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GENERALIZED CHEBYSHEV POLYNOMIALS

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Abstract

Let $h(x)$ be a non constant polynomial with rational coefficients. Our aim is to introduce the $h(x)$-Chebyshev polynomials of the first and second kind T_n and U_n. We show that they are in a $\mathbb{Q}$-vectorial subspace $E_n(x)$ of $\mathbb{Q}[x]$ of dimension n. We establish that the polynomial sequences $(h^k T_{n-k})_k$ and $(h^k U_{n-k})_k$, $(0 \le k \le n-1)$ are two bases of $\mathbb{E}_n(x)$ for which T_n and U_n admit remarkable integer coordinates.

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1. Introduction

The n^{th} Chebyshev polynomials of the first and the second kind are respectively defined by the following second order recurrences

$$T_n = 2xT_{n-1} - T_{n-2} \text{ with } T_0 = 1 \text{ and } T_1 = x,$$
$$U_n = 2xU_{n-1} - U_{n-2} \text{ with } U_0 = 1 \text{ and } U_1 = 2x.$$

For $n \ge 1$, the explicit expressions of T_n and U_n are given (see for instance [4, 3]) by the following identities

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \tag{1}$$

$$U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \binom{n-k}{k} x^{n-2k}. \tag{2}$$

The Chebyshev polynomials of the first and the second kind admit respectively the following exponential generating functions, see Cesarano [8] (we also find in it the ordinary generating functions and some other generalizations, see also [6]),

$$\begin{aligned}
\sum_{n\geq 0} T_n(x) \frac{t^n}{n!} &= \exp(tx) \cos\left(t\sqrt{1-x^2}\right), \\
\sum_{n\geq 0} U_{n-1}(x) \frac{t^n}{n!} &= \frac{\exp(tx)}{\sqrt{1-x^2}} \sin\left(t\sqrt{1-x^2}\right).
\end{aligned}$$

In [7], the integral representations of Chebyshev polynomials of the first and second kind in terms of bivariate Hermite polynomials $H_n(x,y)$ are established (for the used version of Hermite polynomials see for instance Appell and Kampé de Fériet [1]). Cesarano propose [6] a new extension of Chebyshev polynomials via the integral representation as follows, for a real parameter α and the variables x, y,

$$\begin{aligned}
U_n(x,y,\alpha) &= \frac{1}{n!} \int_0^{\infty} \exp(-\alpha t)\, t^n H_n(2x, -y/t)\, dt, \\
T_n(x,y,\alpha) &= \frac{1}{2(n-1)!} \int_0^{\infty} \exp(-\alpha t)\, t^{n-1} H_n(2x, -y/t)\, dt,
\end{aligned}$$

where

$$H_n(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-k)!} \binom{n-k}{k} x^{n-2k} y^k.$$

Identities (1) and (2) give the decomposition of T_n and U_n in the canonical basis of $\mathbb{E}_n[x]$ (with dimension $\lfloor n/2 \rfloor + 1$) of polynomials having the same parity with n and of order $\leq n$. Our aim is to extend the results of Belbachir and Bencherif [2] by considering a non constant polynomial $h(x)$ instead of $2x$. We were led, at the beginning, to consider separately the situation $h(x) = ax$ as a specific distinct case. As an example, we establish that for $(n \geq 2,\ 1 \leq k \leq \lfloor n/2 \rfloor,\ 0 \leq l \leq \lfloor (n-1)/2 \rfloor)$ the family $\mathfrak{B}_n = \left(h^{n-2k}, xh^{n-2l-1}\right)$ is a basis in all situations, excluding the case where $h(x) = ax$. This work is not a generalization of Belbachir and Bencherif work [2], but a complementary situation and an extension to the case $h(x) \neq ax$. We have just to notice that all the results we give are independent of the degree of h.

2. The generalized Chebyshev polynomials

We define, for every integer n, the $h(x)$-Chebyshev polynomials of the first and second kind respectively by the recurrence sequences

$$T_n = h(x)T_{n-1} - T_{n-2} \qquad (T_0 = 1 \text{ and } T_1 = x), \tag{3}$$

$$U_n = h(x)U_{n-1} - U_{n-2} \qquad (U_0 = 1 \text{ and } U_1 = 2x). \tag{4}$$

where $h(x)$ is a non constant polynomial with rational coefficients.

The generating series of $(T_n)_n$ and $(U_n)_n$ are given by

$$g_{T_n}(t) = \frac{1 - t(h(x) - x)}{1 - h(x)t + t^2} \quad \text{and} \quad g_{U_n}(t) = \frac{1 - t(h(x) - 2x)}{1 - h(x)t + t^2}.$$

According to these expressions, we establish the following result

Theorem 1. *For every integer $n \geq 1$,*

(5)

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{k}{n-k} \binom{n-k}{k} h(x)^{n-2k} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{n-2k}{n-k} \binom{n-k}{k} x h(x)^{n-2k-1},$$

(6)

$$U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{k}{n-k} \binom{n-k}{k} h(x)^{n-2k} + 2 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{n-2k}{n-k} \binom{n-k}{k} x h(x)^{n-2k-1}.$$

Proof. We have $\frac{1}{1-ht+t^2} = \sum_{m \geq 0} t^m (h-t)^m = \sum_{m\geq 0} \sum_{k=0}^{m} \binom{m}{k} (-1)^k h^{m-k} t^{m+k}$. Taking $m + k = n$, we obtain $\frac{1}{1-ht+t^2} = \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k h^{n-2k} t^n$. It follows that $1 + \sum_{n\geq 1} T_n t^n = 1 + \sum_{n \geq 1} S t^n$, where $S = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} h^{n-2k} - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-k-1}{k} h^{n-2k} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-k-1}{k} x h^{n-2k-1}$. Using the equality $\binom{n-k-1}{k} = \frac{n-2k}{n-k} \binom{n-k}{k}$, we obtain $S = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{k}{n-k} \binom{n-k}{k} h^{n-2k}$ $+ \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \frac{n-2k}{n-k} \binom{n-k}{k} x h^{n-2k-1}$.

Similarly, we obtain formula (6). ∎

Remark 2. For $h(x) = 2x$, we have for each $n \geq 1$.

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \tag{7}$$

$$U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \binom{n-k}{k} x^{n-2k}. \tag{8}$$

These identities are well known for Chebyshev polynomials of the first and second kind (see for instance [4]).

3. Determining an adequate basis

The aim of this section is to establish that for $h(x)$ not equal to ax, the family of polynomials $\mathfrak{B}_n$ bellow constitutes a basis of the vectorial space $\mathbb{E}_n(x)$.

Let $\mathfrak{B}_1 = \{x\}$ and $\mathfrak{B}_n = \left(h^{n-2k}, xh^{n-2l-1}\right)$ $(n \geq 2)$, with $1 \leq k \leq \lfloor n/2 \rfloor$ and $0 \leq l \leq \lfloor (n-1)/2 \rfloor$. For every $n \geq 1$, the polynomials T_n and U_n are in the $\mathbb{Q}$-vectorial subspace $\mathbb{E}_n(x)$ of $\mathbb{Q}[x]$ generated by the family $\mathfrak{B}_n$.

Theorem 3. *The family $\mathfrak{B}_n$ is a basis of $\mathbb{E}_n(x)$.*

Proof. For $n = 2m$ $(m \geq 2)$ (the case $n = 2$ is trivial), $\mathfrak{B}_{2m} = \{1, xh, h^2, xh^3, \ldots, h^{2m-2}, xh^{2m-1}\}$. From the fact that the degree, with respect to x, of xh^{2m-1} is strictly greater than degrees of h^{2m-1} and xh^{2m-k} $(k \geq 2)$, the polynomial xh^{2m-1} can not be written as linear combination of $1, xh, \ldots, h^{2m-2}$.

Now, for $a_i \in \mathbb{Q}$, $(0 \leq i \leq 2m-3)$, suppose that

$$h^{2m-2} = a_{2m-3}xh^{2m-3} + a_{2m-4}h^{2m-4} + \cdots + a_1 xh + a_0. \tag{9}$$

The degree of h^{2m-2} is strictly greater than the degree of xh^{2m-3} except when $d^\circ h = 1$ for which the two degrees are equal. Then, if $d^\circ h > 1$, relation (9) is not possible. If $d^\circ h = 1$ then $h(x) = ax + b$ with a and b in $\mathbb{Q}$-$\{0\}$. By identification according to (9) we obtain a contradiction. Then $\mathfrak{B}_{2m}$ is a basis of $\mathbb{E}_{2m}(x)$.

The same approach holds for n odd. ∎

The first values of polynomials T_n and U_n over the basis $\mathfrak{B}_n$ are

$$
\begin{aligned}
T_1 &= x,\\
T_2 &= xh - 1,\\
T_3 &= xh^2 - h - x,\\
T_4 &= xh^3 - h^2 - 2xh + 1,\\
T_5 &= xh^4 - h^3 - 3xh^2 + 2h + x,\\
T_6 &= xh^5 - h^4 - 4xh^3 + 3h^2 + 3xh - 1,
\end{aligned}
$$

$$
\begin{aligned}
U_1 &= 2x,\\
U_2 &= 2xh - 1,\\
U_3 &= 2xh^2 - h - 2x,\\
U_4 &= 2xh^3 - h^2 - 4xh + 1,\\
U_5 &= 2xh^4 - h^3 - 6xh^2 + 2h + 2x,\\
U_6 &= 2xh^5 - h^4 - 8xh^3 + 3h^2 + 6xh - 1.
\end{aligned}
$$

4. Two other bases

In this section, we establish that $\mathfrak{T}_n = (h^k T_{n-k})_{0\le k\le n-1}$ and $\mathfrak{U}_n = (h^k U_{n-k})_{0\le k\le n-1}$ for $n \ge 1$, are two bases of $\mathbb{E}_n(x)$. Notice that $\mathfrak{T}_n$ and $\mathfrak{U}_n$ are families of $\mathbb{E}_n(x)$.

Theorem 4. *For any $n \ge 1$, $\mathfrak{T}_n$ and $\mathfrak{U}_n$ are bases of $\mathbb{E}_n(x)$.*

This result follows from the following lemma.

Lemma 5. *For any $n \ge 1$,*

$$
\det\nolimits_{\mathfrak{B}_n}(\mathfrak{T}_n) = \begin{cases} 1 & \text{for } n \text{ even},\\ (-1)^{\lfloor n/2\rfloor} & \text{for } n \text{ odd},\end{cases}
$$

$$
\det\nolimits_{\mathfrak{B}_n}(\mathfrak{U}_n) = \begin{cases} 2^{\lfloor n/2\rfloor} & \text{for } n \text{ even},\\ (-1)^{\lfloor n/2\rfloor}.2^{\lfloor n/2\rfloor+1} & \text{for } n \text{ odd}.\end{cases}
$$

Proof. For any integer $m \ge 1$ and for $1 \le k \le 2m+1$, set $V_k^{(m)} := h^{k-1}T_{2m+1-k}$ and $W_k^{(m)} := h^{k-1}U_{2m+1-k}$. We have $V_{k+1}^{(m)} - V_k^{(m)} = V_k^{(m-1)}$ and $W_{k+1}^{(m)} - W_k^{(m)} = W_k^{(m-1)}$.

Let $\Delta_m := \det_{\mathfrak{B}_{2m}}\left(V_1^{(m)}, V_2^{(m)}, \ldots, V_{2m}^{(m)}\right)$ and $D_m := \det_{\mathfrak{B}_{2m}}\left(W_1^{(m)}, W_2^{(m)}, \ldots, W_{2m}^{(m)}\right)$. We have

$$\begin{aligned}\Delta_m &= \det{}_{\mathfrak{B}_{2m}}\left(V_1^{(m)}, V_2^{(m)} - V_1^{(m)}, \ldots, V_{2m}^{(m)} - V_{2m-1}^{(m)}\right)\\ &= \det{}_{\mathfrak{B}_{2m}}\left(V_1^{(m)}, V_1^{(m-1)}, V_2^{(m-1)}, \ldots, V_{2m-1}^{(m-1)}\right).\end{aligned}$$

Set $d_h^\circ T_n$ the degree of T_n with respect to h according to the basis $\mathfrak{B}_n$. It follows that $d_h^\circ T_0 = 0$ and $d_h^\circ T_n = n-1$ $(n \geq 1)$. Then $d_h^\circ V_k^{(m)} = 2m-1$ $(1 \leq k \leq 2m)$. We have $d_h^\circ V_k^{(m-1)} = 2m-3$ $(1 \leq k \leq 2m-2)$ and $d_h^\circ V_{2m-1}^{(m-1)} = 2m-2$. The dominant coefficient of $V_1^{(m)}$, $V_{2m-1}^{(m-1)}$ and $V_k^{(m-1)}$ $(1 \leq k \leq 2m-2)$ are equal to 1. It follows that $\Delta_m = (-1)^{1+2m-1}\det_{\mathfrak{B}_{2(m-1)}}\left(V_1^{(m-1)}, V_2^{(m-1)}, \ldots, V_{2(m-1)}^{(m-1)}\right) = \Delta_{m-1} = \cdots = \Delta_1 = 1$. Similarly

$$\begin{aligned}D_m &= \det{}_{\mathfrak{B}_{2m}}\left(W_1^{(m)}, W_2^{(m)}, \ldots, W_{2m}^{(m)}\right)\\ &= \det{}_{\mathfrak{B}_{2m}}\left(W_1^{(m)}, W_1^{(m-1)}, W_2^{(m-1)}, \ldots, W_{2m-1}^{(m-1)}\right).\end{aligned}$$

Since $d_h^\circ W_k^{(m)} = 2m-1$ $(1 \leq k \leq 2m)$, $d_h^\circ W_k^{(m-1)} = 2m-3$ $(1 \leq k \leq 2m-2)$ and $d_h^\circ W_{2m-1}^{(m-1)} = 2m-2$ with dominant coefficient of $W_1^{(m)}$ and $W_k^{(m-1)}$ $(1 \leq k \leq 2m-2)$ are equal to 2 and the dominant coefficient of $W_{2m-1}^{(m-1)}$ is equal to 1. Then $D_m = 2(-1)^{2m}D_{m-1} = 2D_{m-1} = \cdots = 2^m$.

Taking $n = 2m+1$, we have

$$\begin{aligned}&\det{}_{\mathfrak{B}_{2m+1}}(T_{2m+1}, hT_{2m}, \ldots, h^{2m}T_1)\\ &= (-1)^{2m}.(-1)^m.\det{}_{\mathfrak{B}_{2m}}(T_{2m}, hT_{2m-1}, \ldots, h^{2m-1}T_1)\\ &= (-1)^m\det{}_{\mathfrak{B}_{2m}}(T_{2m}, hT_{2m-1}, \ldots, h^{2m-1}T_1)\\ &= (-1)^m\det{}_{\mathfrak{B}_{2m}}(V_1^{(m)}, V_2^{(m)}, \ldots, V_{2m}^{(m)})\\ &= (-1)^m\Delta_m\\ &= (-1)^m,\end{aligned}$$

and

$$\begin{aligned}&\det{}_{\mathfrak{B}_{2m+1}}(U_{2m+1}, hU_{2m}, \ldots, h^{2m}U_1)\\ &= 2(-1)^{2m+2}.(-1)^m.\det{}_{\mathfrak{B}_{2m}}(U_{2m}, hU_{2m-1}, \ldots, h^{2m-1}U_1)\\ &= 2(-1)^m\det{}_{\mathfrak{B}_{2m}}(U_{2m}, hU_{2m-1}, \ldots, h^{2m-1}U_1)\\ &= 2(-1)^m D_m\\ &= (-1)^m 2^{m+1}.\end{aligned}$$

■

5. Expressions of T_n and U_n in the bases $\mathfrak{T}_n$ and $\mathfrak{U}_n$

In this section we give the decomposition of T_n and U_n in each of the bases $\mathfrak{T}_n$ and $\mathfrak{U}_n$ respectively. There are height possibilities:

1. T_{2n} over $\mathfrak{T}_{2n}$: T_{2n} is T_{2n};
2. U_{2n} over $\mathfrak{U}_{2n}$: U_{2n} is U_{2n};
3. T_{2n+1} over $\mathfrak{T}_{2n+1}$: T_{2n+1} is T_{2n+1};
4. U_{2n+1} over $\mathfrak{U}_{2n+1}$: U_{2n+1} is U_{2n+1};
5. T_{2n} over $\mathfrak{U}_{2n}$, i.e., expressing T_{2n} in terms of U_{2n}, hU_{2n-1}, h^2U_{2n-2}, $\dots, h^{2n-2}U_2$, $h^{2n-1}U_1$;
6. U_{2n} over $\mathfrak{T}_{2n}$, i.e., expressing U_{2n} in terms of T_{2n}, hT_{2n-1}, h^2T_{2n-2}, $\dots, h^{2n-2}T_2$, $h^{2n-1}T_1$;
7. T_{2n+1} over $\mathfrak{U}_{2n+1}$, i.e., expressing T_{2n+1} in terms of U_{2n+1}, hU_{2n}, h^2U_{2n-1}, $\dots, h^{2n-1}U_2$, $h^{2n}U_1$;
8. U_{2n+1} over $\mathfrak{T}_{2n+1}$, i.e., expressing U_{2n+1} in terms of T_{2n+1}, hT_{2n}, h^2T_{2n-1}, $\dots, h^{2n-1}T_2$, $h^{2n}T_1$.

The first four situations are obvious. The remaining situations are established below.

Theorem 6. *For every integer $n \geq 1$,*

$$2T_{2n} = 2U_{2n} + \sum_{j=1}^{2n-1} (-1)^{j+1} \alpha_{2n-1,j} h^j U_{2n-j}, \tag{10}$$

$$2T_{2n+1} = U_{2n+1} + \sum_{j=1}^{2n} (-1)^{j+1} \alpha_{2n,j} h^j U_{2n+1-j}. \tag{11}$$

Theorem 7. *For every integer $n \geq 1$,*

$$U_{2n} = T_{2n} + \frac{1}{2} \sum_{j=1}^{2n-1} (-1)^{j-1} \left[\binom{2n-1}{j} + \alpha_{2n-2,j-1} \right] h^j T_{2n-j}, \tag{12}$$

$$U_{2n+1} = 2T_{2n+1} + \sum_{j=1}^{2n} (-1)^j \alpha_{2n,j} h^j T_{2n+1-j}, \tag{13}$$

where $(\alpha_{n,j})_n$ is the sequence given by $\alpha_{n,j} = \sum_{k=j}^{n} (-1)^k \binom{k}{j}$ $(0 \leq j \leq n)$ with the first values

$n\backslash j$	0	1	2	3	4	5	6
1	0	-1					
2	1	1	1				
3	0	-2	-2	-1			
4	1	2	4	3	1		
5	0	-3	-6	-7	-4	-1	
6	1	3	9	13	11	5	1

We need the following proposition to establish Theorems 6 and 7.

Proposition 8. *For every integers n and j, the following holds*

1. $\alpha_{n+1,j} + \alpha_{n,j-1} = \alpha_{n-1,j} + \alpha_{n-1,j-1}$,
2. $\alpha_{2n,j-1} + \alpha_{2n-2,j-2} + \alpha_{2n-2,j-1} = \binom{2n}{j-1}$,
3. $-\alpha_{2n+2,j} + \alpha_{2n,j-1} + \alpha_{2n,j} + \frac{1}{2}\alpha_{2n,j-2} = -\frac{1}{2}\binom{2n+1}{j-1}$.

Proof. 1. $\alpha_{n+1,j} + \alpha_{n,j-1} - \alpha_{n-1,j} - \alpha_{n-1,j-1} = (-1)^n\binom{n}{j} + (-1)^n\binom{n}{j-1} + (-1)^{n+1}\binom{n+1}{j} = 0$.

2. $\alpha_{2n,j-1} + \alpha_{2n-2,j-2} + \alpha_{2n,j-1}$

$= 2\sum_{k=j-1}^{2n-2}(-1)^k\binom{k}{j-1} - \binom{2n-1}{j-1} + \binom{2n}{j-1} + \sum_{k=j-2}^{2n-2}(-1)^k\binom{k}{j-2}$

$= \sum_{k=j-1}^{2n-2}(-1)^k\binom{k}{j-1} + \sum_{k=j-1}^{2n-2}(-1)^k\binom{k+1}{j-1} - \binom{2n-1}{j-1} + \binom{2n}{j-1} + (-1)^j\binom{j-2}{j-2}$

$= (-1)^{j-1}\binom{j-1}{j-1} + \binom{2n-1}{j-1} - \binom{2n-1}{j-1} + \binom{2n}{j-1} + (-1)^j\binom{j-2}{j-2} = \binom{2n}{j-1}$.

3. $-\alpha_{2n+2,j} + \alpha_{2n,j-1} + \alpha_{2n,j} + \frac{1}{2}\alpha_{2n,j-2}$

$= \binom{2n+1}{j} - \binom{2n+2}{j} + \frac{1}{2}(-1)^j\binom{j-2}{j-2} + \frac{1}{2}\sum_{k=j-1}^{2n}(-1)^k\binom{k}{j-1} + \frac{1}{2}\sum_{k=j-1}^{2n}(-1)^k\binom{k+1}{j-1}$

$= \binom{2n+1}{j} - \binom{2n+2}{j} + \frac{1}{2}(-1)^j\binom{j-2}{j-2} + \frac{1}{2}(-1)^{j-1}\binom{j-1}{j-1} + \frac{1}{2}\binom{2n+1}{j-1} = -\frac{1}{2}\binom{2n+1}{j-1}$. ∎

Using Theorem 6, we get

$$
\begin{aligned}
2T_1 &= U_1,\\
2T_2 &= 2U_2 - hU_1,\\
2T_3 &= U_3 + hU_2 - h^2U_1,\\
2T_4 &= 2U_4 - 2hU_3 + 2h^2U_2 - h^3U_1,\\
2T_5 &= U_5 + 2hU_4 - 4h^2U_3 + 3h^3U_2 - h^4U_1,\\
2T_6 &= 2U_6 - 3hU_5 + 6h^2U_4 - 7h^3U_3 + 4h^4U_2 - h^5U_1,\\
2T_7 &= U_7 + 3hU_6 - 9h^2U_5 + 13h^3U_4 - 11h^4U_3 + 5h^5U_2 - h^6U_1,
\end{aligned}
$$

and from Theorem 7, we get

$$\begin{aligned}
U_1 &= 2T_2,\\
U_2 &= T_2 + hT_1,\\
U_3 &= 2T_3 - hT_2 + h^2T_1,\\
U_4 &= T_4 + 2hT_3 - 2h^2T_2 + h^3T_1,\\
U_5 &= 2T_5 - 2hT_4 + 4h^2T_3 - 3h^3T_2 + h^4T_1,\\
U_6 &= T_6 + 3hT_5 - 6h^2T_4 + 7h^3T_3 - 4h^4T_2 + h^5T_1,\\
U_7 &= 2T_7 - 3hT_6 + 9h^2T_5 - 13h^3T_4 + 11h^4T_3 - 5h^5T_2 + h^6T_1.
\end{aligned}$$

6. Proof of Theorems

The proofs of the two theorems are essentially based on the induction approach and on the proposition above.

***Proof of Theorem* 6.** The precedent tables show the validity of the first terms. We have

$$2T_{2n+2} = hU_{2n+1} + \sum_{j=1}^{2n}(-1)^{j+1}\alpha_{2n,j}h^{j+1}U_{2n+1-j} - 2U_{2n} - \sum_{j=1}^{2n-1}(-1)^{j+1}\alpha_{2n-1,j}h^jU_{2n-j}.$$

Set $j+1 = j'$ in the first summation and using relation (4), we get

$$\begin{aligned}
2T_{2n+2} &= 2U_{2n+2} - hU_{2n+1} - \alpha_{2n,2n}h^{2n+1}U_1 + \alpha_{2n,2n-1}h^{2n}U_2 + \alpha_{2n-1,1}hU_{2n+1}\\
&\quad - \alpha_{2n-1,2n-1}h^{2n}U_2 + \sum_{j=2}^{2n-1}(-1)^jh^jU_{2n+2-j}\left[\alpha_{2n,j-1} - \alpha_{2n-1,j} - \alpha_{2n-1,j-1}\right],
\end{aligned}$$

Using $\alpha_{n,1} = \frac{1}{4}(-1)^n(2n+1) - \frac{1}{4}, \alpha_{n,n} = (-1)^n, \alpha_{n,n-1} = (-1)^n n + (-1)^{n-1}$, one deduces

$$2T_{2n+2} = 2U_{2n+2} + \sum_{j=1}^{2n+1}(-1)^{j+1}\alpha_{2n+1,j}h^jU_{2n+2-j}.$$

Formula (10) is proved. Let us establish (11), we have

$$\begin{aligned}
2T_{2n+3} &= 2hU_{2n+2} + \sum_{j=1}^{2n+1}(-1)^{j+1}\alpha_{2n+1,j}h^{j+1}U_{2n+2-j} - U_{2n+1}\\
&\quad - \sum_{j=1}^{2n}(-1)^{j+1}\alpha_{2n,j}h^jU_{2n+1-j}\\
&= U_{2n+3} + hU_{2n+2} - \alpha_{2n+1,2n}h^{2n+1}U_2 + \alpha_{2n+1,2n+1}h^{2n+2}U_1 + \alpha_{2n,1}hU_{2n+2}
\end{aligned}$$

$$+\ \alpha_{2n,2n}h^{2n+1}U_2+\sum_{j=2}^{2n}(-1)^{j+1}\left[-\alpha_{2n+1,j-1}+\alpha_{2n,j}+\alpha_{2n,j-1}\right]h^jU_{2n+3-j}.$$
$$=\ U_{2n+3}+\sum_{j=1}^{2n+2}(-1)^{j+1}\alpha_{2n+2,j}h^jU_{2n+3-j}.$$

■

Proof of Theorem 7. Suppose, by induction, that relations (12) and (13) are true until order n. Then, we have

$$\begin{aligned}
&U_{2n+2}\\
&=\ 2hT_{2n+1}+\sum_{j=1}^{2n}(-1)^j\alpha_{2n,j}h^{j+1}T_{2n+1-j}-T_{2n}-\frac{1}{2}\sum_{j=1}^{2n-1}(-1)^{j-1}\left[\binom{2n-1}{j}\right.\\
&+\ \left.\alpha_{2n-2,j-1}\right]h^jT_{2n-j}\\
&=\ T_{2n+2}+hT_{2n+1}+\alpha_{2n,2n}h^{2n+1}T_1-\alpha_{2n,2n-1}h^{2n}T_2-\frac{1}{2}\left[\binom{2n-1}{2n-1}+\alpha_{2n-2,2n-2}\right]\\
&\qquad h^{2n}T_2+\frac{1}{2}\left[\binom{2n-1}{1}+\alpha_{2n-2,0}\right]hT_{2n+1}+\sum_{j=2}^{2n-1}(-1)^{j-1}\left[\alpha_{2n,j-1}+\frac{1}{2}\left(\binom{2n-1}{j-1}\right.\right.\\
&+\ \left.\left.\alpha_{2n-2,j-2}+\binom{2n-1}{j}+\alpha_{2n-2,j-1}\right)\right]h^jT_{2n+2-j}\\
&=\ T_{2n+2}+\frac{1}{2}\sum_{j=1}^{2n+1}(-1)^{j-1}\left[\binom{2n+1}{j}+\alpha_{2n,j-1}\right]h^jT_{2n+2-j}.
\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
&U_{2n+3}\\
&=\ hT_{2n+2}+\frac{1}{2}\sum_{j=1}^{2n+1}(-1)^{j-1}\left[\binom{2n+1}{j}+\alpha_{2n,j-1}\right]h^{j+1}T_{2n+2-j}-2T_{2n+1}\\
&-\ \sum_{j=1}^{2n}(-1)^j\alpha_{2n,j}h^jT_{2n+1-j}\\
&=\ 2T_{2n+3}-hT_{2n+2}+\frac{1}{2}\left[\binom{2n+1}{2n+1}+\alpha_{2n,2n}\right]h^{2n+2}T_1-\frac{1}{2}\left[\binom{2n+1}{2n}+\alpha_{2n,2n-1}\right]\\
&\qquad h^{2n+1}T_2-\alpha_{2n,2n}h^{2n+1}T_2-\alpha_{2n,1}hT_{2n+2}+\sum_{j=2}^{2n}(-1)^jh^jT_{2n+3-j}\left[\frac{1}{2}\left(\binom{2n+1}{j-1}\right.\right.\\
&-\ \left.\left.\alpha_{2n,j-2}\right)+\alpha_{2n,j-1}+\alpha_{2n,j}\right]\\
&=\ 2T_{2n+3}+\sum_{j=1}^{2n+2}(-1)^j\alpha_{2n+2,j}h^jT_{2n+3-j}.
\end{aligned}$$

■

Some perspectives. The extension given here is related to two papers written by Belbachir and Bencherif [4, 3]. As a first perspective, as suggested by Professor Andreas Philippou in a private communication, it is interesting to establish the same results for the multivariate Fibonacci and Lucas polynomials. The second one is to see how can be extend the results given by Prodinger [10] and by Belbachir and Benmezai [5] to the q-analog situation.

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