algebras are an algebraic counterpart of positive implicational calculus. Various type of generalization of algebraic structures were defined in the literature, one of them is g-Hilbert algebra. In this paper, we discuss relationships between g-Hilbert algebras, gi-algebras, implication gruopoid, pre-logics and MV-algebras with BE-algebras. Now, in the following diagram we summarize the results of this paper and the previous results in this filed. The mark $A \to B$ (respectively, $A \xrightarrow{ex} B$) means that A conclude B (respectively, A conclude B with condition "example" briefly "ex").



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