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IDEMPOTENT ELEMENTS OF WEAK PROJECTION GENERALIZED HYPERSUBSTITUTIONS

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Abstract

A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$ is a mapping σ which maps every operation symbol f_i to the term $\sigma(f_i)$ and may not preserve arity. It is the main tool to study strong hyperidentities that are used to classify varieties into collections called strong hypervarieties. Each generalized hypersubstitution can be extended to a mapping $\hat{\sigma}$ on the set of all terms of type τ. A binary operation on $\mathrm{Hyp}_{\mathrm{G}}(\tau)$, the set of all generalized hypersubstitutions of type τ, can be defined by using this extension. The set $\mathrm{Hyp}_{\mathrm{G}}(\tau)$ together with such a binary operation forms a monoid, where a hypersubstitution σ_{id}, which maps f_i to $f_i(x_1, \ldots, x_{n_i})$ for every $i \in I$, is the neutral element of this monoid. A weak projection generalized hypersubstitution of type τ is a generalized hypersubstitution of type τ which maps at least one of the operation symbols to a variable. In semigroup theory, the various types of its elements are widely considered. In this paper, we present the characterizations of idempotent weak projection generalized hypersubstitutions of type (m, n) and give some sufficient conditions for a weak projection generalized hypersubstitution of type (m, n) to be regular, where $m, n \geq 1$.

Keywords: idempotent, regular, generalized hypersubstitution, weak projection generalized hypersubstitution.

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1. Introduction

Let n be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an *alphabet* and its elements are called *variables*. Let $\{f_i : i \in I\}$

be the set of operation symbols, indexed by a nonempty set I. The sets X_n and $\{f_i : i \in I\}$ have to be disjoint. To every operation symbol f_i, we assign a natural number $n_i \geq 1$, called the *arity* of f_i. As in the definition of algebra, the sequence $\tau = (n_i)_{i\in I}$ of all arities is called the *type*. The classes of algebras are described by logical expressions. This formal language is built up by variables from an n-element set. With these notations for operation symbols and variables, we can define the terms of type τ, (see [5, 6, 7]).

Let $n \geq 1$, the *n-ary terms* of type τ are inductively defined as follows:

(i) every variable $x_i \in X_n$ is an n-ary term of type τ;

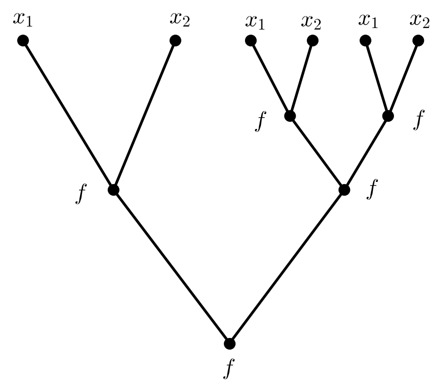
(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

The set $W_\tau(X_n) = W_\tau(\{x_1, \ldots, x_n\})$ of all n-ary terms of type τ is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii). We denote the set of all terms of type τ by

$$W_\tau(X) := \bigcup_{n\geq 1} W_\tau(X_n).$$

Terms can be visualized by tree diagrams, where the vertices are labelled by operation symbols and the leaves are labelled by variables (see [2]). Trees have many applications in mathematics, computer science, linguistic and in other fields. For instance, the following tree corresponds to the term:

$$f(f(x_1, x_2), f(f(x_1, x_2), f(x_1, x_2)))$$



In universal algebra, identities are used to classify algebras of the same type into varieties, and hyperidentities are use to classify varieties of the same type into hypervarieties. The concept of hypersubstitutions was introduced by Denecke, Lau, Pöschel and Schweigert [3] as a way of making precise the concept of hyperidentity and hypervarieties. In [11], Leeratanavalee and Denecke generalized the concepts of hypersubstitutions and hyperidentities to the concepts of generalized hypersubstitutions and strong hyperidentities, respectively. They used the generalized superpositions to study the concept of generalized hypersubstitutions. A *generalized superposition* of terms is a mapping $S^k : (W_\tau(X))^{k+1} \to W_\tau(X)$ which is defined by the following steps.

(i) If $t = x_j \in X_k$ where $1 \leq j \leq k$, then $S^k(x_j, t_1, \ldots, t_k) := t_j$.

(ii) If $t = x_j \in X \setminus X_k$ where $j > k$, then $S^k(x_j, t_1, \ldots, t_k) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$ and assume that $S^k(s_j, t_1, \ldots, t_k)$ are already defined for all $1 \leq j \leq n_i$, then

$$S^k(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_k) := f_i(S^k(s_1, t_1, \ldots, t_k), \ldots, S^k(s_{n_i}, t_1, \ldots, t_k)).$$

A *generalized hypersubstitution* of type τ is a mapping $\sigma : \{f_i : i \in I\} \to W_\tau(X)$ which maps each operation symbol of type τ to a term of the same type which may not preserve arity. We denote the set of all generalized hypersubstitutions of type τ by $\mathrm{Hyp}_\mathrm{G}(\tau)$.

The generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ on the set of all terms of type τ inductively defined as follows:

(i) $\hat{\sigma}[x] := x$ for any variable $x \in X$;

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for every n_i-ary operation symbol f_i and assume that $\hat{\sigma}[t_j]$ is already defined for all $1 \leq j \leq n_i$.

In [11], the authors defined a binary operation $\circ_\mathrm{G}$ on $\mathrm{Hyp}_\mathrm{G}(\tau)$ by $\sigma_1 \circ_\mathrm{G} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ where $\circ$ is usual composition, and showed that the structure $\mathbf{Hyp}_\mathrm{G}(\tau) := (\mathrm{Hyp}_\mathrm{G}(\tau); \circ_\mathrm{G}, \sigma_{\mathrm{id}})$ is a monoid where σ_{id} is an identity hypersubstitution. Moreover, if $\mathrm{Hyp}(\tau)$ denotes the set of all arity-preserving hypersubstitutions of type τ, then $\mathbf{Hyp}(\tau)$ forms a submonoid of $\mathbf{Hyp}_\mathrm{G}(\tau)$ under $\circ_\mathrm{G}$.

In 2000, Leeratanavalee and Denecke used generalized hypersubstitutions as a tool to study strong hyperidentities and used such strong hyperidentities to classify varieties into collections called *strong hypervarieties*. An identity $s \approx t$ of a variety V is called a *strong hyperidentity* if the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ holds in V for every generalized hypersubstitution $\sigma \in \mathrm{Hyp}_\mathrm{G}(\tau)$. If $\mathbf{M} := (M; \circ_\mathrm{G}, \sigma_{\mathrm{id}})$ is a submonoid of $\mathbf{Hyp}_\mathrm{G}(\tau)$, then $s \approx t$ is called an *M-strong hyperidentity* if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities for every $\sigma \in M$. A variety V is called M-strongly solid if every identity satisfied in V is an M-strong hyperidentity, and in case of $\mathbf{M} = \mathbf{Hyp}_\mathrm{G}(\tau)$, we will say V is a *strongly solid variety.* The set of all strongly solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ, (see [8, 9, 10, 11]). Moreover, they used the extensions of generalized hypersubstitutions to define tree transformations. It turns out that the algebraic properties of the set of tree transformations are described by algebraic properties of the set of all generalized hypersubstitutions [4].

These results suggest the importance of studying the particular monoid of generalized hypersubstitutions and its submonoids in both specific choices of τ and in general. In [5], Denecke and Wismath studied M-hyperidentities and M-solid varieties based on submonoids M of the monoid $\mathbf{Hyp}(\tau)$. They defined a number of natural such monoids based on various properties of hypersubstitutions. Later in [9], Leetatanavalee extended these concepts to generalized hypersubstitutions. A number of fairly natural examples of submonoids of the monoid $\mathbf{Hyp}_{\mathrm{G}}(\tau)$ of all generalized hypersubstitutions of a given type τ were given, (see [9]).

In semigroup theory, it is of interest to consider various types of its elements, including regular, idempotent, completely regular, etc. In [14], the authors characterized idempotent elements of $\mathbf{Hyp}_{\mathrm{G}}(2)$ and determined the order of each generalized hypersubstitution of this type. All regular elements of the monoid of all generalized hypersubstitutions of type (2) were studied by Puninagool and Leeratanavalee [10]. The generalized results were also given in [15], in fact, the idempotent and regular elements of $\mathbf{Hyp}_{\mathrm{G}}(n)$ were determined.

In 2007, Puninagool and Leeratanavalee [13] continued in this vein, by studying the semigroup properties of the submonoid pre-generalized hypersubstitutions of $\mathbf{Hyp}_{\mathrm{G}}(2,2)$. Indeed, they characterized idempotent elements of pre-generalized hypersubstitutions with a specific type $(2,2)$. Later in 2016, Lekkoksung and Jampachon gave a generalization of the results of this paper in [12] by considering idempotent elements of pre-generalized hypersubstitutions of type (m,n) where $m,n \geq 1$.

The idempotent elements of the set of all weak projection generalized hypersubstitutions of type $(2,2)$ were characterized in [10] by Leeratanavalee. In the present paper, we generalize the results of the paper given by Leeratanavalee, (see [10]). In fact, we extend his results to the type (m,n), where $m,n \geq 1$. Moreover, we give some sufficient conditions for a weak projection generalized hypersubstitution of type (m,n) to be regular.

2. Weak projection generalized hypersubstitutions of type (m,n)

In this section we provide the definitions of projection generalized hypersubstitutions, weak projection generalized hypersubstitutions and pre-generalized hypersubstitutions of type (m,n).

Definition. Let f and g be operation symbols of arity m and n, respectively. We denote the generalized hypersubstitution σ with $\sigma(f) = t_1$ and $\sigma(g) = t_2$ by σ_{t_1,t_2}.

(i) A generalized hypersubstitution σ of type (m, n) is called a *projection generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are variables. We denote the set of all projection generalized hypersubstitutions of type (m, n) by $\mathrm{P}_{\mathrm{G}}(m, n)$.

(ii) A generalized hypersubstitution σ of type (m, n) is called a *weak projection generalized hypersubstitution* if the term $\sigma(f)$ or $\sigma(g)$ is a variable. We denote the set of all projection generalized hypersubstitutions of type (m, n) by $\mathrm{WP}_{\mathrm{G}}(m, n)$.

(iii) A generalized hypersubstitution σ of type (m, n) is called a *pre-generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are not variables. We denote the set of all pre-generalized hypersubstitutions of type (m, n) by $\mathrm{Pre}_{\mathrm{G}}(m, n)$. That is, $\mathrm{Pre}_{\mathrm{G}}(m, n) := \mathrm{Hyp}_{\mathrm{G}}(m, n) \setminus \mathrm{WP}_{\mathrm{G}}(m, n)$.

Throughout this paper, we let f and g be operation symbols of arity m and n, respectively.

In [9], the author showed that $\mathrm{P}_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ and $\mathrm{WP}_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ are submonoids of $\mathbf{Hyp}_{\mathrm{G}}(\tau)$, moreover, $\mathrm{P}_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$ forms a submonoid of $\mathrm{WP}_{\mathrm{G}}(\tau) \cup \{\sigma_{\mathrm{id}}\}$. It is easy to see that every projection generalized hypersubstitution is idempotent and σ_{id} is also idempotent, (see [9]).

3. IDEMPOTENT WEAK PROJECTION HYPERSUBSTITUTIONS

For any semigroup S, an element e of S is *idempotent* if $ee = e$. This element is called an *idempotent element* of S, (see [1]). The concept of idempotent elements plays an important role in many branches of mathematics, for example, in semigroup theory and semiring theory. In this section, we give some sufficient and necessary conditions for elements of $\mathrm{WP}_{\mathrm{G}}(m, n) \setminus \mathrm{P}_{\mathrm{G}}(m, n)$ to be idempotent. Firstly, we present some notions which are used to prove our results.

Let F be a variable over the two-element alphabet $\{f, g\}$. For an arbitrary non-variable term t of type (m, n), we define semigroup words $P^i(t)$, where $i \in \mathbb{N}$, over $\{f, g\}$ by the following steps. For $t_1, \ldots, t_j \in W_{(m,n)}(X)$, where $j \in \{m, n\}$,

(i) if $t = F(t_1, \ldots, t_j)$ where F has arity j and $j < i$, then $P^i(t) = F$,

(ii) if $t = F(t_1, \ldots, x_i, \ldots, t_j)$ where F has arity j and $x_i \in X$, $1 \leq i \leq j$, then $P^i(t) = F$,

(iii) if $t = F(t_1, \ldots, t_i, \ldots, t_j)$ where F has arity j and $t_i \in W_{(m,n)}(X) \setminus X$, $1 \leq i \leq j$, then $P^i(t) = F(P^i(t_i))$.

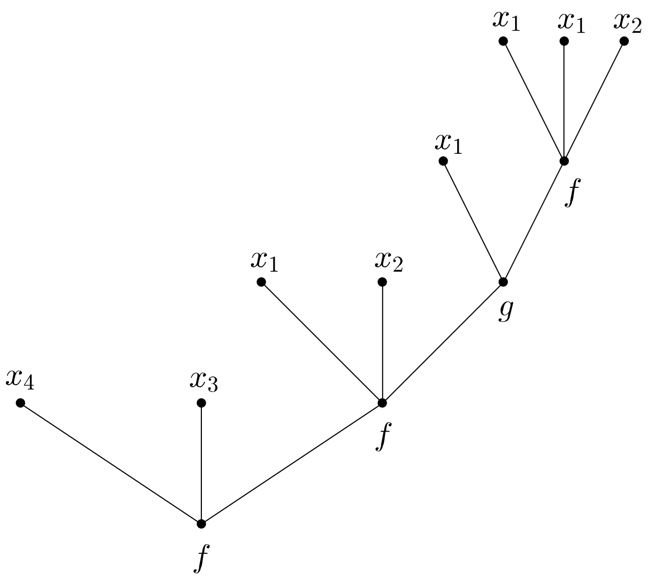
Instead of $F_1(F_2(\cdots F_n)\cdots)$ we will use $F_1 F_2 \cdots F_n \cdots$ for the semigroup words $P^i(t)$.

Let $t = F(t_1, \ldots, t_j)$ where F has arity $j \in \{m, n\}$ and $i \leq \max\{m, n\}$, we define $M^i(t)$ by

(i) if $t_i \in X$, then $M^i(t) = t_i$,

(ii) if $t_i = F'(s_1 \ldots, s_k)$ where F' has arity $k \in \{m, n\}$ and assume that $M^i(s_i)$ are already defined, then $M^i(t) = M^i(s_i)$.

For example, let f and g be operation symbols of type 3 and 2, respectively, and $t = f(x_4, x_3, f(x_1, x_2, g(x_1, f(x_1, x_1, x_2))))$. This term t can be visualized by the tree:



Then

$P^1(t) = f,$ $P^3(t) = ffg,$
$P^4(t) = f,$ $P^1(f(x_1, x_2, g(x_1, f(x_1, x_1, x_2)))) = f,$
$P^3(f(x_1, x_2, g(x_1, f(x_1, x_1, x_2)))) = fg,$ $P^1(g(x_1, f(x_1, x_1, x_2))) = g,$
$P^2(g(x_1, f(x_1, x_1, x_2))) = gf,$ $P^3(g(x_1, f(x_1, x_1, x_2))) = g,$

and

$M^1(t) = x_4,$ $M^2(t) = x_3,$ $M^3(t)$ is not define,
$M^2(g(x_1, f(x_1, x_1, x_2))) = x_1,$ $M^3(f(x_1, x_1, x_2)) = x_2.$

For $i \geq 1$ and $t \in W_{(m,n)}(X)$, we denote:

$$\begin{aligned}
\mathrm{var}(t) &:= \text{the set of all variables occurring in the term } t,\\
\mathrm{op}(t) &:= \text{the number of all operation symbols occurring in the term } t,\\
\mathrm{ops}(P^i(t)) &:= \text{the set of all operation symbols occurring in the semigroup}\\
& \text{word } P^i(t),\\
\mathrm{firstop}(t) &:= \text{the first operation symbol (from the left) occurring in}\\
& \text{the term } t.
\end{aligned}$$

The following proposition is a characterization of idempotent elements of generalized hypersubstitutions of type (m,n).

Proposition 1. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type (m,n). Then the following statements are equivalent:*

(i) *σ_{t_1,t_2} is idempotent;*

(ii) *$\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.*

Proof. (i) $\Rightarrow$ (ii): By the assumption,

$$\hat{\sigma}_{t_1,t_2}[t_1] = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(f)] = (\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_1,t_2})(f) = \sigma_{t_1,t_2}(f) = t_1.$$

Similarly, we obtain $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

(ii) $\Rightarrow$ (i): By our hypothesis, we obtain

$$(\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_1,t_2})(f) = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(f)] = \hat{\sigma}_{t_1,t_2}[t_1] = t_1 = \sigma_{t_1,t_2}(f).$$

Similarly, $(\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_1,t_2})(g) = \sigma_{t_1,t_2}(g)$. Hence, σ_{t_1,t_2} is idempotent. ∎

We will give the exact forms of the terms t_1 and t_2 that a weak projection generalized hypersubstitution σ_{t_1,t_2} is idempotent.

Lemma 2. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$ be idempotent. Then we have the following.*

(i) *If $t_1 \in X$ and $op(t_2) = 1$, then the operation symbol occurring in t_2 is g.*

(ii) *If $t_2 \in X$ and $op(t_1) = 1$, then the operation symbol occurring in t_1 is f.*

Proof. (i) Since $\mathrm{op}(t_2) = 1$, the term t_2 begins with the operation symbol f or g. If $t_2 = f(x_1, \ldots, x_m)$ where $x_1, \ldots, x_m \in X$, then $\hat{\sigma}_{t_1,t_2}[t_2]$ is a variable, this is a contradiction. Thus, $t_2 = g(x_1, \ldots, x_n)$ where $x_1 \ldots, x_n \in X$.

(ii) This is similar to (i). ∎

Next, we give sufficient and necessary conditions for elements of $\mathrm{WP_G}(m,n) \setminus \mathrm{P_G}(m,n)$ to be idempotent, where there is only one operation symbol occurring in one of these terms.

Proposition 3. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, $t_1 \in X$, $t_2 \in W_{(m,n)}(X) \setminus X$ and $t_2 = g(s_1, \ldots, s_n)$ where $op(t_2) = 1$. Then the following statements are equivalent:*

(i) *σ_{t_1,t_2} is idempotent;*

(ii) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j = x_j$.*

Proof. (i) $\Rightarrow$ (ii): Assume that σ_{t_1,t_2} is idempotent. Let $x_j \in \text{var}(t_2)$ where $1 \leq j \leq n$. Suppose that $s_j = x_l$ where $j \neq l$. By the assumption, we consider

$$\begin{aligned}\hat{\sigma}_{t_1,t_2}[t_2] &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], x_l, \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], x_l, \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]).\end{aligned}$$

Thus, we replace x_j in the term t_2 by x_l. It follows that $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$. This is a contradiction. Thus, $s_j = x_j$.

(ii) $\Rightarrow$ (i): It is clear that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. We now consider $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[g(s_1, \ldots, s_n)] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n])$. Then, if $x_j \in \text{var}(t_2)$ where $1 \leq j \leq n$, we replace x_j in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Hence, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. ∎

Similarly, we obtain the following result.

Proposition 4. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, $t_2 \in X$, $t_1 \in W_{(m,n)}(X) \setminus X$ and $t_1 = f(s_1, \ldots, s_m)$ where $op(t_1) = 1$. Then the following statements are equivalent:*

(i) *σ_{t_1,t_2} is idempotent;*

(ii) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j = x_j$.*

Next, we present sufficient and necessary conditions for $\sigma_{t_1,t_2} \in \mathrm{WP_G}(m,n) \setminus \mathrm{P_G}(m,n)$ to be idempotent, where the number of operation symbols occurring in one of these terms is more than one.

Proposition 5. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, where $t_1, t_2 \in W_{(m,n)}(X)$, $t_1 = x_i \in X_m$, $op(t_2) > 1$ and $P^i(t_2) = F_1 \cdots F_k$ where $F_l \in \{f, g\}$, $1 \leq l \leq k$. Then the following statements are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *in $P^i(t_2)$ there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = g$ with the subterm t_2' of t_2 where $t_2' = g(s_1, \ldots, s_n)$ such that $s_p \in W_{(m,n)}(X)$ and $1 \leq p \leq n$, and the following conditions are satisfied:*

   (i) *if $x_j \in var(t_2)$ and $s_j \in X$ where $1 \leq j \leq n$, then $s_j = x_j$;*

(ii) *if* $x_j \in var(t_2)$ *and* $s_j \in W_{(m,n)}(X) \setminus X$ *where* $1 \leq j \leq n$, *then* $ops(P^i(s_j)) = \{f\}$ *and* $M^i(s_j) = x_j$.

Proof. (1) $\Rightarrow$ (2): Since $P^i(t_2) = F_1 \cdots F_k$ where $F_l \in \{f, g\}$ for $1 \leq l \leq k$, then there must exist $q \in \{1, \ldots, k\}$ such that $F_q = g$ since otherwise $\hat{\sigma}_{t_1,t_2}[t_2] \in X$, which is a contradiction. Let $q \in \{1, \ldots, k\}$ be the smallest positive integer such that $F_q = g$ with the subterm t_2' of t_2 where $t_2' = g(s_1, \ldots, s_n)$ and $s_1, \ldots, s_n \in W_{(m,n)}(X)$. Now, $\text{ops}(F_1 \cdots F_{q-1}) = \{f\}$ or $F_1 = F_q = g$. Since σ_{t_1,t_2} is idempotent, we have that

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t_2'] \\ &= \hat{\sigma}_{t_1,t_2}[g(s_1, \ldots, s_n)] \\ &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]). \end{aligned}$$

From the last equality, $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) = t_2$, we consider the following cases.

(i) If $x_j \in \text{var}(t_2)$ and $s_j \in X$ where $1 \leq j \leq n$, then we replace x_j in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_j]$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, we have that $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Thus, $s_j = x_j$.

(ii) If $x_j \in \text{var}(t_2)$ and $s_j \in W_{(m,n)}(X) \setminus X$ where $1 \leq j \leq n$, then $s_j = f(r_1, \ldots, r_m)$ or $s_j = g(r_1', \ldots, r_n')$, where $r_1, \ldots, r_m, r_1', \ldots, r_n' \in W_{(m,n)}(X)$.

Suppose that $s_j = g(r_1', \ldots, r_n')$, where $r_1', \ldots, r_n' \in W_{(m,n)}(X)$. Then, we have that

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t_2'] \\ &= \hat{\sigma}_{t_1,t_2}[g(s_1, \ldots, s_j, \ldots, s_n)] \\ &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]). \end{aligned}$$

Since $x_j \in \text{var}(t_2)$, we replace x_j in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_j]$. We see that the term $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n])$ is longer than the term t_2 since $s_j = g(r_1', \ldots, r_n')$ is a subterm of t_2'. Hence, $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$, this is a contradiction. Thus, $s_j = f(r_1, \ldots, r_m)$ where $r_1, \ldots, r_m \in W_{(m,n)}(X)$. Next, we show that $\text{ops}(P^i(s_j)) = \{f\}$. Suppose that $\text{ops}(P^i(s_j)) \neq \{f\}$. By the definition of $P^i(t)$, we have that $\text{ops}(P^i(s_j)) \neq \emptyset$. Thus, there is $P^i(s_j) = F_1 \cdots F_k$ for some $k \in \mathbb{N}$ such that $F_q = g$ for some $1 \leq q \leq k$. Let l be the smallest positive integer such that $F_l = g$ with the subterm t_3' of s_j where $t_3' = g(s_1', \ldots, s_n')$ and

$s'_1, \ldots, s'_n \in W_{(m,n)}(X)$. Now, $\text{ops}(F_1 \cdots F_{q-1}) = \{f\}$ or $F_1 = F_q = g$. Since σ_{t_1,t_2} is idempotent, we have that

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t'_2] \\ &= \hat{\sigma}_{t_1,t_2}[g(s_1, \ldots, s_j, \ldots, s_n)] \\ &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]). \end{aligned}$$

Since $x_j \in \text{var}(t_2)$, we replace x_j in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_j]$. We see that the term $S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n])$ is longer than the term t_2 since $t'_3 = g(s'_1, \ldots, s'_n)$ is a subterm of s_j. Hence, $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$, this is a contradiction. Thus, $\text{ops}(P^i(s_j)) = \{f\}$. Finally, we assume that $M^i(s_j) = x_l$ where $l \neq j$. Since $\text{ops}(P^i(s_j)) = \{f\}$, then we have that

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= \hat{\sigma}_{t_1,t_2}[t'_2] \\ &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], \\ &\quad S^n(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[r_1], \ldots, \hat{\sigma}_{t_1,t_2}[r_{i-1}], x_l, \hat{\sigma}_{t_1,t_2}[r_{i+1}], \ldots, \hat{\sigma}_{t_1,t_2}[r_m]), \\ &\quad \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], x_l, \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]). \end{aligned}$$

Since $x_j \in \text{var}(t_2)$, we replace x_j in the term t_2 by x_l. This is a contradiction since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Therefore, $M^i(s_j) = x_j$.

(2) $\Rightarrow$ (1): In $P^i(t_2)$, let l be the smallest positive integer such that $F_l = g$ where $1 \leq l \leq k$ with the subterm t'_3 of t_2 where $t'_3 = g(\bar{s}_1, \ldots, \bar{s}_n)$. Then $\text{ops}(F_1 \cdots F_{q-1}) = \{f\}$ or $F_1 = F_q = g$ and

$$\begin{aligned} \hat{\sigma}_{t_1,t_2}[t_2] &= \hat{\sigma}_{t_1,t_2}[t'_3] \\ &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[\bar{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\bar{s}_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[\bar{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\bar{s}_n]). \end{aligned}$$

We consider these two cases.

(i) If $x_j \in \text{var}(t_2)$ and $\bar{s}_j \in X$ where $1 \leq j \leq n$, then we replace x_j in the term t_2 by x_j.

(ii) If $x_j \in \text{var}(t_2)$ where $1 \leq j \leq n$ and $s_j \in W_{(m,n)}(X) \setminus X$, then we replace x_j in the term t_2 by x_j since $\text{ops}(P^i(s_j)) = \{f\}$ and $M^i(s_j) = x_j$.

It follows that $S^n(t_2, \hat{\sigma}_{t_1,t_2}[\bar{s}_1], \ldots, \hat{\sigma}_{t_1,t_2}[\bar{s}_n]) = t_2$. Thus, σ_{t_1,t_2} is idempotent. ∎

On the other hand, we obtain the following proposition.

Proposition 6. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, where $t_1, t_2 \in W_{(m,n)}(X)$, $t_2 = x_i \in X_n$, $op(t_1) > 1$ and $P^i(t_1) = F_1 \cdots F_k$ where $F_l \in \{f, g\}$, $1 \leq l \leq k$. Then the following statements are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *in $P^i(t_1)$ there exists the smallest positive integer $l \in \{1, \ldots, k\}$ such that $F_l = f$ with the subterm t_1' of t_1 where $t_1' = f(s_1, \ldots, s_m)$ such that $s_p \in W_{(m,n)}(X)$ and $1 \leq p \leq m$, and the following conditions are satisfied:*

  (i) *if $x_j \in var(t_1)$ and $s_j \in X$ where $1 \leq j \leq m$, then $s_j = x_j$;*

  (ii) *if $x_j \in var(t_1)$ and $s_j \in W_{(m,n)}(X) \setminus X$ where $1 \leq j \leq m$, then $ops(P^i(s_j)) = \{g\}$ and $M^i(s_j) = x_j$.*

Lemma 7. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, where $t_1, t_2 \in W_{(m,n)}(X)$. If $t_1 = x_i \in X \setminus X_m$, $op(t_2) > 1$ and σ_{t_1,t_2} is idempotent, then firstop$(t_2) = g$.*

Proof. Suppose that firstop$(t_2) = f$. Then $t_2 = f(s_1, \ldots, s_m)$ where $s_1, \ldots, s_m \in W_{(m,n)}(X)$ and they are not all variables. Consider

$$\begin{aligned} t_2 &= \hat{\sigma}_{t_1,t_2}[t_2] \\ &= S^m(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]) \\ &= S^m(x_i, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_m]) \\ &= x_i. \end{aligned}$$

This is a contradiction. Thus, firstop$(t_2) = g$. ∎

Proposition 8. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, where $t_1, t_2 \in W_{(m,n)}(X)$. Let $t_1 = x_i \in X \setminus X_m$ and $op(t_2) > 1$. Then the following statements are equivalent:*

(1) *σ_{t_1,t_2} is idempotent;*

(2) *$t_2 = g(s_1, \ldots, s_n)$ where $s_r \in W_{(m,n)}(X)$, $1 \leq r \leq n$, and if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $s_j \in X$ and $s_j = x_j$.*

Proof. (1) $\Rightarrow$ (2): By Lemma 7, we have that $t_2 = g(s_1, \ldots, s_n)$ where $s_r \in W_{(m,n)}(X)$ and $1 \leq r \leq n$. Assume that $x_j \in \text{var}(t_2)$ where $1 \leq j \leq n$. Let $s_j \notin X$. Then either $s_j = f(s_1', \ldots, s_m')$ or $s_j = g(s_1'', \ldots, s_n'')$ where $s_k', s_l'' \in W_{(m,n)}(X)$, $1 \leq k \leq m$ and $1 \leq l \leq n$.

• If $s_j = f(s'_1, \ldots, s'_m)$ where $s'_k \in W_{(m,n)}(X)$ and $1 \leq k \leq m$, then we consider

$$\begin{aligned}\hat{\sigma}_{t_1,t_2}[t_2] &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, S^m(\sigma_{t_1,t_2}(f), \hat{\sigma}_{t_1,t_2}[s'_1], \ldots, \hat{\sigma}_{t_1,t_2}[s'_m]), \\ &\quad \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], x_i, \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]).\end{aligned}$$

Since $x_j \in \mathrm{var}(t_2)$, then we replace x_j in the term t_2 by x_i. It follows that $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$.

• If $s_j = g(s''_1, \ldots, s''_n)$ where $s''_l \in W_{(m,n)}(X)$ and $1 \leq l \leq n$, then we consider

$$\begin{aligned}\hat{\sigma}_{t_1,t_2}[t_2] &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], \\ &\quad S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s''_1], \ldots, \hat{\sigma}_{t_1,t_2}[s''_n]), \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]).\end{aligned}$$

Since $x_j \in \mathrm{var}(t_1)$, we replace x_j in the term t_2 by

$$t = S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s''_1], \ldots, \hat{\sigma}_{t_1,t_2}[s''_n]).$$

After replacing, the term

$$S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], t, \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n])$$

must be longer than the term t_2. Thus, $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$.

Altogether, we have $s_j \in X$. If $s_j = x_l$ where $l \neq j$, then we consider

$$\begin{aligned}\hat{\sigma}_{t_1,t_2}[t_2] &= S^n(\sigma_{t_1,t_2}(g), \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_j], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]) \\ &= S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_{j-1}], x_l, \hat{\sigma}_{t_1,t_2}[s_{j+1}], \ldots, \hat{\sigma}_{t_1,t_2}[s_n]).\end{aligned}$$

Since $x_j \in \mathrm{var}(t_2)$, we replace x_j in the term t_2 by x_l. This implies that $\hat{\sigma}_{t_1,t_2}[t_2] \neq t_2$. This is a contradiction. Therefore, $x_j = s_j$.

(2) $\Rightarrow$ (1): It is clear that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. We now consider $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[g(s_1, \ldots, s_n)] = S^n(t_2, \hat{\sigma}_{t_1,t_2}[s_1], \ldots, \hat{\sigma}_{t_1,t_2}[s_n])$. If $x_j \in \mathrm{var}(t_2)$ where $1 \leq j \leq n$, we replace x_j in the term t_2 by $\hat{\sigma}_{t_1,t_2}[s_j] = x_j$. Hence, $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. ■

Similarly, we obtain the following lemma and proposition.

Lemma 9. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, where $t_1, t_2 \in W_{(m,n)}(X)$. If $t_2 = x_i \in X \setminus X_n$, $op(t_1) > 1$ and σ_{t_1,t_2} is idempotent, then firstop$(t_1) = f$.*

Proposition 10. *Let $\sigma_{t_1,t_2} \in WP_G(m,n) \setminus P_G(m,n)$, where $t_1, t_2 \in W_{(m,n)}(X)$. Let $t_2 = x_i \in X \setminus X_n$ and $op(t_1) > 1$. Then the following statements are equivalent:*

1. *σ_{t_1,t_2} is idempotent;*
2. *$t_1 = f(s_1, \ldots, s_m)$ where $s_r \in W_{(m,n)}(X)$, $1 \leq r \leq m$, and if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $s_j \in X$ and $s_j = x_j$.*

4. REGULAR ELEMENTS OF WEAK PROJECTION HYPERSUBSTITUTIONS

An element a of a semigroup S is *regular* [1] if there is $x \in S$ such that $a = axa$. In this section, we give some sufficient conditions for a weak projection generalized hypersubstitution of type (m,n) to be regular.

Proposition 11. *Let σ_{t_1,t_2} be a weak projection generalized hypersubstitution of type (m,n). Then the following statements hold.*

(i) *If $t_1 \in X$ and $op(t_2) = 1$, then σ_{t_1,t_2} is regular.*

(ii) *If $t_2 \in X$ and $op(t_1) = 1$, then σ_{t_1,t_2} is regular.*

Proof. We prove only (i). For (ii) can be proved using similar arguments. Assume that $t_1 \in X$ and $\mathrm{op}(t_2) = 1$. If $t_2 = f(s_1, \ldots, s_m)$ and $x_j \in \mathrm{var}(t_2)$, $1 \leq j \leq m$, then we put $t_4 = t_1$ and $t_3 = g(\bar{s}_1, \ldots, \bar{s}_n)$ where $\bar{s}_j = s_k$ if $x_j = s_k$ for $1 \leq j, k \leq m$ and $\bar{s}_{j'} = x_{j'}$ if $x_{j'} \notin \mathrm{var}(t_2)$ for $1 \leq j' \leq n$. By simple calculation, we obtain $\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_3,t_4} \circ_{\mathrm{G}} \sigma_{t_1,t_2} = \sigma_{t_1,t_2}$. If $t_2 = g(s_1, \ldots, s_n)$ and $x_j \in \mathrm{var}(t_2)$, $1 \leq j \leq n$, then we put $t_3 = t_1$ and $t_4 = g(\bar{s}_1, \ldots, \bar{s}_n)$ where $\bar{s}_j = x_k$ if $x_j = s_k$ for $1 \leq j, k \leq n$ and $\bar{s}_{j'} = x_{j'}$ if $x_{j'} \notin \mathrm{var}(t_2)$ for $1 \leq j' \leq n$. By simple calculation, we obtain $\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_3,t_4} \circ_{\mathrm{G}} \sigma_{t_1,t_2} = \sigma_{t_1,t_2}$. That is, σ_{t_1,t_2} is regular. ■

Proposition 12. *Let σ_{t_1,t_2} be a weak projection generalized hypersubstitution of type (m,n) where $t_1 = x \in X$, $op(t_2) > 1$ and $t_2 = g(s_1, \ldots, s_n)$ for $s_p \in W_{(m,n)}(X)$, $1 \leq p \leq n$. Then σ_{t_1,t_2} is regular if one of the following conditions are satisfied:*

(i) *if $x_j \in var(t_2)$, $1 \leq j \leq n$, then $s_k = x_j$ for some $1 \leq k \leq n$;*

(ii) *if $x_j \in var(t_2)$ where $1 \leq j \leq n$, then $ops(P^{fix}(s_k)) = \{f\}$ and $M^{fix}(s_k) = x_j$ for some $1 \leq k \leq n$ and $fix \in \mathbb{N}$.*

Proof. To find $\sigma_{t_3,t_4} \in \mathrm{WP_G}(m,n)$ such that $\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_3,t_4} \circ_{\mathrm{G}} \sigma_{t_1,t_2} = \sigma_{t_1,t_2}$. It is clear that $(\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_3,t_4} \circ_{\mathrm{G}} \sigma_{t_1,t_2})(f) = \sigma_{t_1,t_2}(f)$ for all $\sigma_{t_3,t_4} \in \mathrm{WP_G}(m,n)$ since t_1 is a variable. Thus, we have to find $\sigma_{t_3,t_4} \in \mathrm{WP_G}(m,n)$ such that $(\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_3,t_4} \circ_{\mathrm{G}} \sigma_{t_1,t_2})(g) = \sigma_{t_1,t_2}(g)$. By assumption, we put $t_3 = x_{\mathrm{fix}}$ and

$t_4 = g(\bar{s}_1, \ldots, \bar{s}_n)$ where $\bar{s}_j = x_k$ and $\bar{s}_{j'} = x_{j'}$ if $x_{j'} \notin \mathrm{var}(t_2)$, $1 \leq j' \leq n$. It follows that

$$(\sigma_{t_1,t_2} \circ_{\mathrm{G}} \sigma_{t_3,t_4} \circ_{\mathrm{G}} \sigma_{t_1,t_2})(g) = \hat{\sigma}_{t_1,t_2}[\hat{\sigma}_{t_3,t_4}[t_2]] = \hat{\sigma}_{t_1,t_2}[g(x_1, \ldots, x_n)] = t_2.$$

Therefore, σ_{t_1,t_2} is a regular. ■

Proposition 13. *Let σ_{t_1,t_2} be a weak projection generalized hypersubstitution of type (m, n) where $t_1 = x \in X$, $op(t_2) > 1$ and $t_2 = f(s_1, \ldots, s_m)$ for $s_p \in W_{(m,n)}(X)$, $1 \leq p \leq m$. Then σ_{t_1,t_2} is regular if one of the following conditions are satisfied:*

(i) *if $x_j \in var(t_2)$, $1 \leq j \leq n$, then $s_k = x_j$ for some $1 \leq k \leq m$;*

(ii) *if $x_j \in var(t_2)$ where $1 \leq j \leq m$, then $ops(P^{fix}(s_k)) = \{g\}$ and $M^{fix}(s_k) = x_j$ for some $1 \leq k \leq m$ and $fix \in \mathbb{N}$.*

Similarly, we obtain the following propositions.

Proposition 14. *Let σ_{t_1,t_2} be a weak projection generalized hypersubstitution of type (m, n) where $t_2 = x \in X$, $op(t_1) > 1$ and $t_1 = f(s_1, \ldots, s_m)$ for $s_p \in W_{(m,n)}(X)$, $1 \leq p \leq m$. Then σ_{t_1,t_2} is regular if one of the following conditions are satisfied:*

(i) *if $x_j \in var(t_1)$, $1 \leq j \leq m$, then $s_k = x_j$ for some $1 \leq k \leq m$;*

(ii) *if $x_j \in var(t_1)$ where $1 \leq j \leq m$, then $ops(P^{fix}(s_k)) = \{g\}$ and $M^{fix}(s_k) = x_j$ for some $1 \leq k \leq m$ and $fix \in \mathbb{N}$.*

Proof. The proof is similar to Proposition 12 by choosing $t_3 = f(\bar{s}_1, \ldots, \bar{s}_m)$ where $\bar{s}_j = x_k$ and $\bar{s}_{j'} = x_{j'}$ if $x_{j'} \notin \mathrm{var}(t_2)$, $1 \leq j' \leq m$ and $t_4 = \mathrm{fix}$. ■

Proposition 15. *Let σ_{t_1,t_2} be a weak projection generalized hypersubstitution of type (m, n) where $t_2 = x \in X$, $op(t_1) > 1$ and $t_1 = g(s_1, \ldots, s_n)$ for $s_p \in W_{(m,n)}(X)$, $1 \leq p \leq n$. Then σ_{t_1,t_2} is regular if one of the following conditions are satisfied:*

(i) *if $x_j \in var(t_1)$, $1 \leq j \leq n$, then $s_k = x_j$ for some $1 \leq k \leq n$;*

(ii) *if $x_j \in var(t_1)$ where $1 \leq j \leq n$, then $ops(P^{fix}(s_k)) = \{f\}$ and $M^{fix}(s_k) = x_j$ for some $1 \leq k \leq n$ and $fix \in \mathbb{N}$.*

5. Conclusion and open problems

The theory of strong hyperidentities and strong hypervarieties is based on the monoid $\mathbf{Hyp}_G(\tau)$ of a fixed type τ. Moreover, the algebraic properties of the monoid $\mathbf{Hyp}_G(\tau)$ of a fixed type τ can be used to describe the algebraic properties of tree transformations, (see [4, 11]). Theses reasons demonstrate the importance of studying the monoid properties of $\mathbf{Hyp}_G(\tau)$ and its submonoids. For our future research, we will determine all maximal idempotent submonoids of the monoid of weak projection generalized hypersubstitutions of type (m, n), where $m, n \geq 1$. We pose the following open problems.

1. Give the characterization of a weak projection generalized hypersubstitutions of type (m, n) being regular.

2. Determine Green's relations on $\mathbf{Hyp}_G(m, n)$.

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