

SPECTRA OF R -VERTEX JOIN AND R -EDGE JOIN OF TWO GRAPHS

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Abstract

The R -graph $\mathcal{R}(G)$ of a graph G is the graph obtained from G by introducing a new vertex u_e for each $e \in E(G)$ and making u_e adjacent to both the end vertices of e . In this paper, we determine the adjacency, Laplacian and signless Laplacian spectra of R -vertex join and R -edge join of a connected regular graph with an arbitrary regular graph in terms of their eigenvalues. Moreover, applying these results we construct some non-regular A -cospectral, L -cospectral and Q -cospectral graphs, and find the number of spanning trees.

Keywords: spectrum, cospectral graphs, R -vertex join, R -edge join.

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1. INTRODUCTION

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *adjacency matrix* of G , denoted by $A(G)$, is an $n \times n$ symmetric matrix such that $A(u, v) = 1$ if and only if vertex u is adjacent to vertex v and 0 otherwise. If $D(G)$ is the diagonal matrix of vertex degrees of G , then the *Laplacian matrix* $L(G)$ and *signless Laplacian matrix* $Q(G)$ are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ respectively. For a given matrix M of size n , we denote the characteristic polynomial $\det(xI_n - M)$ of M by $f_M(x)$. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ and $\nu_1(G) \leq \nu_2(G) \leq \cdots \leq \nu_n(G)$ respectively

and the multiset of these eigenvalues is called as adjacency spectrum, Laplacian spectrum and signless Laplacian spectrum respectively. Two graphs are said to be A -cospectral, L -cospectral and Q -cospectral if they have the same A -spectrum, L -spectrum and Q -spectrum respectively.

Many works have already done on different kinds of graph operations. One of this is join of two graphs. The *join* [5] of two graphs is their disjoint union together with all the edges that connect all the vertices of the first graph with all the vertices of the second graph. The R -graph $\mathcal{R}(G)$ [2] of a graph G is the graph obtained from G by adding a vertex u_e and joining u_e to the end vertices of e for each $e \in E(G)$. The set of such new vertices is denoted by $I(G)$ i.e., $I(G) = V(\mathcal{R}(G)) \setminus V(G)$. In this paper we are interested on finding adjacency, Laplacian and signless Laplacian spectrum of some R -joins of graphs, which are defined below.

Definition. Let G_1 and G_2 be two vertex-disjoint graphs with number of vertices n_1 and n_2 , and edges m_1 and m_2 , respectively. Then

- (i) The R -vertex join [7] of G_1 and G_2 , denoted by $G_1 \langle v \rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining each vertex of $V(G_1)$ with every vertex of $V(G_2)$. The graph $G_1 \langle v \rangle G_2$ has $n_1 + n_2 + m_1$ vertices and $3m_1 + n_1n_2 + m_2$ edges.
- (ii) The R -edge join [7] of G_1 and G_2 , denoted by $G_1 \langle e \rangle G_2$, is the graph obtained from $\mathcal{R}(G_1)$ and G_2 by joining each vertex of $I(G_1)$ with every vertex of $V(G_2)$. The graph $G_1 \langle e \rangle G_2$ has $n_1 + n_2 + m_1$ vertices and $m_1(3 + n_2) + m_2$ edges.

In [6], Indulal computed adjacency spectra of subdivision-vertex join and subdivision-edge join for two regular graph in terms of their spectra. In [8], Liu and Zhang generalized the result by determining the A -spectra, L -spectra and Q -spectra of subdivision-vertex and subdivision-edge join for a regular graph and an arbitrary graph and also constructed infinite family of new integral graphs. In [7], Liu *et al.* formulated the resistance distances and Kirchhoff index of $G_1 \langle v \rangle G_2$ and $G_1 \langle e \rangle G_2$ respectively. Motivated by these works, here we determine the adjacency, Laplacian and signless Laplacian spectrum of $G_1 \langle v \rangle G_2$ and $G_1 \langle e \rangle G_2$ for a connected regular graph G_1 and an arbitrary regular graph G_2 in terms of the corresponding eigenvalues of G_1 and G_2 . Some non-regular A -cospectral, L -cospectral and Q -cospectral graphs are also exhibited in Example 21 in Section 2.

Our results are based upon Lemma 1 and 2 stated below.

Lemma 1 (Schur Complement [3]). *Suppose that the order of all four matrices M , N , P and Q satisfy the rules of operations on matrices. Then we have,*

$$\begin{aligned} \begin{vmatrix} M & N \\ P & Q \end{vmatrix} &= |Q||M - NQ^{-1}P|, \text{ if } Q \text{ is a non-singular square matrix,} \\ &= |M||Q - NM^{-1}P|, \text{ if } M \text{ is a non-singular square matrix.} \end{aligned}$$

Lemma 2 [8]. Let A be an $n \times n$ real matrix, and $J_{s \times t}$ denote the $s \times t$ matrix with all entries equal to one. Then

$$\det(A + \alpha J_{n \times n}) = \det(A) + \alpha \mathbf{1}_n^T \text{adj}(A) \mathbf{1}_n,$$

where α is a real number and $\text{adj}(A)$ is the adjugate matrix of A .

For a graph G with n vertices and m edges, the *vertex-edge incidence matrix* $R(G)$ [4] is a matrix of order $n \times m$, with entry $r_{ij} = 1$ if the i^{th} vertex is incident to the j^{th} edge, and 0 otherwise. In particular, if G is an r -regular graph then $R(G)R(G)^T = A(G) + rI_n$ and $R(G)^T R(G) = A(l(G)) + 2I_m$, where $l(G)$ is the line graph.

Let $t(G)$ denote the number of spanning trees of G . It is well known [2] that if G is a connected graph on n vertices with Laplacian spectrum $0 = \mu_1(G) \leq \mu_2(G) \leq \dots \leq \mu_n(G)$, then $t(G) = \frac{\mu_2(G) \cdots \mu_n(G)}{n}$.

2. OUR RESULTS

Throughout the paper for any integer k , I_k denotes the identity matrix of size k , $\mathbf{1}_k$ denotes the column vector of size k whose all entries are 1 and $J_{n_1 \times n_2}$ denotes $n_1 \times n_2$ matrix whose all entries are 1.

Definition [1, 9]. The M -coronal $\Gamma_M(x)$ of an $n \times n$ matrix M is defined as the sum of the entries of the matrix $(xI_n - M)^{-1}$ (if exists), that is,

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1} \mathbf{1}_n.$$

The following Lemma is straightforward.

Lemma 3 [1]. If M is an $n \times n$ matrix with each row sum equal to a constant t , then $\Gamma_M(x) = \frac{n}{x-t}$.

Let G_i be a graph with n_i vertices and m_i edges. Let $V(G_1) = \{v_1, v_2, \dots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \dots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \dots, u_{n_2}\}$. Then $V(G_1) \cup I(G_1) \cup V(G_2)$ is a partition of $V(G_1 \langle v \rangle G_2)$ and $V(G_1 \langle e \rangle G_2)$.

2.1. Spectra of R -vertex join

The degree of the vertices of $G_1 \langle v \rangle G_2$ are $d_{G_1 \langle v \rangle G_2}(v_i) = 2d_{G_1}(v_i) + n_2$, $d_{G_1 \langle v \rangle G_2}(e_i) = 2$ and $d_{G_1 \langle v \rangle G_2}(u_i) = d_{G_2}(u_i) + n_1$.

2.1.1. A -spectra of R -vertex join

Let G_i be a graph on n_i vertices and m_i edges. Then the adjacency matrix of $G_1 \langle v \rangle G_2$ can be written as:

$$A(G_1 \langle v \rangle G_2) = \begin{pmatrix} A(G_1) & R(G_1) & J_{n_1 \times n_2} \\ R(G_1)^T & O_{m_1} & O_{m_1 \times n_2} \\ J_{n_2 \times n_1} & O_{n_2 \times m_1} & A(G_2) \end{pmatrix}.$$

Theorem 4. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $G_1 \langle v \rangle G_2$ consists of:

- (i) The eigenvalue $\lambda_j(G_2)$ for every eigenvalue λ_j ($j = 2, 3, \dots, n_2$) of $A(G_2)$,
- (ii) The eigenvalue 0 with multiplicity $m_1 - n_1$,
- (iii) Two roots of the equation $x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1) = 0$ for each eigenvalue λ_i ($i = 2, 3, \dots, n_1$) of $A(G_1)$,
- (iv) Three roots of the equation $x^3 - (r_1 + r_2)x^2 - (2r_1 + n_1n_2 - r_1r_2)x + 2r_1r_2 = 0$.

Proof. The adjacency characteristic polynomial of $G_1 \langle v \rangle G_2$ is

$$\begin{aligned} f_{A(G_1 \langle v \rangle G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - A(G_1 \langle v \rangle G_2)) \\ &= \det \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & -J_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & O_{m_1 \times n_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= \det(xI_{n_2} - A(G_2)) \det(S) = \prod_{i=1}^{n_2} \{x - \lambda_j(G_2)\} \det(S), \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} -J_{n_1 \times n_2} \\ O_{m_1 \times n_2} \end{pmatrix} (xI_{n_2} - A(G_2))^{-1} \begin{pmatrix} -J_{n_2 \times n_1} & O_{n_2 \times m_1} \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1 \times n_1} & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix}. \end{aligned}$$

$$\begin{aligned}
\det(S) &= x^{m_1} \det(xI_{n_1} - A(G_1) - \Gamma_{A(G_2)}(x)J_{n_1 \times n_1} - \frac{1}{x}R(G_1)R(G_1)^T) \\
&= x^{m_1} \left[\det(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T) \right. \\
&\quad \left. - \Gamma_{A(G_2)}(x)\mathbf{1}_{n_1}^T \operatorname{adj}\{xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T\} \mathbf{1}_{n_1} \right] \\
&= x^{m_1} \det(xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T) \\
&\quad \left[1 - \Gamma_{A(G_2)}(x)\mathbf{1}_{n_1}^T \{xI_{n_1} - A(G_1) - \frac{1}{x}R(G_1)R(G_1)^T\}^{-1} \mathbf{1}_{n_1} \right] \\
&= x^{m_1} \det(xI_{n_1} - A(G_1) - \frac{1}{x}(r_1 I_{n_1} + A(G_1))) \\
&\quad \left[1 - \Gamma_{A(G_2)}(x)\Gamma_{A(G_1) + \frac{1}{x}R(G_1)R(G_1)^T}(x) \right] \\
&= x^{m_1} \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{1}{x}(r_1 + \lambda_i(G_1)) \right\} \left[1 - \frac{n_2}{(x-r_2)} \frac{n_1}{(x-r_1 - \frac{2r_1}{x})} \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
f_{A(G_1 \langle v \rangle G_2)}(x) &= x^{m_1} \prod_{j=1}^{n_2} \left\{ x - \lambda_j(G_2) \right\} \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{1}{x}(r_1 + \lambda_i(G_1)) \right\} \\
&\quad \left[1 - \frac{n_2}{(x-r_2)} \frac{n_1}{(x-r_1 - \frac{2r_1}{x})} \right] \\
&= x^{m_1-n_1} \prod_{j=2}^{n_2} \{x - \lambda_j(G_2)\} \prod_{i=2}^{n_1} \{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\} \\
&\quad \{x^3 - (r_1 + r_2)x^2 - (2r_1 + n_1n_2 - r_1r_2)x + 2r_1r_2\}. \quad \blacksquare
\end{aligned}$$

Corollary 5. *Let G be an r -regular graph with n vertices and m edges. Then the adjacency spectrum of $G \langle v \rangle K_{p,q}$ consists of:*

- (i) *The eigenvalue 0 with multiplicity $m - n + p + q - 2$.*
- (ii) *Two roots of the equation $x^2 - \lambda_i(G)x - r - \lambda_i(G) = 0$ for each eigenvalue λ_i ($i = 2, 3, \dots, n$) of $A(G)$.*
- (iii) *Four roots of the equation $x^4 - rx^3 - (pq + pn + qn + 2r)x^2 + (pqr - 2pqn)x + 2pqr = 0$.*

Corollary 6. (a) *If H_1 and H_2 are A -cospectral regular graphs, and H is a regular graph, then $H_1 \langle v \rangle H$ and $H_2 \langle v \rangle H$; and $H \langle v \rangle H_1$ and $H \langle v \rangle H_2$ are A -cospectral.*

- (b) If F_1 and F_2 ; and H_1 and H_2 are A -cospectral regular graphs, then $F_1\langle v \rangle H_1$ and $F_2\langle v \rangle H_2$ are A -cospectral.

2.1.2. L -spectra of R -vertex join

Let G_i be a graph on n_i vertices and m_i edges. Then the Laplacian matrix of $G_1\langle v \rangle G_2$ is given by [7]:

$$L(G_1\langle v \rangle G_2) = \begin{pmatrix} (r_1 + n_2)I_{n_1} + L(G_1) & -R(G_1) & -J_{n_1 \times n_2} \\ -R(G_1)^T & 2I_{m_1} & O_{m_1 \times n_2} \\ -J_{n_2 \times n_1} & O_{n_2 \times m_1} & n_1 I_{n_2} + L(G_2) \end{pmatrix}.$$

Theorem 7. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then the Laplacian spectrum of $G_1\langle v \rangle G_2$ consists of:

- (i) The eigenvalue $n_1 + \mu_j(G_2)$ for every eigenvalue μ_j ($j = 2, 3, \dots, n_2$) of $L(G_2)$.
- (ii) The eigenvalue 2 with multiplicity $m_1 - n_1$.
- (iii) Two roots of the equation $x^2 - (2 + r_1 + n_2 + \mu_i(G_1))x + 2n_2 + 3\mu_i(G_1) = 0$ for each eigenvalue μ_i ($i = 2, 3, \dots, n_1$) of $L(G_1)$.
- (iv) Three roots of the equation $x^3 - (2 + r_1 + n_1 + n_2)x^2 - (2n_1 + 2n_2 + r_1 n_1)x = 0$.

Proof. The proof of the theorem is similar to that of Theorem 4. ■

Corollary 8. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then

$$t(G_1\langle v \rangle G_2) = \frac{2^{m_1 - n_1} \cdot (2n_1 + 2n_2 + r_1 n_1) \cdot \prod_{i=2}^{n_1} (2n_2 + 3\mu_i(G_1)) \cdot \prod_{j=2}^{n_2} (n_1 + \mu_j(G_2))}{n_1 + n_2 + m_1}.$$

Corollary 9. (a) If H_1 and H_2 are L -cospectral regular graphs, and H is a regular graph, then $H_1\langle v \rangle H$ and $H_2\langle v \rangle H$; and $H\langle v \rangle H_1$ and $H\langle v \rangle H_2$ are L -cospectral.

- (b) If F_1 and F_2 ; and H_1 and H_2 are L -cospectral regular graphs, then $F_1\langle v \rangle H_1$ and $F_2\langle v \rangle H_2$ are L -cospectral.

2.1.3. Q -spectra of R -vertex join

Let G_i be a graph on n_i vertices and m_i edges. Then the signless Laplacian matrix of $G_1\langle v \rangle G_2$ can be obtained as:

$$Q(G_1 \langle v \rangle G_2) = \begin{pmatrix} (r_1 + n_2)I_{n_1} + Q(G_1) & R(G_1) & J_{n_1 \times n_2} \\ R(G_1)^T & 2I_{m_1} & O_{m_1 \times n_2} \\ J_{n_2 \times n_1} & O_{n_2 \times m_1} & n_1 I_{n_2} + Q(G_2) \end{pmatrix}.$$

Theorem 10. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then the signless Laplacian spectrum of $G_1 \langle v \rangle G_2$ consists of:

- (i) The eigenvalue $n_1 + \nu_j(G_2)$ for every eigenvalue ν_j ($j = 1, 2, \dots, n_2 - 1$) of $Q(G_2)$.
- (ii) The eigenvalue 2 with multiplicity $m_1 - n_1$.
- (iii) Two roots of the equation $x^2 - (2 + r_1 + n_2 + \nu_i(G_1))x + 2r_1 + 2n_2 + \nu_i(G_1) = 0$ for each eigenvalue ν_i ($i = 1, 2, \dots, n_1 - 1$) of $Q(G_1)$.
- (iv) Three roots of the equation $x^3 - (2 + 3r_1 + 2r_2 + n_1 + n_2)x^2 + (4r_1 + 4r_2 + 2n_1 + 2n_2 + 2r_2n_2 + 3r_1n_1 + 6r_1r_2)x - 8r_1r_2 - 4r_1n_1 - 4r_2n_2 = 0$.

Corollary 11. (a) If H_1 and H_2 are Q -cospectral regular graphs, and H is a regular graph, then $H_1 \langle v \rangle H$ and $H_2 \langle v \rangle H$; and $H \langle v \rangle H_1$ and $H \langle v \rangle H_2$ are Q -cospectral.

(b) If F_1 and F_2 ; and H_1 and H_2 are Q -cospectral regular graphs, then $F_1 \langle v \rangle H_1$ and $F_2 \langle v \rangle H_2$ are Q -cospectral.

2.2. Spectra of R -edge join

The degree of the vertices of $G_1 \langle e \rangle G_2$ are $d_{G_1 \langle e \rangle G_2}(v_i) = 2d_{G_1}(v_i)$, $d_{G_1 \langle e \rangle G_2}(e_i) = 2 + n_2$ and $d_{G_1 \langle e \rangle G_2}(u_i) = d_{G_2}(u_i) + m_1$.

Lemma 12. For any real numbers $c, d > 0$, we have

$$(cI_n - dJ_{n \times n})^{-1} = \frac{1}{c}I_n + \frac{d}{c(c - nd)}J_{n \times n}.$$

Proof. $(cI_n - dJ_{n \times n})^{-1} = \frac{\text{adj}(cI_n - dJ_{n \times n})}{\det(cI_n - dJ_{n \times n})} = \frac{c^{n-2}(c - nd)I_n + c^{n-2}dJ_{n \times n}}{c^{n-1}(c - nd)} = \frac{1}{c}I_n + \frac{d}{c(c - nd)}J_{n \times n}.$ ■

2.2.1. A -spectra of R -edge join

Let G_i be a graph on n_i vertices and m_i edges. Then the adjacency matrix of $G_1 \langle e \rangle G_2$ can be written as:

$$A(G_1 \langle e \rangle G_2) = \begin{pmatrix} A(G_1) & R(G_1) & O_{m_1 \times n_2} \\ R(G_1)^T & O_{m_1} & J_{n_1 \times n_2} \\ O_{n_2 \times m_1} & J_{n_2 \times n_1} & A(G_2) \end{pmatrix}.$$

Theorem 13. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then the adjacency spectrum of $G_1 \langle e \rangle G_2$ consists of:

- (i) The eigenvalue $\lambda_j(G_2)$, for every eigenvalue λ_j ($j = 2, 3, \dots, n_2$) of $A(G_2)$.
- (ii) The eigenvalue 0 with multiplicity $m_1 - n_1$.
- (iii) Two roots of the equation $x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1) = 0$ for each eigenvalue λ_i ($i = 2, 3, \dots, n_1$) of $A(G_1)$.
- (iv) Three roots of the equation $x^3 - (r_1 + r_2)x^2 - (2r_1 + m_1n_2 - r_1r_2)x + 2r_1r_2 + r_1m_1n_2 = 0$.

Proof. The adjacency characteristic polynomial of $G_1 \langle e \rangle G_2$ is

$$\begin{aligned} f_{A(G_1 \langle e \rangle G_2)}(x) &= \det(xI_{n_1+n_2+m_1} - A(G_1 \langle e \rangle G_2)) \\ &= \det \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) & O_{n_1 \times n_2} \\ -R(G_1)^T & xI_{m_1} & -J_{m_1 \times n_2} \\ O_{n_2 \times n_1} & -J_{n_2 \times m_1} & xI_{n_2} - A(G_2) \end{pmatrix} \\ &= \det(xI_{n_2} - A(G_2)) \det(S) = \prod_{j=1}^{n_2} \{x - \lambda_j(G_2)\} \det(S), \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} \end{pmatrix} - \begin{pmatrix} O_{n_1 \times n_2} \\ -J_{m_1 \times n_2} \end{pmatrix} (xI_{n_2} - A(G_2))^{-1} (O_{n_2 \times n_1} - J_{n_2 \times m_1}) \\ &= \begin{pmatrix} xI_{n_1} - A(G_1) & -R(G_1) \\ -R(G_1)^T & xI_{m_1} - \Gamma_{A(G_2)}(x)J_{m_1 \times m_1} \end{pmatrix}. \end{aligned}$$

$$\det(S) = \det(xI_{m_1} - \Gamma_{A(G_2)}(x)J_{m_1 \times m_1})$$

$$\det(xI_{n_1} - A(G_1) - R(G_1)(xI_{m_1} - \Gamma_{A(G_2)}(x)J_{m_1 \times m_1})^{-1}R(G_1)^T)$$

$$\begin{aligned}
&= x^{m_1} \left(1 - \Gamma_{A(G_2)}(x) \frac{m_1}{x}\right) \det \left[xI_{n_1} - A(G_1) \right. \\
&\quad \left. - R(G_1) \left\{ \frac{1}{x} I_{m_1} + \frac{\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))} J_{m_1 \times m_1} \right\} R(G_1)^T \right] \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2)}(x) \frac{m_1}{x}\right) \det \left(xI_{n_1} - A(G_1) - \frac{1}{x} R(G_1) R(G_1)^T \right. \\
&\quad \left. - \frac{\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))} R(G_1) J_{m_1 \times m_1} R(G_1)^T \right) \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2)}(x) \frac{m_1}{x}\right) \det \left(xI_{n_1} - A(G_1) - \frac{1}{x} R(G_1) R(G_1)^T \right. \\
&\quad \left. - \frac{\Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))} r_1^2 J_{n_1 \times n_1} \right) \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2)}(x) \frac{m_1}{x}\right) \left[\det \left(xI_{n_1} - A(G_1) - \frac{1}{x} R(G_1) R(G_1)^T \right) \right. \\
&\quad \left. - \frac{r_1^2 \Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))} \mathbf{1}_{n_1}^T \operatorname{adj} \left(xI_{n_1} - A(G_1) - \frac{1}{x} R(G_1) R(G_1)^T \right) \mathbf{1}_{n_1} \right] \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2)}(x) \frac{m_1}{x}\right) \det \left(xI_{n_1} - A(G_1) - \frac{1}{x} R(G_1) R(G_1)^T \right) \\
&\quad \left[1 - \frac{r_1^2 \Gamma_{A(G_2)}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))} \mathbf{1}_{n_1}^T \left(xI_{n_1} - A(G_1) - \frac{1}{x} R(G_1) R(G_1)^T \right)^{-1} \mathbf{1}_{n_1} \right] \\
&= x^{m_1} \left(1 - \Gamma_{A(G_2)}(x) \frac{m_1}{x}\right) \det \left(xI_{n_1} - A(G_1) - \frac{1}{x} (r_1 I_{n_1} + A(G_1)) \right) \\
&\quad \left[1 - \frac{r_1^2 \Gamma_{A(G_2)}(x) \Gamma_{A(G_1) + \frac{1}{x} R(G_1) R(G_1)^T}(x)}{x(x-m_1\Gamma_{A(G_2)}(x))} \right] \\
&= x^{m_1} \left(1 - \frac{m_1 n_2}{x(x-r_2)}\right) \prod_{i=1}^{n_1} \left\{ x - \lambda_i(G_1) - \frac{1}{x} (r_1 + \lambda_i(G_1)) \right\} \\
&\quad \left[1 - \frac{r_1^2 n_1 n_2}{x(x-r_2) \left(x - \frac{m_1 n_2}{x-r_2}\right) \left(x - r_1 - \frac{2r_1}{x}\right)} \right] \\
&= x^{m_1-n_1} \left(\frac{x^2 - r_2 x - m_1 n_2}{x(x-r_2)} \right) \prod_{i=1}^{n_1} \left\{ x^2 - \lambda_i(G_1) x - r_1 - \lambda_i(G_1) \right\} \\
&\quad \left[1 - \frac{r_1^2 n_1 n_2}{(x^2 - r_2 x - m_1 n_2)(x^2 - r_1 x - 2r_1)} \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
& f_{A(G_1 \langle e \rangle G_2)}(x) \\
&= x^{m_1 - n_1} \left(\frac{x^2 - r_2 x - m_1 n_2}{x(x - r_2)} \right) \prod_{j=1}^{n_2} \{x - \lambda_j(G_2)\} \prod_{i=1}^{n_1} \{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\} \\
&\quad \left[1 - \frac{r_1^2 n_1 n_2}{(x^2 - r_2 x - m_1 n_2)(x^2 - r_1 x - 2r_1)} \right] \\
&= x^{m_1 - n_1} \left(\frac{x^2 - r_2 x - m_1 n_2}{x(x - r_2)} \right) \prod_{j=1}^{n_2} \{x - \lambda_j(G_2)\} \prod_{i=1}^{n_1} \{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\} \\
&\quad \left[\frac{(x^2 - r_2 x - m_1 n_2)(x^2 - r_1 x - 2r_1) - r_1^2 n_1 n_2}{(x^2 - r_2 x - m_1 n_2)(x^2 - r_1 x - 2r_1)} \right] \\
&= x^{m_1 - n_1} \prod_{j=2}^{n_2} \{x - \lambda_j(G_2)\} \prod_{i=2}^{n_1} \{x^2 - \lambda_i(G_1)x - r_1 - \lambda_i(G_1)\} \\
&\quad \{x^3 - (r_1 + r_2)x^2 - (2r_1 + m_1 n_2 - r_1 r_2)x + 2r_1 r_2 + r_1 m_1 n_2\}. \quad \blacksquare
\end{aligned}$$

Corollary 14. *Let G be an r -regular graph with n vertices and m edges. Then the adjacency spectrum of $G \langle e \rangle K_{p,q}$ consists of:*

- (i) *The eigenvalue 0 with multiplicity $m - n + p + q - 2$,*
- (ii) *Two roots of the equation $x^2 - \lambda_i(G)x - r - \lambda_i(G) = 0$ for each eigenvalue λ_i ($i = 2, 3, \dots, n$) of $A(G)$,*
- (iii) *Four roots of the equation $x^4 - rx^3 - (pq + pm + qm + 2r)x^2 + (pqr + pmr + qmr - 2pqm)x + 2pqr + 2pqrm = 0$.*

Corollary 15. (a) *If H_1 and H_2 are A -cospectral regular graphs, and H is a regular graph, then $H_1 \langle e \rangle H$ and $H_2 \langle e \rangle H$; and $H \langle e \rangle H_1$ and $H \langle e \rangle H_2$ are A -cospectral.*

(b) *If F_1 and F_2 ; and H_1 and H_2 are A -cospectral regular graphs, then $F_1 \langle e \rangle H_1$ and $F_2 \langle e \rangle H_2$ are A -cospectral.*

2.2.2. L -spectra of R -edge join

Let G_i be a graph on n_i vertices and m_i edges. Then the Laplacian matrix of $G_1 \langle e \rangle G_2$ is given by [7]:

$$L(G_1 \langle e \rangle G_2) = \begin{pmatrix} r_1 I_{n_1} + L(G_1) & -R(G_1) & O_{m_1 \times n_2} \\ -R(G_1)^T & (2 + n_2)I_{m_1} & -J_{n_1 \times n_2} \\ O_{n_2 \times m_1} & -J_{n_2 \times n_1} & m_1 I_{n_2} + L(G_2) \end{pmatrix}.$$

Theorem 16. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then the Laplacian spectrum of $G_1 \langle e \rangle G_2$ consists of:

- (i) The eigenvalue $m_1 + \mu_j(G_2)$ for every eigenvalue μ_j ($j = 2, 3, \dots, n_2$) of $L(G_2)$,
- (ii) The eigenvalue $2 + n_2$ with multiplicity $m_1 - n_1$,
- (iii) Two roots of the equation $x^2 - (2 + r_1 + n_2 + \mu_i(G_1))x + r_1n_2 + 3\mu_i(G_1) + n_2\mu_i(G_1) = 0$ for each eigenvalue μ_i ($i = 2, 3, \dots, n_1$) of $L(G_1)$,
- (iv) Three roots of the equation $x^3 - (2 + r_1 + m_1 + n_2)x^2 + (2m_1 + r_1n_2 + r_1m_1)x = 0$.

Proof. The proof of the theorem is similar to that of Theorem 13. ■

Corollary 17. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then

$$t(G_1 \langle e \rangle G_2) = \frac{(2+n_2)^{m_1-n_1} \cdot (2m_1+r_1n_2+r_1m_1) \cdot \prod_{i=2}^{n_1} (r_1n_2+3\mu_i(G_1)+n_2\mu_i(G_1)) \cdot \prod_{j=2}^{n_2} (m_1+\mu_j(G_2))}{n_1+n_2+m_1}.$$

Corollary 18. (a) If H_1 and H_2 are L -cospectral regular graphs, and H is a regular graph, then $H_1 \langle e \rangle H$ and $H_2 \langle e \rangle H$; and $H \langle e \rangle H_1$ and $H \langle e \rangle H_2$ are L -cospectral.

(b) If F_1 and F_2 ; and H_1 and H_2 are L -cospectral regular graphs, then $F_1 \langle e \rangle H_1$ and $F_2 \langle e \rangle H_2$ are L -cospectral.

2.2.3. Q -spectra of R -edge join

Let G_i be a graph on n_i vertices and m_i edges. Then the signless Laplacian matrix of $G_1 \langle e \rangle G_2$ can be obtained as:

$$Q(G_1 \langle e \rangle G_2) = \begin{pmatrix} r_1I_{n_1} + Q(G_1) & R(G_1) & O_{m_1 \times n_2} \\ R(G_1)^T & (2 + n_2)I_{m_1} & J_{n_1 \times n_2} \\ O_{n_2 \times m_1} & J_{n_2 \times n_1} & m_1I_{n_2} + Q(G_2) \end{pmatrix}.$$

Theorem 19. For $i = 1, 2$, let G_i be an r_i -regular graph with n_i vertices and m_i edges. Then the signless Laplacian spectrum of $G_1 \langle e \rangle G_2$ consists of:

- (i) The eigenvalue $m_1 + \nu_j(G_2)$ for every eigenvalue ν_j ($j = 1, 2, \dots, n_2 - 1$) of $Q(G_2)$,
- (ii) The eigenvalue $2 + n_2$ with multiplicity $m_1 - n_1$,
- (iii) Two roots of the equation $x^2 - (2 + r_1 + n_2 + \nu_i(G_1))x + 2r_1 + r_1n_2 + 3\nu_i(G_1) + n_2\nu_i(G_1) = 0$ for each eigenvalue ν_i ($i = 1, 2, \dots, n_1 - 1$) of $Q(G_1)$,

- (iv) Three roots of the equation $x^3 - (2 + 3r_1 + 2r_2 + m_1 + n_2)x^2 + (4r_1 + 4r_2 + 2m_1 + 3r_1n_2 + 2r_2n_2 + 3r_1m_1 + 6r_1r_2)x - 4r_1m_1 - 8r_1r_2 - 6r_1r_2n_2 = 0$.

Corollary 20. (a) If H_1 and H_2 are Q -cospectral regular graphs, and H is a regular graph, then $H_1\langle e \rangle H$ and $H_2\langle e \rangle H$; and $H\langle e \rangle H_1$ and $H\langle e \rangle H_2$ are Q -cospectral.

- (b) If F_1 and F_2 ; and H_1 and H_2 are Q -cospectral regular graphs, then $F_1\langle e \rangle H_1$ and $F_2\langle e \rangle H_2$ are Q -cospectral.

Example 21. Let us consider A -cospectral regular graphs H_1 and H_2 [10] as given in Figure 1. These graphs are also L -cospectral and Q -cospectral, because they are regular graphs. In Figure 2 we present R -graphs of H_1 and H_2 .

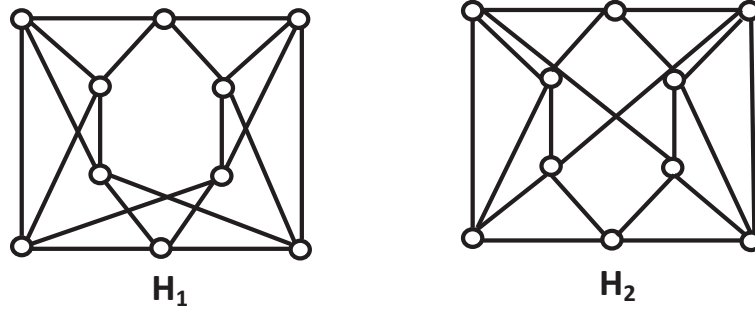


Figure 1. Two A -cospectral regular graphs.

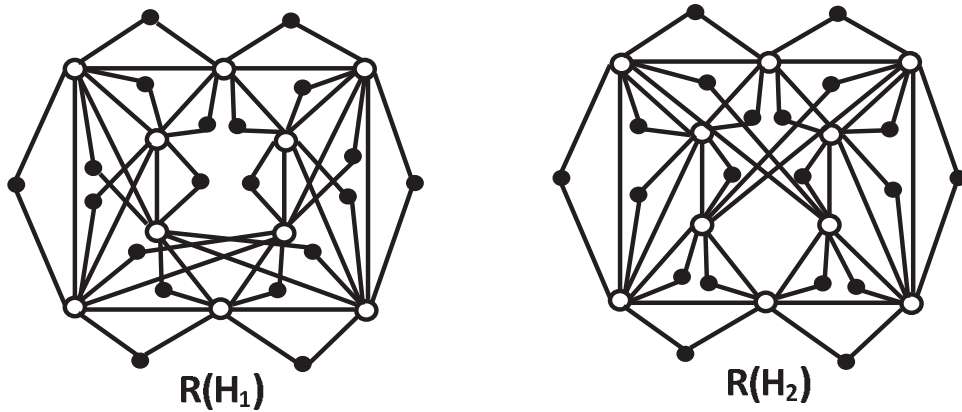


Figure 2. R -graph of H_1 and H_2 .

If we consider any regular graph G then $H_1\langle v\rangle G$ and $H_2\langle v\rangle G$ (respectively $H_1\langle e\rangle G$ and $H_2\langle e\rangle G$) are simultaneously A -cospectral, L -cospectral and Q -cospectral. In particular if $G = K_2$ with $V(K_2) = \{x, y\}$, then $H_1\langle v\rangle K_2$ (respectively $H_1\langle e\rangle K_2$) is obtained by making all unfilled (respectively filled) vertices of $\mathcal{R}(H_1)$ with both x and y . Similarly $H_2\langle v\rangle K_2$ and $H_2\langle e\rangle K_2$ can be obtained.

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