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# SPECTRA OF *R*-VERTEX JOIN AND *R*-EDGE JOIN OF TWO GRAPHS

Arpita Das and Pratima Panigrahi

*Department of Mathematics*
*Indian Institute of Technology Kharagpur*
*India, 721302*

**e-mail:** arpita.das1201@gmail.com
pratima@maths.iitkgp.ernet.in

**Abstract**

The $R$-graph $\mathcal{R}(G)$ of a graph $G$ is the graph obtained from $G$ by introducing a new vertex $u_e$ for each $e \in E(G)$ and making $u_e$ adjacent to both the end vertices of $e$. In this paper, we determine the adjacency, Laplacian and signless Laplacian spectra of $R$-vertex join and $R$-edge join of a connected regular graph with an arbitrary regular graph in terms of their eigenvalues. Moreover, applying these results we construct some non-regular $A$-cospectral, $L$-cospectral and $Q$-cospectral graphs, and find the number of spanning trees.

**Keywords:** spectrum, cospectral graphs, $R$-vertex join, $R$-edge join.

**2010 Mathematics Subject Classification:** 05C50.

## 1. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *adjacency matrix* of $G$, denoted by $A(G)$, is an $n \times n$ symmetric matrix such that $A(u, v) = 1$ if and only if vertex $u$ is adjacent to vertex $v$ and 0 otherwise. If $D(G)$ is the diagonal matrix of vertex degrees of $G$, then the *Laplacian matrix* $L(G)$ and *signless Laplacian matrix* $Q(G)$ are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ respectively. For a given matrix $M$ of size $n$, we denote the characteristic polynomial $det(xI_n - M)$ of $M$ by $f_M(x)$. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ and $\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_n(G)$ respectively and the multiset of these eigenvalues is called as adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum respectively. Two graphs are said to be $A$-cospectral, $L$-cospectral and $Q$-cospectral if they have the same $A$-spectrum, $L$-spectrum and $Q$-spectrum respectively.

Many works have already done on different kinds of graph operations. One of this is join of two graphs. The *join* [5] of two graphs is their disjoint union together with all the edges that connect all the vertices of the first graph with all the vertices of the second graph. The *R-graph* $\mathcal{R}(G)$ [2] of a graph $G$ is the graph obtained from $G$ by adding a vertex $u_e$ and joining $u_e$ to the end vertices of $e$ for each $e \in E(G)$. The set of such new vertices is denoted by $I(G)$ i.e., $I(G) = V(\mathcal{R}(G))\backslash V(G)$. In this paper we are interested on finding adjacency, Laplacian and signless Laplacian spectrum of some $R$-joins of graphs, which are defined below.

**Definition.** Let $G_1$ and $G_2$ be two vertex-disjoint graphs with number of vertices $n_1$ and $n_2$, and edges $m_1$ and $m_2$, respectively. Then

(i) The *R-vertex join* [7] of $G_1$ and $G_2$, denoted by $G_1\langle \mathrm{v}\rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and $G_2$ by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The graph $G_1\langle \mathrm{v}\rangle G_2$ has $n_1+n_2+m_1$ vertices and $3m_1+n_1n_2+m_2$ edges.

(ii) The *R-edge join* [7] of $G_1$ and $G_2$, denoted by $G_1\langle \mathrm{e}\rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and $G_2$ by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. The graph $G_1\langle \mathrm{e}\rangle G_2$ has $n_1+n_2+m_1$ vertices and $m_1(3+n_2)+m_2$ edges.

In [6], Indulal computed adjacency spectra of subdivision-vertex join and subdivision-edge join for two regular graph in terms of their spectra. In [8], Liu and Zhang generalized the result by determining the $A$-spectra, $L$-spectra and $Q$-spectra of subdivision-vertex and subdivision-edge join for a regular graph and an arbitrary graph and also constructed infinite family of new integral graphs. In [7], Liu *et al.* formulated the resistance distances and Kirchhoff index of $G_1\langle \mathrm{v}\rangle G_2$ and $G_1\langle \mathrm{e}\rangle G_2$ respectively. Motivated by these works, here we determine the adjacency, Laplacian and signless Laplacian spectrum of $G_1\langle \mathrm{v}\rangle G_2$ and $G_1\langle \mathrm{e}\rangle G_2$ for a connected regular graph $G_1$ and an arbitrary regular graph $G_2$ in terms of the corresponding eigenvalues of $G_1$ and $G_2$. Some non-regular $A$-cospectral, $L$-cospectral and $Q$-cospectral graphs are also exhibited in Example 21 in Section 2.

Our results are based upon Lemma 1 and 2 stated below.

**Lemma 1** (Schur Complement [3])**.** *Suppose that the order of all four matrices $M$, $N$, $P$ and $Q$ satisfy the rules of operations on matrices. Then we have,*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |Q||M - NQ^{-1}P|, \textit{if } Q \textit{ is a non-singular square matrix,}$$
$$= |M||Q - NM^{-1}P|, \textit{if } M \textit{ is a non-singular square matrix.}$$

**Lemma 2** [8]. *Let $A$ be an $n \times n$ real matrix, and $J_{s\times t}$ denote the $s \times t$ matrix with all entries equal to one. Then*

$$det(A + \alpha J_{n\times n}) = det(A) + \alpha \mathbf{1}_n^T adj(A)\mathbf{1}_n,$$

*where $\alpha$ is an real number and $adj(A)$ is the adjugate matrix of $A$.*

For a graph $G$ with $n$ vertices and $m$ edges, the *vertex-edge incidence matrix* $R(G)$ [4] is a matrix of order $n \times m$, with entry $r_{ij} = 1$ if the $i^{th}$ vertex is incident to the $j^{th}$ edge, and 0 otherwise. In particular, if $G$ is an $r$-regular graph then $R(G)R(G)^T = A(G) + rI_n$ and $R(G)^T R(G) = A(l(G)) + 2I_m$, where $l(G)$ is the line graph.

Let $t(G)$ denote the number of spanning trees of $G$. It is well known [2] that if $G$ is a connected graph on $n$ vertices with Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$, then $t(G) = \frac{\mu_2(G)\cdots\mu_n(G)}{n}$.

## 2. Our results

Throughout the paper for any integer $k$, $I_k$ denotes the identity matrix of size $k$, $\mathbf{1}_k$ denotes the column vector of size $k$ whose all entries are 1 and $J_{n_1\times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1.

**Definition** [1, 9]. The *$M$-coronal* $\Gamma_M(x)$ of an $n \times n$ matrix $M$ is defined as the sum of the entries of the matrix $(xI_n - M)^{-1}$(if exists), that is,

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1}\mathbf{1}_n.$$

The following Lemma is straightforward.

**Lemma 3** [1]. *If $M$ is an $n \times n$ matrix with each row sum equal to a constant $t$, then $\Gamma_M(x) = \frac{n}{x-t}$.*

Let $G_i$ be a graph with $n_i$ vertices and $m_i$ edges. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. Then $V(G_1) \cup I(G_1) \cup V(G_2)$ is a partition of $V(G_1\langle \mathrm{v}\rangle G_2)$ and $V(G_1\langle \mathrm{e}\rangle G_2)$.

### 2.1. Spectra of $R$-vertex join

The degree of the vertices of $G_1\langle \mathrm{v}\rangle G_2$ are $d_{G_1\langle \mathrm{v}\rangle G_2}(v_i) = 2d_{G_1}(v_i) + n_2$, $d_{G_1\langle \mathrm{v}\rangle G_2}(e_i) = 2$ and $d_{G_1\langle \mathrm{v}\rangle G_2}(u_i) = d_{G_2}(u_i) + n_1$.

### 2.1.1. *A*-spectra of *R*-vertex join

Let $G_i$ be a graph on $n_i$ vertices and $m_i$ edges. Then the adjacency matrix of $G_1\langle\mathrm{v}\rangle G_2$ can be written as:

$$A(G_1\langle\mathrm{v}\rangle G_2)=\begin{pmatrix} A(G_1) & R(G_1) & J_{n_1\times n_2}\\ R(G_1)^T & O_{m_1} & O_{m_1\times n_2}\\ J_{n_2\times n_1} & O_{n_2\times m_1} & A(G_2)\end{pmatrix}.$$

**Theorem 4.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the adjacency spectrum of $G_1\langle\mathrm{v}\rangle G_2$ consists of:*

(i) *The eigenvalue $\lambda_j(G_2)$ for every eigenvalue $\lambda_j$ $(j=2,3,\dots,n_2)$ of $A(G_2)$,*

(ii) *The eigenvalue $0$ with multiplicity $m_1-n_1$,*

(iii) *Two roots of the equation $x^2-\lambda_i(G_1)x-r_1-\lambda_i(G_1)=0$ for each eigenvalue $\lambda_i$ $(i=2,3,\dots,n_1)$ of $A(G_1)$,*

(iv) *Three roots of the equation $x^3-(r_1+r_2)x^2-(2r_1+n_1n_2-r_1r_2)x+2r_1r_2=0$.*

***Proof.*** The adjacency characteristic polynomial of $G_1\langle\mathrm{v}\rangle G_2$ is

$$\begin{aligned} f_{A(G_1\langle\mathrm{v}\rangle G_2)}(x) &= \det(xI_{n_1+n_2+m_1}-A(G_1\langle\mathrm{v}\rangle G_2))\\ &=\det\begin{pmatrix} xI_{n_1}-A(G_1) & -R(G_1) & -J_{n_1\times n_2}\\ -R(G_1)^T & xI_{m_1} & O_{m_1\times n_2}\\ -J_{n_2\times n_1} & O_{n_2\times m_1} & xI_{n_2}-A(G_2)\end{pmatrix}\\ &=\det(xI_{n_2}-A(G_2))\det(S)=\prod_{i=1}^{n_2}\{x-\lambda_j(G_2)\}\det(S),\end{aligned}$$

where

$$S=\begin{pmatrix} xI_{n_1}-A(G_1) & -R(G_1)\\ -R(G_1)^T & xI_{m_1}\end{pmatrix}-\begin{pmatrix}-J_{n_1\times n_2}\\ O_{m_1\times n_2}\end{pmatrix}(xI_{n_2}-A(G_2))^{-1}\begin{pmatrix}-J_{n_2\times n_1} & O_{n_2\times m_1}\end{pmatrix}$$

$$=\begin{pmatrix} xI_{n_1}-A(G_1)-\Gamma_{A(G_2)}(x)J_{n_1\times n_1} & -R(G_1)\\ -R(G_1)^T & xI_{m_1}\end{pmatrix}.$$

$$
\begin{aligned}
\det(S) &= x^{m_1}\det\big(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1\times n_1} - \tfrac{1}{x}R(G_1)R(G_1)^T\big)\\
&= x^{m_1}\Big[\det\big(xI_{n_1} - A(G_1) - \tfrac{1}{x}R(G_1)R(G_1)^T\big)\\
&\qquad - \Gamma_{A(G_2)}(x)\mathbf{1}_{n_1}^T\mathrm{adj}\big\{xI_{n_1} - A(G_1) - \tfrac{1}{x}R(G_1)R(G_1)^T\big\}\,\mathbf{1}_{n_1}\Big]\\
&= x^{m_1}\det\big(xI_{n_1} - A(G_1) - \tfrac{1}{x}R(G_1)R(G_1)^T\big)\\
&\qquad \Big[1 - \Gamma_{A(G_2)}(x)\mathbf{1}_{n_1}^T\,\big\{xI_{n_1} - A(G_1) - \tfrac{1}{x}R(G_1)R(G_1)^T\big\}^{-1}\mathbf{1}_{n_1}\Big]\\
&= x^{m_1}\det(xI_{n_1} - A(G_1) - \tfrac{1}{x}(r_1I_{n_1} + A(G_1)))\\
&\qquad \Big[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}R(G_1)R(G_1)^T}(x)\Big]\\
&= x^{m_1}\prod_{i=1}^{n_1}\big\{x - \lambda_i(G_1) - \tfrac{1}{x}(r_1 + \lambda_i(G_1))\big\}\left[1 - \tfrac{n_2}{(x-r_2)}\tfrac{n_1}{(x-r_1-\frac{2r_1}{x})}\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_{A(G_1\langle \mathrm{v}\rangle G_2)}(x) &= x^{m_1}\prod_{j=1}^{n_2}\big\{x - \lambda_j(G_2)\big\}\prod_{i=1}^{n_1}\big\{x - \lambda_i(G_1) - \tfrac{1}{x}(r_1 + \lambda_i(G_1))\big\}\\
&\qquad \left[1 - \tfrac{n_2}{(x-r_2)}\tfrac{n_1}{(x-r_1-\frac{2r_1}{x})}\right]\\
&= x^{m_1-n_1}\prod_{j=2}^{n_2}\{x - \lambda_j(G_2)\}\prod_{i=2}^{n_1}\{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\}\\
&\qquad \big\{x^3 - (r_1 + r_2)x^2 - (2r_1 + n_1n_2 - r_1r_2)x + 2r_1r_2\big\}. \qquad \blacksquare
\end{aligned}
$$

**Corollary 5.** *Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then the adjacency spectrum of $G\langle \mathrm{v}\rangle K_{p,q}$ consists of:*

(i) *The eigenvalue $0$ with multiplicity $m - n + p + q - 2$.*

(ii) *Two roots of the equation $x^2 - \lambda_i(G)x - r - \lambda_i(G) = 0$ for each eigenvalue $\lambda_i$ $(i = 2, 3, \dots, n)$ of $A(G)$.*

(iii) *Four roots of the equation $x^4 - rx^3 - (pq + pn + qn + 2r)x^2 + (pqr - 2pqn)x + 2pqr = 0$.*

**Corollary 6.** (a) *If $H_1$ and $H_2$ are $A$-cospectral regular graphs, and $H$ is a regular graph, then $H_1\langle \mathrm{v}\rangle H$ and $H_2\langle \mathrm{v}\rangle H$; and $H\langle \mathrm{v}\rangle H_1$ and $H\langle \mathrm{v}\rangle H_2$ are $A$-cospectral.*

(b) *If $F_1$ and $F_2$; and $H_1$ and $H_2$ are A-cospectral regular graphs, then $F_1\langle \mathrm{v}\rangle H_1$ and $F_2\langle \mathrm{v}\rangle H_2$ are A-cospectral.*

#### 2.1.2. $L$-spectra of $R$-vertex join

Let $G_i$ be a graph on $n_i$ vertices and $m_i$ edges. Then the Laplacian matrix of $G_1\langle \mathrm{v}\rangle G_2$ is given by [7]:

$$L(G_1\langle \mathrm{v}\rangle G_2) = \begin{pmatrix} (r_1+n_2)I_{n_1}+L(G_1) & -R(G_1) & -J_{n_1\times n_2} \\ -R(G_1)^T & 2I_{m_1} & O_{m_1\times n_2} \\ -J_{n_2\times n_1} & O_{n_2\times m_1} & n_1 I_{n_2}+L(G_2) \end{pmatrix}.$$

**Theorem 7.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the Laplacian spectrum of $G_1\langle \mathrm{v}\rangle G_2$ consists of:*

(i) *The eigenvalue $n_1+\mu_j(G_2)$ for every eigenvalue $\mu_j$ $(j=2,3,\ldots,n_2)$ of $L(G_2)$.*

(ii) *The eigenvalue 2 with multiplicity $m_1-n_1$.*

(iii) *Two roots of the equation $x^2-(2+r_1+n_2+\mu_i(G_1))x+2n_2+3\mu_i(G_1)=0$ for each eigenvalue $\mu_i$ $(i=2,3,\ldots,n_1)$ of $L(G_1)$.*

(iv) *Three roots of the equation $x^3-(2+r_1+n_1+n_2)x^2-(2n_1+2n_2+r_1n_1)x=0$.*

***Proof.*** The proof of the theorem is similar to that of Theorem 4. ■

**Corollary 8.** *For $i=1,2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then*

$$t(G_1\langle \mathrm{v}\rangle G_2) = \frac{2^{m_1-n_1}\cdot(2n_1+2n_2+r_1n_1)\cdot\prod_{i=2}^{n_1}(2n_2+3\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(n_1+\mu_j(G_2))}{n_1+n_2+m_1}.$$

**Corollary 9.** (a) *If $H_1$ and $H_2$ are L-cospectral regular graphs, and $H$ is a regular graph, then $H_1\langle \mathrm{v}\rangle H$ and $H_2\langle \mathrm{v}\rangle H$; and $H\langle \mathrm{v}\rangle H_1$ and $H\langle \mathrm{v}\rangle H_2$ are L-cospectral.*

(b) *If $F_1$ and $F_2$; and $H_1$ and $H_2$ are L-cospectral regular graphs, then $F_1\langle \mathrm{v}\rangle H_1$ and $F_2\langle \mathrm{v}\rangle H_2$ are L-cospectral.*

#### 2.1.3. $Q$-spectra of $R$-vertex join

Let $G_i$ be a graph on $n_i$ vertices and $m_i$ edges. Then the signless Laplacian matrix of $G_1\langle \mathrm{v}\rangle G_2$ can be obtained as:

$$Q(G_1\langle \mathrm{v}\rangle G_2) = \begin{pmatrix} (r_1+n_2)I_{n_1}+Q(G_1) & R(G_1) & J_{n_1\times n_2} \\ R(G_1)^T & 2I_{m_1} & O_{m_1\times n_2} \\ J_{n_2\times n_1} & O_{n_2\times m_1} & n_1 I_{n_2}+Q(G_2) \end{pmatrix}.$$

**Theorem 10.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the signless Laplacian spectrum of $G_1\langle \mathrm{v}\rangle G_2$ consists of:*

(i) *The eigenvalue $n_1 + \nu_j(G_2)$ for every eigenvalue $\nu_j$ $(j = 1, 2, \dots, n_2 - 1)$ of $Q(G_2)$.*

(ii) *The eigenvalue 2 with multiplicity $m_1 - n_1$.*

(iii) *Two roots of the equation $x^2-(2+r_1+n_2+\nu_i(G_1))x+2r_1+2n_2+\nu_i(G_1) = 0$ for each eigenvalue $\nu_i$ $(i = 1, 2, \dots, n_1 - 1)$ of $Q(G_1)$.*

(iv) *Three roots of the equation $x^3 - (2 + 3r_1 + 2r_2 + n_1 + n_2)x^2 + (4r_1 + 4r_2 + 2n_1 + 2n_2 + 2r_2n_2 + 3r_1n_1 + 6r_1r_2)x - 8r_1r_2 - 4r_1n_1 - 4r_2n_2 = 0$.*

**Corollary 11.** (a) *If $H_1$ and $H_2$ are $Q$-cospectral regular graphs, and $H$ is a regular graph, then $H_1\langle \mathrm{v}\rangle H$ and $H_2\langle \mathrm{v}\rangle H$; and $H\langle \mathrm{v}\rangle H_1$ and $H\langle \mathrm{v}\rangle H_2$ are $Q$-cospectral.*

(b) *If $F_1$ and $F_2$; and $H_1$ and $H_2$ are $Q$-cospectral regular graphs, then $F_1\langle \mathrm{v}\rangle H_1$ and $F_2\langle \mathrm{v}\rangle H_2$ are $Q$-cospectral.*

## 2.2. Spectra of $R$-edge join

The degree of the vertices of $G_1\langle \mathrm{e}\rangle G_2$ are $d_{G_1\langle \mathrm{e}\rangle G_2}(v_i) = 2d_{G_1}(v_i)$, $d_{G_1\langle \mathrm{e}\rangle G_2}(e_i) = 2 + n_2$ and $d_{G_1\langle \mathrm{e}\rangle G_2}(u_i) = d_{G_2}(u_i) + m_1$.

**Lemma 12.** *For any real numbers $c, d > 0$, we have*

$$(cI_n - dJ_{n\times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}.$$

***Proof.*** $(cI_n - dJ_{n\times n})^{-1} = \frac{adj(cI_n-dJ_{n\times n})}{det(cI_n-dJ_{n\times n})} = \frac{c^{n-2}(c-nd)I_n+c^{n-2}dJ_{n\times n}}{c^{n-1}(c-nd)} = \frac{1}{c}I_n + \frac{d}{c(c-nd)}J_{n\times n}$. ■

### 2.2.1. $A$-spectra of $R$-edge join

Let $G_i$ be a graph on $n_i$ vertices and $m_i$ edges. Then the adjacency matrix of $G_1\langle \mathrm{e}\rangle G_2$ can be written as:

$$A(G_1\langle \mathrm{e}\rangle G_2) = \begin{pmatrix} A(G_1) & R(G_1) & O_{m_1\times n_2} \\ R(G_1)^T & O_{m_1} & J_{n_1\times n_2} \\ O_{n_2\times m_1} & J_{n_2\times n_1} & A(G_2) \end{pmatrix}.$$

**Theorem 13.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the adjacency spectrum of $G_1\langle \mathrm{e}\rangle G_2$ consists of:*

(i) *The eigenvalue $\lambda_j(G_2)$, for every eigenvalue $\lambda_j$ $(j = 2, 3, \ldots, n_2)$ of $A(G_2)$.*

(ii) *The eigenvalue 0 with multiplicity $m_1 - n_1$.*

(iii) *Two roots of the equation $x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1) = 0$ for each eigenvalue $\lambda_i$ $(i = 2, 3, \ldots, n_1)$ of $A(G_1)$.*

(iv) *Three roots of the equation $x^3 - (r_1 + r_2)x^2 - (2r_1 + m_1n_2 - r_1r_2)x + 2r_1r_2 + r_1m_1n_2 = 0$.*

***Proof.*** The adjacency characteristic polynomial of $G_1\langle \mathrm{e}\rangle G_2$ is

$$\begin{aligned} f_{A(G_1\langle \mathrm{e}\rangle G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - A(G_1\langle \mathrm{e}\rangle G_2)) \\ &= \det \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1\times n_2} \\ -R(G_1)^T & xI_{m_1} & -J_{m_1\times n_2} \\ O_{n_2\times n_1} & -J_{n_2\times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= \det(xI_{n_2} - A(G_2))\det(S) = \prod_{j=1}^{n_2}\{x - \lambda_j(G_2)\}\det(S), \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} O_{n_1\times n_2} \\ -J_{m_1\times n_2} \end{pmatrix} (xI_{n_2} - A(G_2))^{-1} \begin{pmatrix} O_{n_2\times n_1} & -J_{n_2\times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - \Gamma_{A(G_2)}(x)J_{m_1\times m_1} \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \det(S) = {} & \det\big(xI_{m_1} - \Gamma_{A(G_2)}(x)J_{m_1\times m_1}\big) \\ & \det\big(xI_{n_1} - A(G_1) - R(G_1)(xI_{m_1} - \Gamma_{A(G_2)}(x)J_{m_1\times m_1})^{-1}R(G_1)^T\big) \end{aligned}$$

$$= x^{m_1}\left(1-\Gamma_{A(G_2)}(x)\tfrac{m_1}{x}\right)\det\Big[xI_{n_1}-A(G_1)$$

$$-R(G_1)\Big\{\tfrac{1}{x}I_{m_1}+\tfrac{\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))}J_{m_1\times m_1}\Big\}R(G_1)^T\Big]$$

$$= x^{m_1}\left(1-\Gamma_{A(G_2)}(x)\tfrac{m_1}{x}\right)\det\Big(xI_{n_1}-A(G_1)-\tfrac{1}{x}R(G_1)R(G_1)^T$$

$$-\tfrac{\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))}R(G_1)J_{m_1\times m_1}R(G_1)^T\Big)$$

$$= x^{m_1}\left(1-\Gamma_{A(G_2)}(x)\tfrac{m_1}{x}\right)\det\Big(xI_{n_1}-A(G_1)-\tfrac{1}{x}R(G_1)R(G_1)^T$$

$$-\tfrac{\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))}r_1^2J_{n_1\times n_1}\Big)$$

$$= x^{m_1}(1-\Gamma_{A(G_2)}(x)\tfrac{m_1}{x})\Big[\det\big(xI_{n_1}-A(G_1)-\tfrac{1}{x}R(G_1)R(G_1)^T\big)$$

$$-\tfrac{r_1^2\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))}\mathbf{1}_{n_1}^T\operatorname{adj}\big(xI_{n_1}-A(G_1)-\tfrac{1}{x}R(G_1)R(G_1)^T\big)\,\mathbf{1}_{n_1}\Big]$$

$$= x^{m_1}\left(1-\Gamma_{A(G_2)}(x)\tfrac{m_1}{x}\right)\det\big(xI_{n_1}-A(G_1)-\tfrac{1}{x}R(G_1)R(G_1)^T\big)$$

$$\Big[1-\tfrac{r_1^2\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))}\mathbf{1}_{n_1}^T\left(xI_{n_1}-A(G_1)-\tfrac{1}{x}R(G_1)R(G_1)^T\right)^{-1}\mathbf{1}_{n_1}\Big]$$

$$= x^{m_1}\left(1-\Gamma_{A(G_2)}(x)\tfrac{m_1}{x}\right)\det\big(xI_{n_1}-A(G_1)-\tfrac{1}{x}(r_1I_{n_1}+A(G_1))\big)$$

$$\Big[1-\tfrac{r_1^2\Gamma_{A(G_2)}(x)\Gamma_{A(G_1)+\frac{1}{x}R(G_1)R(G_1)^T}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))}\Big]$$

$$= x^{m_1}\left(1-\tfrac{m_1n_2}{x(x-r_2)}\right)\prod_{i=1}^{n_1}\left\{x-\lambda_i(G_1)-\tfrac{1}{x}(r_1+\lambda_i(G_1))\right\}$$

$$\Big[1-\tfrac{r_1^2n_1n_2}{x(x-r_2)(x-\frac{m_1n_2}{x-r_2})(x-r_1-\frac{2r_1}{x})}\Big]$$

$$= x^{m_1-n_1}\left(\tfrac{x^2-r_2x-m_1n_2}{x(x-r_2)}\right)\prod_{i=1}^{n_1}\left\{x^2-\lambda_i(G_1)x-r_1-\lambda_i(G_1)\right\}$$

$$\Big[1-\tfrac{r_1^2n_1n_2}{(x^2-r_2x-m_1n_2)(x^2-r_1x-2r_1)}\Big].$$

Therefore

$$
\begin{aligned}
&f_{A(G_1\langle \mathrm{e}\rangle G_2)}(x)\\
&= x^{m_1-n_1}\left(\tfrac{x^2-r_2x-m_1n_2}{x(x-r_2)}\right)\prod_{j=1}^{n_2}\{x-\lambda_j(G_2)\}\prod_{i=1}^{n_1}\left\{x^2-\lambda_i(G_1)x-r_1-\lambda_i(G_1)\right\}\\
&\quad\left[1-\tfrac{r_1^2n_1n_2}{(x^2-r_2x-m_1n_2)(x^2-r_1x-2r_1)}\right]\\
&= x^{m_1-n_1}\left(\tfrac{x^2-r_2x-m_1n_2}{x(x-r_2)}\right)\prod_{j=1}^{n_2}\{x-\lambda_j(G_2)\}\prod_{i=1}^{n_1}\left\{x^2-\lambda_i(G_1)x-r_1-\lambda_i(G_1)\right\}\\
&\quad\left[\tfrac{(x^2-r_2x-m_1n_2)(x^2-r_1x-2r_1)-r_1^2n_1n_2}{(x^2-r_2x-m_1n_2)(x^2-r_1x-2r_1)}\right]\\
&= x^{m_1-n_1}\prod_{j=2}^{n_2}\{x-\lambda_j(G_2)\}\prod_{i=2}^{n_1}\left\{x^2-\lambda_i(G_1)x-r_1-\lambda_i(G_1)\right\}\\
&\quad\left\{x^3-(r_1+r_2)x^2-(2r_1+m_1n_2-r_1r_2)x+2r_1r_2+r_1m_1n_2\right\}. \qquad \blacksquare
\end{aligned}
$$

**Corollary 14.** *Let $G$ be an $r$-regular graph with $n$ vertices and $m$ edges. Then the adjacency spectrum of $G\langle \mathrm{e}\rangle K_{p,q}$ consists of:*

(i) *The eigenvalue $0$ with multiplicity $m-n+p+q-2$,*

(ii) *Two roots of the equation $x^2-\lambda_i(G)x-r-\lambda_i(G)=0$ for each eigenvalue $\lambda_i$ $(i=2,3,\dots,n)$ of $A(G)$,*

(iii) *Four roots of the equation $x^4-rx^3-(pq+pm+qm+2r)x^2+(pqr+pmr+qmr-2pqm)x+2pqr+2pqrm=0$.*

**Corollary 15.** (a) *If $H_1$ and $H_2$ are $A$-cospectral regular graphs, and $H$ is a regular graph, then $H_1\langle \mathrm{e}\rangle H$ and $H_2\langle \mathrm{e}\rangle H$; and $H\langle \mathrm{e}\rangle H_1$ and $H\langle \mathrm{e}\rangle H_2$ are $A$-cospectral.*

(b) *If $F_1$ and $F_2$; and $H_1$ and $H_2$ are $A$-cospectral regular graphs, then $F_1\langle \mathrm{e}\rangle H_1$ and $F_2\langle \mathrm{e}\rangle H_2$ are $A$-cospectral.*

#### 2.2.2. *L*-spectra of *R*-edge join

Let $G_i$ be a graph on $n_i$ vertices and $m_i$ edges. Then the Laplacian matrix of $G_1\langle \mathrm{e}\rangle G_2$ is given by [7]:

$$
L(G_1\langle \mathrm{e}\rangle G_2)=\begin{pmatrix} r_1I_{n_1}+L(G_1) & -R(G_1) & O_{m_1\times n_2}\\ -R(G_1)^T & (2+n_2)I_{m_1} & -J_{n_1\times n_2}\\ O_{n_2\times m_1} & -J_{n_2\times n_1} & m_1I_{n_2}+L(G_2)\end{pmatrix}.
$$

**Theorem 16.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the Laplacian spectrum of $G_1\langle \mathrm{e}\rangle G_2$ consists of:*

(i) *The eigenvalue $m_1 + \mu_j(G_2)$ for every eigenvalue $\mu_j$ $(j = 2, 3, \ldots, n_2)$ of $L(G_2)$,*

(ii) *The eigenvalue $2 + n_2$ with multiplicity $m_1 - n_1$,*

(iii) *Two roots of the equation $x^2 - (2 + r_1 + n_2 + \mu_i(G_1))x + r_1 n_2 + 3\mu_i(G_1) + n_2\mu_i(G_1) = 0$ for each eigenvalue $\mu_i$ $(i = 2, 3, \ldots, n_1)$ of $L(G_1)$,*

(iv) *Three roots of the equation $x^3 - (2 + r_1 + m_1 + n_2)x^2 + (2m_1 + r_1 n_2 + r_1 m_1)x = 0$.*

***Proof.*** The proof of the theorem is similar to that of Theorem 13. ■

**Corollary 17.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then*

$$t(G_1\langle \mathrm{e}\rangle G_2) = \frac{(2+n_2)^{m_1-n_1}\cdot(2m_1+r_1n_2+r_1m_1)\cdot\prod_{i=2}^{n_1}(r_1n_2+3\mu_i(G_1)+n_2\mu_i(G_1))\cdot\prod_{j=2}^{n_2}(m_1+\mu_j(G_2))}{n_1+n_2+m_1}.$$

**Corollary 18.** (a) *If $H_1$ and $H_2$ are $L$-cospectral regular graphs, and $H$ is a regular graph, then $H_1\langle \mathrm{e}\rangle H$ and $H_2\langle \mathrm{e}\rangle H$; and $H\langle \mathrm{e}\rangle H_1$ and $H\langle \mathrm{e}\rangle H_2$ are $L$-cospectral.*

(b) *If $F_1$ and $F_2$; and $H_1$ and $H_2$ are $L$-cospectral regular graphs, then $F_1\langle \mathrm{e}\rangle H_1$ and $F_2\langle \mathrm{e}\rangle H_2$ are $L$-cospectral.*

### 2.2.3. *Q*-spectra of *R*-edge join

Let $G_i$ be a graph on $n_i$ vertices and $m_i$ edges. Then the signless Laplacian matrix of $G_1\langle \mathrm{e}\rangle G_2$ can be obtained as:

$$Q(G_1\langle \mathrm{e}\rangle G_2) = \begin{pmatrix} r_1 I_{n_1} + Q(G_1) & R(G_1) & O_{m_1\times n_2} \\ R(G_1)^T & (2+n_2)I_{m_1} & J_{n_1\times n_2} \\ O_{n_2\times m_1} & J_{n_2\times n_1} & m_1 I_{n_2} + Q(G_2) \end{pmatrix}.$$

**Theorem 19.** *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. Then the signless Laplacian spectrum of $G_1\langle \mathrm{e}\rangle G_2$ consists of:*

(i) *The eigenvalue $m_1 + \nu_j(G_2)$ for every eigenvalue $\nu_j$ $(j = 1, 2, \ldots, n_2 - 1)$ of $Q(G_2)$,*

(ii) *The eigenvalue $2 + n_2$ with multiplicity $m_1 - n_1$,*

(iii) *Two roots of the equation $x^2 - (2+r_1+n_2+\nu_i(G_1))x + 2r_1 + r_1 n_2 + 3\nu_i(G_1) + n_2\nu_i(G_1) = 0$ for each eigenvalue $\nu_i$ $(i = 1, 2, \ldots, n_1 - 1)$ of $Q(G_1)$,*

(iv) *Three roots of the equation* $x^3 - (2 + 3r_1 + 2r_2 + m_1 + n_2)x^2 + (4r_1 + 4r_2 + 2m_1 + 3r_1n_2 + 2r_2n_2 + 3r_1m_1 + 6r_1r_2)x - 4r_1m_1 - 8r_1r_2 - 6r_1r_2n_2 = 0$.

**Corollary 20.** (a) *If $H_1$ and $H_2$ are $Q$-cospectral regular graphs, and $H$ is a regular graph, then $H_1\langle\mathrm{e}\rangle H$ and $H_2\langle\mathrm{e}\rangle H$; and $H\langle\mathrm{e}\rangle H_1$ and $H\langle\mathrm{e}\rangle H_2$ are $Q$-cospectral.*

(b) *If $F_1$ and $F_2$; and $H_1$ and $H_2$ are $Q$-cospectral regular graphs, then $F_1\langle\mathrm{e}\rangle H_1$ and $F_2\langle\mathrm{e}\rangle H_2$ are $Q$-cospectral.*

**Example 21.** Let us consider $A$-cospectral regular graphs $H_1$ and $H_2$ [10] as given in Figure 1. These graphs are also $L$-cospectral and $Q$-cospectral, because they are regular graphs. In Figure 2 we present $R$-graphs of $H_1$ and $H_2$.

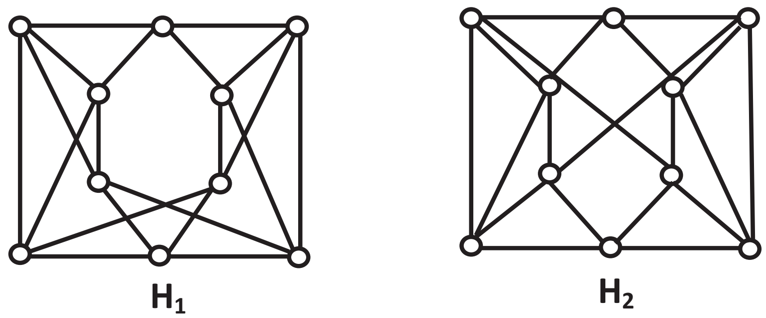


Figure 1. Two $A$-cospectral regular graphs.

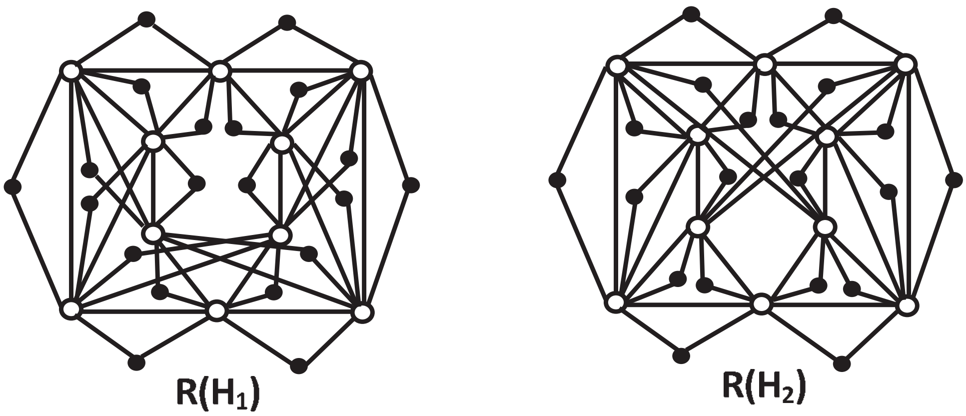


Figure 2. $R$-graph of $H_1$ and $H_2$.

If we consider any regular graph $G$ then $H_1\langle\mathrm{v}\rangle G$ and $H_2\langle\mathrm{v}\rangle G$ (respectively $H_1\langle\mathrm{e}\rangle G$ and $H_2\langle\mathrm{e}\rangle G$) are simultaneously $A$-cospectral, $L$-cospectral and $Q$-cospectral. In particular if $G = K_2$ with $V(K_2) = \{x, y\}$, then $H_1\langle\mathrm{v}\rangle K_2$ (respectively $H_1\langle\mathrm{e}\rangle K_2$) is obtained by making all unfilled (respectively filled) vertices of $\mathcal{R}(H_1)$ with both $x$ and $y$. Similarly $H_2\langle\mathrm{v}\rangle K_2$ and $H_2\langle\mathrm{e}\rangle K_2$ can be obtained.

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