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ON Γ-SEMIRING WITH IDENTITY

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Abstract

In this paper we study the properties of structures of the semigroup $(M, +)$ and the Γ-semigroup M of Γ-semiring M and regular Γ-semiring M satisfying the identity $a + a\alpha b = a$ or $a\alpha b + a = a$ or $a + a\alpha b + b = a$ or $a + 1 = 1$, for all $a \in M$, $\alpha \in \Gamma$. We also study the properties of Γ-semiring with unity 1 which is also an additive identity.

Keywords: Γ-semigroup, Γ-semiring, regular Γ-semiring.

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1. Introduction

In 1995, Murali Krishna Rao [5, 6, 7] introduced the notion of a Γ-semiring as a generalization of Γ-ring, ring, ternary semiring and semiring. The notion of a semiring was first introduced by Vandiver [12] in 1934 but semirings had appeared in earlier studies on the theory of ideals of rings. A universal algebra $S = (S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, i.e., $a(b + c) = ab + ac$, $(a + b)c = ac + bc$, for all $a, b, c \in S$. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line, then $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the mutiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings In structure, semirings lie between semigroups and rings. The study of rings shows that multiplicative structure of ring is independent of additive structure whereas in semiring multiplicative structure of semiring is not independent of additive stricture of semiring. It is well known that ideals play an important role in the study of any algebraic structures, in particular semirings.

Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. Henriksen defined k-ideals in semirings to obtain analogues of ring results for semiring. Hanumanthachari and Venuraju [4] studied the additive semigroup structure of semiring. M. Satyanarayana [10] studied the additive semigroup of ordered semirings. Vasanthi *et al.* [13, 14] studied the properties of semiring with identity. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semirings are useful in the areas of theoretical computer science as well as in the solutions of graph theory and optimization theory in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a Γ-ring was introduced by N. Nobusawa [9] in 1964. As a generalization of ring, semiring was introduced by Vandiver [12] in 1934. In 1981 Sen [11] introduced the notion of Γ-semigroup as a generalization of semigroup. The notion of Ternary algebraic system was introduced by Lehmer [2] in 1932, Lister [3] introduced ternary ring. Dutta and Kar [1] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a Γ-semiring where $\Gamma = Z$. The important reason for the development of Γ-semiring is a generalization of results of rings, Γ-rings, semirings, semigroups and ternary semirings. Murali Krishna Rao and Venkateswarlu [8] introduced the notion of Γ-incline and field Γ-semiring and studied properties of regular Γ-incline and field Γ-semiring.

In this paper we study the properties of additive structure $(M, +)$ and Γ-semigroup structure of Γ-semiring M satisfying the identity $a + a\alpha b = a$, or $a\alpha b + a = a$ or $a + a\alpha b + b = a$, for all $a, b \in M$ and $\alpha \in \Gamma$ or $a + 1 = 1$, for all $a \in M$, and we also study the properties of Γ-semiring with unity 1 which is also an additive identity.

2. Preliminaries

In this section we recall some important definitions introduced by pioneers in this field earlier that will be required to this paper.

Definition 2.1. A semigroup M is a non-empty set equipped with a binary operation $\cdot$, which is associative.

Definition 2.2. A semigroup $(M, +)$ is said to be band if $a + a = a$, for all $a \in M$.

Definition 2.3. A semigroup $(M, +)$ is said to be rectangular band if $a+b+a = a$, for all $a, b \in M$.

Definition 2.4. A semigroup $(M, +)$ is said to be left (right) singular if $a + b = a(a + b = b)$, for all $a, b \in M$.

Definition 2.5. A semigroup $(M, \cdot)$ is said to be left (right) singular if $xy = x(xy = y)$, for all $x, y \in M$.

Definition 2.6. A semiring $(M, +, \cdot)$ is an algebra with two binary operation $+$ and $\cdot$ such that $(M, +)$ and $(M, \cdot)$ are semigroups and the following distributive laws hold.
$x(y + z) = xy + xz$
$(x + y)z = xz + yz$, for all $x, y, z \in M$.

Definition 2.7. A semiring $(M, +, \cdot)$ is said to be mono semiring if $a + b = ab$, for all $a, b \in M$.

Definition 2.8. A semiring $(M, +, \cdot)$ with zero element is said to be zero sum free semiring if $x + x = 0$, for all $x \in M$.

Definition 2.9. A semiring $(M, +, \cdot)$ is said to be Boolean semiring if $a = a^2$, for all $a \in M$.

Definition 2.10. Let M and Γ be two non-empty sets. Then M is called a Γ-semigroup if it satisfies

(i) $x\alpha y \in M$

(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.11. A Γ-semigroup M is said to be left (right) singular if for each $a \in M$ there exists $\alpha \in \Gamma$ such that $a\alpha b = a(b\alpha a = a)$, for all $b \in M$.

Definition 2.12. A Γ-semigroup M is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 2.13. Let M be a Γ-semigroup. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.

Definition 2.14. Let M be a Γ-semigroup. If every element of M is an idempotent of M, then Γ-semigroup M is said to be band.

Definition 2.15. Let M be a Γ-semigroup. An element $a \in M$, is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.16. Let M be a Γ-semigroup. Every element of M is a regular element of M, then M is said to be regular Γ-semigroup M.

Definition 2.17. A Γ-semigroup M is called a rectangular band if for every pair $a, b \in M$ there exist $\alpha, \beta \in \Gamma$ such that $a\alpha b\beta a = a$.

Definition 2.18. A Γ-semigroup M is called Γ-semiring M, if $(M,+),(\Gamma,+)$ are semigroups and satisfies the following conditions.

(i) $a\alpha(b+c) = a\alpha b + a\alpha c$

(ii) $(a+b)\alpha c = a\alpha c + b\alpha c$

(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$ for all $a,b,c \in M, \alpha,\beta \in \Gamma$.

Definition 2.19. A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = 0x\alpha = 0$, for all $x \in M$ and $\alpha \in \Gamma$.

Definition 2.20. A Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x, x + y = y + x$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.21. Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.22. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1(a\alpha b = 1)$.

Definition 2.23. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.24. In a Γ-semiring M, an element $u \in M$ is said to be unit if there exist $a \in M$ and $\alpha \in \Gamma$ such that $a\alpha u = 1 = u\alpha a$.

Definition 2.25. Let M be a Γ-semiring.If every element of M is an idempotent of M, then M is said to be idempotent Γ-semiring M.

Definition 2.26. Let M be a Γ-semiring. An element $a \in M$, is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.27. Let M be a Γ-semiring. If every element of M, is a regular element of M, then M is said to be regular Γ-semiring M.

3. Γ-semiring satisfying the identity $a + a\alpha b = a$ or $a\alpha b + a = a$

In this section we study the properties of additive semigroup structure and Γ-semigroup structure of Γ-semiring satisfying the identity $a+a\alpha b = a$ or $a\alpha b+a = a$, for all $a, b \in M, \alpha \in \Gamma$.

Theorem 3.1. *Let M be a Γ-semiring with unity satisfying the identity $a\alpha b+a = a$, for all $a, b \in M, \alpha \in \Gamma$. Then semigroup $(M,+)$ is a band.*

Proof. We have $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$. If $b = 1$ then $a\gamma 1 + a = a \Rightarrow a + a = a$. Hence semigroup $(M, +)$ is a band. ■

Theorem 3.2. *Let M be a regular Γ-semiring satisfying $a + a\alpha b = a, a\alpha b + a = a$ for all $a, b \in M, \alpha \in \Gamma$. Then the following are true.*

(i) *Γ-semigroup M is a left singular,*

(ii) *semigroup $(M, +)$ is a band.*

Proof. Let M be a regular Γ-semiring satisfying $a + a\alpha b = a, a\alpha b + a = a$ for all $a, b \in M, \alpha \in \Gamma$.

(i) Let $a \in M$. Since a is a regular, there exist $\alpha, \beta \in \Gamma, b \in M$ such that $a = a\alpha b\beta a$. We have $a + a\alpha b = a, a\alpha b + a = a$, for all $a, b \in M, \alpha, \beta \in \Gamma$. Therefore

$$\begin{aligned} a\alpha b &= a\alpha(b\beta a + b) = a\alpha b\beta a + a\alpha b \\ &= a + a\alpha b = a. \end{aligned}$$

(ii) We have $a + a\alpha b = a \Rightarrow a + a = a$. Hence semigroup $(M, +)$ is a band. ■

Theorem 3.3. *Let M be a Γ-semiring with unity satisfying $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a rectangular band and commutative then there exist $\alpha, \beta \in \Gamma$ such that $a\alpha b = a$ and $b\beta a = b$.*

Proof. Suppose semigroup $(M, +)$ is a commutative rectangular band and $a \in M$. Since $a \in M$, there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} &\text{We have } a + a\alpha b = a, \text{ for all } \alpha \in \Gamma \\ &\Rightarrow\ a + a\gamma b = a \\ &\Rightarrow\ a\gamma b + a\gamma 1 + a\gamma b = a\gamma b + a \\ &\Rightarrow\ a\gamma(b + 1 + b) = a \\ &\Rightarrow\ a\gamma b = a \\ &\text{Since } b \in M, \text{ there exists } \alpha \in \Gamma,\ b\alpha 1 = b \\ &b\alpha a + b = b \\ &\Rightarrow\ b\alpha a + b + b\alpha a = b + b\alpha a \\ &\Rightarrow\ b\alpha a + b\alpha 1 + b\alpha a = b + b\alpha a \\ &\Rightarrow\ b\alpha(a + 1 + a) = b + b\alpha a \\ &\Rightarrow\ b\alpha a = b. \end{aligned}$$

Hence the theorem. ■

Theorem 3.4. *Let M be a Γ-semiring satisfying identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a commutative rectangular band then $a + b\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$.*

Proof. We have $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned} &\Rightarrow a + (a + b + a)\alpha b = a \\ &\Rightarrow a + a\alpha b + b\alpha b + a\alpha b = a \\ &\Rightarrow a + b\alpha b + a\alpha b = a \\ &\Rightarrow a + a\alpha b + b\alpha b = a \\ &\Rightarrow a + b\alpha b = a. \end{aligned}$$

Hence the theorem. ∎

Theorem 3.5. *Let M be a Γ-semiring satisfying the identity $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$. Then the following are true.*

(i) *If Γ-semigroup M is a band, then semigroup $(M, +)$ is a band.*

(ii) *If semigroup $(M, +)$ is a band and a right cancellative, then Γ-semigroup M is a band.*

Proof. (i) Let $a \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha a = a$.

$$\begin{aligned} &\text{Given } a\alpha b + a = a, \text{ for all } a, b \in M, \ \alpha \in \Gamma. \\ &\Rightarrow \ a\alpha a + a = a \\ &\Rightarrow \ a + a = a. \text{ Hence} (M, +) \text{ is a band.} \end{aligned}$$

(ii) Suppose that semigroup $(M, +)$ is a band.

$$\begin{aligned} &\text{We have } a\alpha a + a = a, \text{ for all } a \in M, \ \alpha \in \Gamma \\ &\Rightarrow \ a\alpha a + a = a + a \\ &\Rightarrow \ a\alpha a = a, \text{since } (M, +) \text{ is a right cancellative.} \end{aligned}$$

Hence Γ-semigroup M is a band. ∎

Theorem 3.6. *Let M be a Γ-semiring with unity, satisfying $a\alpha b + b = b$, for all $a, b \in M, \alpha \in \Gamma$. If Γ-semigroup M is a left singular then semigroup $(M, +)$ is a right singular.*

Proof. Let Γ-semigroup M be a left singular and $a \in M$.

$$\begin{aligned} &\text{Suppose } a\alpha b = a, \text{ for all } b \in M, \ \alpha \in \Gamma. \\ &\Rightarrow \ a\alpha b + b = a + b \\ &\Rightarrow \ b = a + b. \end{aligned}$$

Hence the theorem. ∎

Theorem 3.7. *Let M be a Γ-semiring with unity which is also an additive identity. If $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$, then Γ-semigroup M is a left singular.*

Proof. Let M be a Γ-semiring. Suppose $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} & a + a\gamma b = a \\ \Rightarrow\ & a\gamma 1 + a\gamma b = a \\ \Rightarrow\ & a\gamma(1 + b) = a \\ \Rightarrow\ & a\gamma b = a. \end{aligned}$$

Hence the theorem. ∎

Theorem 3.8. *Let M be a Γ-semiring with unity satisfying the identity $a+a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a right cancellative then M is an idempotent Γ-semiring.*

Proof. We have $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} & a + a\gamma 1 = a \\ \Rightarrow\ & a + a = a. \\ \text{We have } & a + a\alpha a = a, \alpha \in \Gamma \\ \Rightarrow\ & a + a\alpha a = a + a \\ \Rightarrow\ & a\alpha a = a. \end{aligned}$$

Hence M is an idempotent Γ-semiring. ∎

Theorem 3.9. *If M is a commutative idempotent Γ-semiring satisfying $a+a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$ and $a+x = x$, for all $a \in M$ then x is the unity element of Γ-semiring M.*

Proof. Let $x \in M$ and $a + x = x$, for all $a \in M$. There exists $\alpha \in \Gamma$ such that $a\alpha a = a$.

$$\begin{aligned} & a + x = x, \text{ for all } a \in M \\ \Rightarrow\ & a\alpha(a + x) = a\alpha x, \text{ for all } \alpha \in \Gamma, a \in M \\ \Rightarrow\ & a\alpha a + a\alpha x = a\alpha x, \text{ for all } \alpha \in \Gamma, a \in M \\ \Rightarrow\ & a + a\alpha x = a\alpha x, \text{ for all } \alpha \in \Gamma, a \in M \\ \Rightarrow\ & a = a\alpha x, \text{ for all } \alpha \in \Gamma, a \in M \\ \Rightarrow\ & a = a\alpha x = x\alpha a, \text{ for all } \alpha \in \Gamma, a \in M. \end{aligned}$$

Hence x is the unity element of Γ-semiring M. ∎

Theorem 3.10. *Let M be a regular Γ-semiring satisfying the identity $a+a\alpha b = a$ and $a\alpha b + b = b$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a right singular then Γ-semigroup M is a left singular.*

Proof. Let M be a regular Γ-semiring and semigroup $(M, +)$ be a right singular and $a, b \in M$. Since $a \in M$ there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

$$\begin{aligned}
& a + a\alpha x = a\alpha x\beta a + a\alpha x = a\alpha x \\
\Rightarrow\ & a = a\alpha x \\
& x\beta a + a = x\beta a + a\alpha x\beta a = x\beta a \\
\Rightarrow\ & a = x\beta a \\
\text{Hence}\ \ & a = a\alpha x = x\beta a \\
& a = a\alpha x\beta a = a\alpha a \\
& \ \ = a\beta a.
\end{aligned}$$

Hence a is an α idempotent and also β idempotent.

$$\begin{aligned}
\text{We have}\ & a + b = b \\
\Rightarrow\ & a\alpha(a + b) = a\alpha b \\
\Rightarrow\ & a\alpha a + a\alpha b = a\alpha b \\
\Rightarrow\ & a + a\alpha b = a\alpha b \\
\Rightarrow\ & a = a\alpha b.
\end{aligned}$$

Therefore $a = a\alpha a = a\alpha b = a\beta a$. Hence the theorem. ■

Theorem 3.11. *Let M be a Γ-semiring satisfying the identity $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a band and right cancellative then M is a regular Γ-semiring.*

Proof. Let $a \in M$. We have $a\alpha a + a = a$, for all $\alpha \in \Gamma$.

$$\begin{aligned}
\Rightarrow\ & a\alpha a + a = a + a \\
\Rightarrow\ & a\alpha a = a.
\end{aligned}$$

Hence M is an idempotent Γ-semiring.

$$\begin{aligned}
\text{Let}\ \ & a \in M. \\
\Rightarrow\ & a = a\alpha a, \ \text{for all } \alpha \in \Gamma \\
& \ \ = a\alpha(a\alpha a), \ \text{for all } \alpha \in \Gamma.
\end{aligned}$$

Therefore every element of M is a regular. Hence M is a regular Γ-semiring. ■

Theorem 3.12. *Let M be a Γ-semiring with unity 1 which is also an additive identity 1. If $a + a\alpha b = a$, for all $a, b \in M$, $\alpha \in \Gamma$, then Γ-semigroup M is a left singular.*

Proof. Let M be a Γ-semiring. Suppose $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} & a + a\alpha b = a, \text{ for all } \alpha \in \Gamma \\ \Rightarrow\ & a\gamma 1 + a\gamma b = a \\ \Rightarrow\ & a\gamma(1 + b) = a \\ \Rightarrow\ & a\gamma b = a. \end{aligned}$$

Therefore Γ-semigroup M is a left singular. ■

Theorem 3.13. *Let M be a regular Γ-semiring satisfying the identity $a+a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a commutative then Γ-semigroup M is a left singular and band.*

Proof. Let $a, b \in M, \alpha, \beta \in \Gamma$ such that $a\alpha b\beta a = a$.

$$\begin{aligned} a\alpha b &= a\alpha(b + b\beta a) \\ &= a\alpha(b\beta a + b) \\ &= a\alpha b\beta a + a\alpha b \\ &= a + a\alpha b \\ &= a. \end{aligned}$$

Obviously Γ-semigroup M is a band, since it is a left singular. Hence the theorem. ■

Theorem 3.14. *Let M be a Γ-semiring satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. If Γ-semigroup M is a right singular then semigroup $(M, +)$ is a left singular.*

Proof. Let M be a Γ-semiring satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Suppose Γ-semigroup M is a right singular. Let $a \in M$. There exists $\alpha \in \Gamma$ such that $a\alpha b = b$, for all $b \in M$. Then

$$\begin{aligned} & a\alpha b = b \\ \Rightarrow\ & a + a\alpha b = a + b \\ \Rightarrow\ & a = a + b. \end{aligned}$$

Hence the theorem. ■

Theorem 3.15. *Let M be a Γ-semiring satisfying the identity $a + a\alpha b = a$, for all $a, b \in M$, $\alpha \in \Gamma$. If Γ-semigroup M is a regular then semigroup $(M, +)$ is a band.*

Proof. Let Γ-semigroup M be a regular and $a \in M$. Then there exist $b \in M$ $\alpha, \beta \in \Gamma$ such that $a\alpha b\beta a = a$.

$$\begin{aligned}
& a + a\alpha b = a \\
\Rightarrow\ & a\beta a + a\alpha b\beta a = a\beta a \\
\Rightarrow\ & a\beta a + a = a\beta a \\
\Rightarrow\ & a + a\beta a + a = a + a\beta a \\
\Rightarrow\ & a + a = a.
\end{aligned}$$

Hence semigroup $(M, +)$ is a band. ∎

Theorem 3.16. *Let M be an idempotent Γ-semiring satisfying the identity $a\alpha b + a = a$, for all $\alpha \in \Gamma, a, b, \in M$. Then semigroup $(M, +)$ is a band.*

Proof. Let M be an idempotent Γ-semiring, $a \in M$ and $a\alpha b + a = a$, for all $\alpha \in \Gamma$. Since M is an idempotent Γ-semiring, there exists $\gamma \in \Gamma$ such that

$$\begin{aligned}
& a\gamma a = a \\
\Rightarrow\ & a\gamma a + a = a + a \\
\Rightarrow\ & a + a = a.
\end{aligned}$$

Hence semigroup $(M, +)$ is a band. ∎

Theorem 3.17. *Let M be a zero square Γ-semiring with zero element. Then $a\alpha b + a = a$ if and only if $a\alpha b = 0$, for all $a, b \in M, \alpha \in \Gamma$.*

Proof.

$$\begin{aligned}
\text{Suppose }\ & a\alpha b + a = a, \text{ for all } a, b \in M, \alpha \in \Gamma. \\
\Rightarrow\ & (a\alpha b)\alpha a + a\alpha a = a\alpha a \\
\Rightarrow\ & (a\alpha b)\alpha a + 0 = 0 \\
\Rightarrow\ & (a\alpha b)\alpha a = 0. \\
\Rightarrow\ & a\alpha b = 0. \\
\text{Conversely suppose that }\ & a\alpha b = 0, \text{ for all } a, b \in M, \alpha \in \Gamma. \\
\Rightarrow\ & a\alpha b + a = 0 + a \\
\Rightarrow\ & a\alpha b + a = a \\
\text{Hence }\ & a\alpha b + a = a, \text{ for all } a, b \in M, \alpha \in \Gamma.
\end{aligned}$$

∎

Theorem 3.18. *Let M be a Boolean Γ-semiring satisfying $a + a\alpha b = a$, for all $\alpha \in \Gamma, a, b \in M$. Then semigroup $(M, +)$ is a band.*

Proof. Let M be a Boolean Γ-semiring and $a \in M$. Then $a\alpha a = a$, for all $\alpha \in \Gamma$. $a + a = a + a\alpha a = a$. Hence semigroup $(M, +)$ is a band. ∎

Theorem 3.19. *Let M be a Γ-semiring. If semigroup $(M, +)$ is a right singular then $(M, +)$ is a rectangular band.*

Proof. Let M be a Γ-semiring. Suppose $a + b = b$, for all $a, b \in M$.

$$\begin{aligned} \Rightarrow a + b + a &= b + a \\ &= a. \end{aligned}$$

Hence semigroup $(M, +)$ is a rectangular band. ∎

Theorem 3.20. *Let M be a Boolean Γ-semiring with unity. Then 1 is the only invertible element.*

Proof. Let $x \in M$ be an invertible element. There exists $\alpha \in \Gamma, y \in M$ such that $x\alpha y = 1$ and there exists $\beta \in \Gamma$ such that $x\beta 1 = x$.

$$\begin{aligned} x &= x\beta 1 \\ &= x\beta(x\alpha y) \\ &= (x\beta x)\alpha y \\ &= x\alpha y \\ &= 1. \end{aligned}$$

Hence the theorem. ∎

Theorem 3.21. *Let M be a Boolean Γ-semiring satisfying $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a left singular, then $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$.*

Proof. Let $b \in M$. Then $b\alpha b = b, \alpha \in \Gamma$

$$\begin{aligned} & a + a\alpha b + b = a, \text{ for all } a, b \in M, \alpha \in \Gamma. \\ \Rightarrow\ & a + a\alpha b + b\alpha b = a \\ \Rightarrow\ & a + (a + b)\alpha b = a \\ \Rightarrow\ & a + a\alpha b = a. \end{aligned}$$

Hence the theorem. ∎

Theorem 3.22. *Let M be a zero square Γ-semiring. If M satisfies the identity $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$. Then $a\alpha b\beta a = 0$ and $b\alpha a\beta b = 0$, for all $a, b \in M, \alpha, \beta \in \Gamma$.*

Proof. We have $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$. Let $\beta \in \Gamma$.

$$\begin{aligned} & a\alpha b\beta a + a\beta a = a\beta a \\ \Rightarrow\ & a\alpha b\beta a + 0 = 0 \\ \Rightarrow\ & a\alpha b\beta a = 0. \end{aligned}$$

Similarly $b\alpha a\beta b = 0$. Hence the theorem. ■

Theorem 3.23. *Let M be a zero sum free Γ-semiring. If $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$, then $M\Gamma M = \{0\}$.*

Proof. Suppose $a\alpha b + a = a$, for all $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned} \Rightarrow\ & a\alpha b + a + a = a + a \\ \Rightarrow\ & a\alpha b + 0 = 0 \\ \Rightarrow\ & a\alpha b = 0. \end{aligned}$$

Hence the theorem. ■

Theorem 3.24. *Let M be a Γ-semiring with unity 1. If $1+1 = 1$ then semigroup $(M, +)$ is a band.*

Proof. Let $a \in M$. There exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} a &= a\gamma 1 \\ &= a\gamma(1 + 1) \\ &= a\gamma 1 + a\gamma 1 \\ &= a + a. \end{aligned}$$

Hence semigroup $(M, +)$ is a band. ■

Theorem 3.25. *If M is a Γ-semiring satisfying $a + b\alpha a = a$ and $b\alpha a + a = a$ for all $a, b \in M, \alpha \in \Gamma$ in which Γ-semigroup M is a rectangular band then Γ-semigroup M is a right singular.*

Proof. Let M be a Γ-semiring in which Γ-semigroup M is a rectangular band and $a, b \in M$. Since M is a rectangular band, there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$.

$$\begin{aligned} \text{Now}\ \ & b + a\alpha b = b, \text{ for all } a, b \in M, \alpha \in \Gamma \\ \Rightarrow\ & (b + a\alpha b)\beta a = b\beta a \\ \Rightarrow\ & b\beta a + a\alpha b\beta a = b\beta a \\ \text{Therefore}\ \ & a = b\beta a. \end{aligned}$$

Hence Γ-semigroup M is a right singular. ■

Theorem 3.26. *Let M be a Γ-semiring with unity 1 satisfying $a+1=1$ for all $a \in M$. Then additive semigroup $(M,+)$ is a band.*

Proof. Let $a \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$.

$$\begin{aligned} a = a\alpha 1 = {} & a\alpha(1+1) \\ = {} & a\alpha 1 + a\alpha 1 \\ = {} & a + a. \end{aligned}$$

Hence an additive semigroup $(M,+)$ of Γ-semiring M is a band. ■

Theorem 3.27. *Let M be a Γ-semiring with unity 1 satisfying $1+a=1$, for all $a \in M$. Then $a = a + a\alpha x\beta a$, for all $\alpha, \beta \in \Gamma, x \in M$.*

Proof. Let $a, x \in M, \alpha, \beta \in \Gamma$. Since $a \in M$ there exists $\delta \in \Gamma$ such that $a\delta 1 = a$.

$$\begin{aligned} a + a\alpha x\beta a &= a\delta 1 + a\delta 1\alpha x\beta a \\ &= a\delta(1 + 1\alpha x\beta a) \\ &= a\delta 1 = a. \end{aligned}$$

Hence $a = a + a\alpha x\beta a$, for all $\alpha, \beta \in \Gamma, x \in M$. ■

Theorem 3.28. *Let M be a Γ-semiring with unity 1 satisfying $a+1=1$, for all $a \in M$. If $a\alpha b = b$ for all $a, b \in M$, $\alpha \in \Gamma$ then semigroup $(M,+)$ is a right singular and a rectangular band.*

Proof. Suppose Γ-semigroup of Γ-semiring M is a left singular and $a, b \in M$. Since $b \in M$ there exists $\gamma \in \Gamma$ such that $1\gamma b = b$.

$$\begin{aligned} \text{Now } a + b &= a + 1\gamma b \\ &= a\gamma b + 1\gamma b \\ &= (a+1)\gamma b \\ &= 1\gamma b \\ &= b. \end{aligned}$$

Therefore $(M,+)$ is a right singular. Let $a, b \in M$.

$$\begin{aligned} \text{Then } \ & a + b = b \\ \Rightarrow \ & a + b + a = b + a \\ \Rightarrow \ & a + b + a = a. \end{aligned}$$

Hence semigroup $(M,+)$ is a rectangular band. ■

Theorem 3.29. *Let M be a Γ-semiring with unity 1. If semigroup $(M,+)$ is a right singular then $a+a\alpha b+b=b$, for all $a,b\in M$, for some $\gamma\in\Gamma$.*

Proof. Let M be a Γ-semiring with unity 1 satisfying $a+1=1$ for all $a\in M$. Suppose semigroup $(M,+)$ is a left singular and $a,b\in M$. Since $b\in M$ there exists $\gamma\in\Gamma$ such that $1\gamma b=b$.

$$\begin{aligned}\text{Now } a+a\gamma b+b&=a+a\gamma b+1\gamma b\\&=a+(a+1)\gamma b\\&=a+1\gamma b\\&=a+b=b.\end{aligned}$$

Hence $a+a\gamma b+b=a$, for all $a,b\in M$. ∎

Theorem 3.30. *Let M be a Γ-semiring with unity 1 satisfying $a+1=1$ for all $a\in M$. Then for each $a\in M$, there exists $\alpha\in\Gamma$ such that $a\alpha b+a=a$, for all $b\in M$.*

Proof. Let $a\in M$. Then there exists $\alpha\in\Gamma$ such that $a\alpha 1=a$. Suppose $b\in M$.

$$\begin{aligned}a&=a\alpha 1\\&=a\alpha(b+1)\\&=a\alpha b+a\alpha 1\\&=a\alpha b+a,\ \text{ for all } b\in M.\end{aligned}$$

Hence the theorem. ∎

Theorem 3.31. *Let M be a Γ-semiring with unity 1, satisfying the identities $a+a\alpha b=a$ and $a+1=a$, for all $a,b\in M,\alpha\in\Gamma$. If a semigroup $(M,+)$ is a left cancellative semigroup, then Γ-semigroup M is a right singular.*

Proof. Let M be a Γ-semiring with unity 1. We have $a+a\alpha b=a$ and $a+1=a$, for all $a,b\in M,\alpha\in\Gamma$, there exists $\gamma\in\Gamma$ such that $1\gamma b=b$ and $a+a\alpha b=a$, for all $a,b\in M$, $\alpha\in\Gamma$.

$$\begin{aligned}&\text{Now }\ a+a\alpha b=a,\ \text{ for all } \alpha\in\Gamma\\&\Rightarrow a+a\alpha b+b=a+b\\&\Rightarrow a+a\gamma b+1\gamma b=a+b\\&\Rightarrow a+(a+1)\gamma b=a+b\\&\Rightarrow a+a\gamma b=a+b\\&\Rightarrow a\gamma b=b.\end{aligned}$$

Hence Γ-semigroup M is a right singular. ∎

4. Γ-SEMIRING SATISFYING THE IDENTITY $a + a\alpha b + b = a$

In this section we study the properties of additive semigroup structure and Γ-semigroup structure of Γ-semiring satisfying the identity $a + a\alpha b + b = a$ for all $a, b \in M, \alpha \in \Gamma$.

Definition 4.1. A Γ-semiring M with zero element is said to be zero sum free Γ-semiring if $x + x = 0$, for all $x \in M$.

Definition 4.2. A Γ-semiring M is said to be Boolean Γ-semiring if $a = a\alpha a$, for all $a \in M, \alpha \in \Gamma$.

Definition 4.3. A Γ-semiring M is said to be zero square Γ-semiring M if $x\alpha x = 0$, for all $\alpha \in \Gamma, x \in M$.

Theorem 4.4. *Let M be a Γ-semiring with unity satisfying $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. If unity 1 of M is also an additive identity then*

(i) *semigroup $(M, +)$ is a band,*

(ii) *Γ-semigroup M is a band,*

(iii) *semigroup $(M, +)$ is a left singular,*

(iv) *semigroup $(M, +)$is a rectangular band.*

Proof. (i) Let $a, b \in M$. Then there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.
We have $a + a\gamma b + b = a$.

$$\begin{aligned}
&\Rightarrow a\gamma 1 + a\gamma b + b = a\\
&\Rightarrow a\gamma(1 + b) + b = a\\
&\Rightarrow a\gamma b + b = a\\
&\Rightarrow a + a\gamma b + b = a + a\\
&\Rightarrow a = a + a. \text{ Hence } (M, +) \text{ is a band.}
\end{aligned}$$

(ii) Let $a \in M$. Then there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.
We have $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned}
&\Rightarrow a\gamma 1 + a\alpha a + a = a\\
&\Rightarrow a\gamma 1 + a\alpha a + a\gamma 1 = a\\
&\Rightarrow a\gamma(1 + a) + a\gamma 1 = a\\
&\Rightarrow a\gamma a + a\gamma 1 = a\\
&\Rightarrow a\gamma a\gamma 1 + a\gamma 1 = a\gamma 1\\
&\Rightarrow a\gamma(a\gamma 1 + 1) = a\\
&\Rightarrow a\gamma(a\gamma 1) = a\\
&\Rightarrow a\gamma a = a.
\end{aligned}$$

Hence Γ-semigroup M is a band.

(iii) We have $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $b \in M$ there exists $\gamma \in \Gamma$ such that $1\gamma b = b$.

$$\begin{aligned}
&\Rightarrow a + a\gamma b + 1\gamma b = a \\
&\Rightarrow a + (a+1)\gamma b = a \\
&\Rightarrow a + a\gamma b = a \\
&\Rightarrow a + a\gamma b + b = a + b \\
&\Rightarrow a = a + b. \text{ Hence } M \text{ is a left singular.}
\end{aligned}$$

(iv) Let $a, b \in M$. There exists $\gamma \in \Gamma$ such that $1\gamma b = b$.

$$\begin{aligned}
a + b + a &= a + a\gamma b + b + b + a \\
&= a + a\gamma b + 1\gamma b + b + a \\
&= a + (a+1)\gamma b + b + a \\
&= a + a\gamma b + b + a \\
&= a + a \\
&= a \quad \text{from (i).}
\end{aligned}$$

Hence semigroup $(M, +)$ is a rectangular band. ■

Theorem 4.5. *Let M be a Γ-semiring with unity satisfies $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. If unity 1 is also an additive identity then Γ-semigroup M is a left singular.*

Proof. Let 1 be the unity as well as an additive identity of Γ-semiring M. We have $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $a, b \in M$ there exist $\gamma, \beta \in \Gamma$ such that $a\gamma 1 = a, 1\beta b = b$. Therefore $a + a\gamma b + b = a$.

$$\begin{aligned}
&\Rightarrow a\gamma 1 + a\gamma b + b = a \\
&\Rightarrow a\gamma(1 + b) + b = a \\
&\Rightarrow a\gamma b + b = a \\
&\Rightarrow a\gamma 1\beta b + 1\beta b = a \\
&\Rightarrow (a\gamma 1 + 1)\beta b = a \\
&\Rightarrow a\gamma 1\beta b = a \\
&\Rightarrow a\gamma b = a.
\end{aligned}$$

Hence the theorem. ■

Theorem 4.6. *Let M be a Γ-semiring satisfying $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is a band then*

(i) *semigroup* $(M,+)$ *is a left singular.*

(ii) *If semigroup* $(M,+)$ *is a commutative then* $a+a\alpha b=a$, *for all* $a,b\in M,\alpha\in\Gamma$.

Proof. (i) Suppose $a+a\alpha b+b=a$, for all $a,b\in M,\alpha\in\Gamma$.

$$\begin{aligned}&\Rightarrow\ a+a\alpha b+b+b=a+b\\&\Rightarrow\ a+a\alpha b+b=a+b\\&\Rightarrow\ a=a+b.\end{aligned}$$

(ii) Suppose $a+a\alpha b+b=a$, for all $a,b\in M,\alpha\in\Gamma$

$$\begin{aligned}&\Rightarrow\ a+a\alpha(b+b)+b=a\\&\Rightarrow\ a+a\alpha b+a\alpha b+b=a\\&\Rightarrow\ a+a\alpha b+b+a\alpha b=a\\&\Rightarrow\ a+a\alpha b=a.\end{aligned}$$

Hence the theorem. ∎

Theorem 4.7. *Let* M *be a* Γ*-semiring with an unity* 1 *which is also an additive identity. If semigroup* $(M,+)$ *is a left singular then* $a+a\gamma b+b=a,\gamma\in\Gamma,a,b\in M$.

Proof. Let $a,b\in M$. There exists $\gamma\in\Gamma$ such that $a\gamma 1=a$.

We have $1+b=1$.

$$\begin{aligned}&\Rightarrow a\gamma 1+a\gamma b=a\gamma 1\\&\Rightarrow a+a\gamma b=a\gamma 1=a\\&\Rightarrow a+a\gamma b+b=a+b\\&\Rightarrow a+a\gamma b+b=a.\end{aligned}$$

Hence the theorem. ∎

Theorem 4.8. *Let* M *be a Boolean* Γ*-semiring. If semigroup* $(M,+)$ *is a left singular, then* $a+a\alpha b+b=a$ *for all* $\alpha\in\Gamma$.

Proof. Let M be a Boolean Γ-semiring and $a,b\in M$. Then $a\alpha a=a$, for all $\alpha\in\Gamma$.

$$\begin{aligned}a+a\alpha b+b&=a\alpha a+a\alpha b+b\\&=a\alpha(a+b)+b\\&=a\alpha(a)+b\\&=a+b\\&=a.\end{aligned}$$

Hence the theorem. ∎

Theorem 4.9. *Let M be a zero square Γ-semiring. If M satisfies the identity $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$. Then $M\Gamma M = \{0\}$.*

Proof. Let $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned}
&\Rightarrow a + a\alpha b + b = a \\
&\Rightarrow a\alpha a + a\alpha(a\alpha b) + a\alpha b = a\alpha a \\
&\Rightarrow 0 + 0\alpha b + a\alpha b = 0 \\
&\Rightarrow a\alpha b = 0, \text{ for all } \alpha \in \Gamma.
\end{aligned}$$

Hence $M\Gamma M = \{0\}$. ■

Theorem 4.10. *Let M be a zero sum free Γ-semiring. $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$ if and only if $a\alpha b = b$, for all $a, b \in M, \alpha \in \Gamma$.*

Proof. Suppose $a + a\alpha b + b = a$, for all $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned}
&\Rightarrow \; a + a\alpha b + b + b = a + b \\
&\Rightarrow \; a + a\alpha b = a + b \\
&\Rightarrow \; a + a + a\alpha b = a + a + b \\
&\Rightarrow \; a\alpha b = b.
\end{aligned}$$

Conversely suppose that $a\alpha b = b$, for all $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned}
&\Rightarrow \; a + a\alpha b = a + b \\
&\Rightarrow \; a + a\alpha b + b = a + b + b \\
&\Rightarrow \; a + a\alpha b + b = a. \text{ for all } a, b \in M, \alpha \in \Gamma.
\end{aligned}$$

Hence the theorem. ■

5. Conclusion

we studied the properties of structures of the semigroup $(M, +)$ and the Γ-semigroup M of Γ-semiring M and we concluded that structures of the semigroup $(M, +)$ and the Γ-semigroup M of Γ-semiring M are dependent and additive and multiplicative structures of a Γ-semiring M play an important role in determining the structure of a Γ-semiring M.

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