Discussiones Mathematicae
General Algebra and Applications 37 (2017) 127–136
doi:10.7151/dmgaa.1275

LEFT ZEROID AND RIGHT ZEROID ELEMENTS OF Γ-SEMIRINGS

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Abstract

In this paper we introduce the notion of a left zeroid and a right zeroid of Γ-semirings. We prove that, a left zeroid of a simple Γ-semiring M is regular if and only if M is a regular Γ-semiring.

Keywords: left zeroid, right zeroid, idempotent, Γ-semiring, division Γ-semiring.

2010 Mathematics Subject Classification: 16Y60, 03G25.

1. Introduction

The notion of a semiring is an algebraic structure with two associative binary operations where one distributes over the other, was first introduced by Vandiver [14] in 1934 but semirings had appeared in earlier studies on the theory of ideals of rings. In structure, semirings lie between semigroups and rings. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [13] in 1964. In 1981, Sen [11] introduced the notion of a Γ-semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [5] in 1932, Lister [6] introduced the notion of a ternary ring. Neumann [12] studied regular rings. In 1995, Murali Krishna Rao [7, 8, 9, 10] introduced the notion of a Γ-semiring as a generalization of Γ-ring, ring, ternary semiring and semiring. The set of all negative integers Z is not a semiring with respect to usual addition and multiplication but Z forms a Γ-semiring where $\Gamma = Z$. The important reason for the development of Γ-semiring is a generalization of results of rings, Γ-rings, semirings, ternary semirings and semigroups.

Clifford and Miller [3] studied zeroid elements in semigroups. Dawson [4] studied semigroups having left or right zeroid elements. The zeroid of a semiring was introduced by Bourne and Zassenhaus [2]. In this paper,we extend the concept of left or right zeroid elements of semigroup to Γ-semiring. We prove that, a left zeroid μ of a simple Γ-semiring M is regular if and only if M is a regular Γ-semiring.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions necessary for this paper.

Definition 2.1. An element u of a semigroup M is called a zeroid element of M if, for each element a of M, there exist x and y in M such that $ax = ya = u$.

Definition 2.2. [1] A semiring $(M, +, \cdot)$ is an algebra with two binary operations " + " and " $\cdot$ " such that $(M, +)$ and $(M, \cdot)$ are semigroups and the following distributive laws hold.
$x(y + z) = xy + xz$
$(x + y)z = xz + yz,$ for all $x, y, z \in M$.

Definition 2.3. A semiring $(M, +, \cdot)$ is said to be division semiring if $(M \setminus \{0\}, \cdot)$ is a group.

Definition 2.4. Let M and Γ be two non-empty sets. Then M is called a Γ-semigroup if it satisfies

(i) $x\alpha y \in M$
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$.

Definition 2.5. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. A Γ-semigroup M is said to be Γ-semiring M if it satisfies the following axioms, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.

Every semiring M is a Γ-semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication

Definition 2.6. A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$ and $\alpha \in \Gamma$.

Definition 2.7. Let M be a Γ-semiring and A be a non-empty subset of M. A is called a Γ-subsemiring of Γ-semiring M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition 2.8. Let M be a Γ-semiring. A subset A of M is called a left (right) ideal of Γ-semiring M if A is closed under addition and $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$. A is called an ideal of M if it is both a left ideal and a right ideal of M.

Definition 2.9. A Γ-semiring M with zero element 0 is said to be hold cancellation laws if $a + b = a + c, b + a = c + a$, where $a, b, c \in M$, then $b = c$.

Definition 2.10. A Γ-semiring M is said to be simple Γ-semiring if it has no proper ideals of M.

Definition 2.11. Let M be a Γ-semiring. An element $a \in M$ is said to be α-idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α-idempotent.

Definition 2.12. Let M be a Γ-semiring. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.13. Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.14. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M$, $\alpha \in \Gamma$ such that $b\alpha a = 1(a\alpha b = 1)$.

Definition 2.15. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.16. A Γ-semiring with unity 1 is said to be division Γ-semiring M, if every nonzero element of Γ-semigroup M has inverse element.

3. A left zeroid and a right zeroid elements in Γ-semirings

In this section we introduce the notion of a left zeroid and a right zeroid elements in Γ-semirings and we study their properties.

Definition 3.1. An element x of a Γ-semiring M is called a left zeroid (right zeroid) if for each $y \in M$, $\alpha \in \Gamma$, there exists $a \in M$ such that $a\alpha y = x$ ($y\alpha a = x$).

Example 3.1. Let $M = \{0, 1\}$ and $\Gamma = \{\alpha, \beta\}$. We define operations with the following tables:

+	0	1
0	0	0
1	0	1

+	α	β
α	α	α
β	α	β

α	0	1
0	0	0
1	1	1

β	0	1
0	0	0
1	1	1

Then $(M, +), (\Gamma, +)$ are semigroups and M is a Γ-semiring. Here 1 and 0 are left zeroids and 1 and 0 are not right zeroids.

Definition 3.2. Let M be a Γ-semiring and $\alpha \in \Gamma$. Define binary operaton " $*$ " on M by $a * b = a\alpha b$ for all $a, b \in M$. Then $(M, +, *)$ is a semiring. It is denoted by M_α.

Theorem 3.1. *Let M be a Γ-semiring with a left zeroid element x and a α-idempotent e. Then $x\alpha e = x$.*

Proof. Let x be a left zeroid element and e be a α-idempotent. Then there exists $a \in M$ such that $a\alpha e = x$. Therefore

$$\begin{aligned} x\alpha e &= a\alpha e\alpha e \\ &= a\alpha e \\ &= x. \end{aligned}$$

■

Corollary 3.2. *Let M be a Γ-semiring with a right zeroid element x and α-idempotent e. Then $e\alpha x = x$.*

Theorem 3.3. *Let M be a Γ-semiring and e be a left zeroid element of M. Then $x\alpha e$ is a left zeroid of M for all $x \in M$, $\alpha \in \Gamma$.*

Proof. Let $y \in M, \alpha \in \Gamma$. Then there exists $t \in M$ such that $t\alpha y = e$, since e is a left zeroid of M.

$\Rightarrow x\alpha t\alpha y = x\alpha e$.

Hence $x\alpha e$ is a left zeroid of M. ■

Corollary 3.4. *Let M be a Γ-semiring, e be a left zeroid of M. Then every element of $M\Gamma e$ is a left zeroid of M.*

Theorem 3.5. *Let M be a Γ-semiring. Then the following are equvivalent.*

(i) *e is a left zeroid of M.*

(ii) *e is a left zeroid of a semiring M_α, for some $\alpha \in \Gamma$.*

(iii) *e is left zeroid of a semiring M_β, for all $\beta \in \Gamma$.*

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii) Suppose e is a left zeroid of a semiring M_α, $\alpha, \beta \in \Gamma$, $x \in M$. Therefore $x\beta x \in M$. Then there exists $z \in M$ such that $z\alpha(x\beta x) = e$, since e is a left zeroid of a semiring M_α,

$\Rightarrow (z\alpha x)\beta x = e$.

Hence e is a left zeroid of a semiring M_β. Therefore e is a left zeroid of a semiring M_β for all $\beta \in \Gamma$.

(iii)$\Rightarrow$(i) is obvious.

Hence the Theorem. ■

Definition 3.3. Let M be a Γ-semiring and $a \in M$. If there exists $b \in M$ such that $b + a = b$ $(a + b = b)$, then a is said to be additively left (right) zeroid of M.

Theorem 3.6. *Let M be a Γ-semiring with $a + a\alpha b = a$ for all $a, b \in M, \alpha \in \Gamma$. If x is a left zeroid of M, then x is an additively left zeroid of M.*

Proof. Suppose $x \in M$ is a left zeroid, $c \in M$ and $\alpha \in \Gamma$. Then there exists $b \in M$ such that $b\alpha c = x$.

$\Rightarrow b + b\alpha c = b + x$

$\Rightarrow b = b + x$.

Therefore x is an additively left zeroid. Hence the Theorem. ■

Theorem 3.7. *Let M be a Γ-semiring with $a + a\alpha b = a$ for all $a, b \in M$, $\alpha \in \Gamma$ and $(M, +)$ be left cancellative. If x is an additively left zeroid of M, then x is a left zeroid of M.*

Proof. Suppose x is an additively left zeroid of M and $\alpha \in \Gamma$. Then there exists $b \in M$ such that $b = b + x$.

$\Rightarrow b + b\alpha c = b + x$, where $c \in M$

$\Rightarrow b\alpha c = x$.

Hence the Theorem. ■

Theorem 3.8. *Let M be a Γ-semiring. If Γ-semiring M has both a left zeroid and a right zeroid. Then every left or right zeroid of M is a zeroid of M.*

Proof. Suppose μ and μ' are a left zeroid and a right zeroid of M respectively, $x \in M$ and $\alpha \in \Gamma$.

Then there exist $s, t \in M$ such that $s\alpha\mu\alpha\mu = \mu$ and $\mu'\alpha\mu'\alpha t = \mu'$. Since $s\alpha\mu$ and $\mu'\alpha t$ are left and right zeroids respectively, there exist $g, h \in M$ such that $g\alpha\mu' = s\alpha\mu$ and $\mu\alpha h = \mu'\alpha t$. Therefore

$$\begin{aligned} s\alpha\mu &= g\alpha\mu' \\ &= g\alpha\mu'\alpha\mu'\alpha t \\ &= g\alpha\mu'\alpha\mu\alpha h \\ &= s\alpha\mu\alpha\mu\alpha h \\ &= (s\alpha\mu\alpha\mu)\alpha h \\ &= \mu\alpha h \\ &= \mu'\alpha t. \end{aligned}$$

Hence

$$\mu = s\alpha\mu\alpha\mu = \mu'\alpha t\alpha\mu.$$

Thus μ is a zeroid of M. Similarly we can prove μ' is a zeroid of M. Hence the Theorem. ■

Theorem 3.9. *If e is a α-idempotent, $\alpha \in \Gamma$, then e is the left identity of $e\Gamma M$ and e is the right identity of $M\Gamma e$.*

Proof. Let $e\gamma x \in e\Gamma M$. Then $e\alpha e\gamma x = e\gamma x$. Hence e is the left identity of $e\Gamma M$. Similarly we can prove e is the right identity of $M\Gamma e$. ■

Theorem 3.10. *If e is a α-idempotent left zeroid of a Γ-semiring M, then $e\Gamma M$ is a division Γ-semiring.*

Proof. Obviously $e\Gamma M$ is a Γ-subsemiring of M and e is the left identity of $e\Gamma M$. Suppose $e\gamma b \in e\Gamma M$ there exists $c \in M$ such that $c\alpha(e\gamma b) = e$.
$\Rightarrow (e\alpha c)\alpha(e\gamma b) = e\alpha e$.
Therefore $(e\alpha c)\alpha(e\gamma b) = e$. Hence e is the left zeroid of $e\Gamma M$ and $e\alpha c$ is the left inverse of $e\Gamma M$. Thus $e\Gamma M$ is a division Γ-semiring. ■

Theorem 3.11. *Let U be a non empty set of all left zeroids of Γ-semiring M. Then U is a left ideal of M.*

Proof. Let $x_1, x_2 \in U$, $a \in M$ and $\alpha \in \Gamma$. Suppose $x \in M$. Then there exist $y, z \in M$ such that $y\alpha x = x_1$ and $z\alpha x = x_2$.
$\Rightarrow (y + z)\alpha x = x_1 + x_2$.
Therefore $x_1 + x_2$ is a left zeroid of M. By Corollary [3.4], $a\alpha x_1$ is a left zeroid of M. Hence U is a left ideal of M. ■

Theorem 3.12. *Let M be a Γsemiring and e be a α-idempotent left zeroid of M, $\alpha \in \Gamma$. Then a mapping $f : M \to (e\alpha M)_\alpha$, defined by $f(x) = e\alpha x$, is an onto homomorphism.*

Proof. Let $x_1, x_2 \in M$. Then

$$\begin{aligned} f(x_1 + x_2) &= e\alpha(x_1 + x_2) \\ &= e\alpha x_1 + e\alpha x_2 \\ &= f(x_1) + f(x_2) \\ f(x_1\alpha x_2) &= e\alpha(x_1\alpha x_2) \\ &= (e\alpha x_1)\alpha x_2 \\ &= e\alpha(e\alpha x_1)\alpha x_2 \\ &= [(e\alpha x_1)\alpha e]\alpha x_2 \\ &= (e\alpha x_1)\alpha(e\alpha x_2) \\ &= f(x_1)\alpha f(x_2). \end{aligned}$$

Hence f is a homomophism from M to $(e\alpha M)_\alpha$. Obviously f is onto. Hence the Theorem. ∎

Theorem 3.13. *Let e be a α-idempotent zeroid of Γ-semiring M and U be set of all zeroids of M. Then U is a commutattive division Γ-semiring with identity.*

Proof. Let U be the set of all zeroids of Γ-semiring and e be a α-idempotent, $\alpha \in \Gamma$. Then $U = e\Gamma M{=}M\Gamma e$. Then U is a division Γ-semiring with identity e by Theorem [3.10].

$$\text{if } x \in M, \quad \text{then} \quad \begin{aligned} e\alpha x &= e\alpha(e\alpha x) \\ &= (e\alpha x)\alpha e \\ &= e\alpha(x\alpha e) \\ &= x\alpha e. \end{aligned}$$

Hence the Theorem. ∎

Corollary 3.14. *Let M be a Γ-semiring. If M has a left zeroid and a right zeroid. Then U is an ideal of M.*

Corollary 3.15. *Let M be a simple Γ-semiring. If M has a left zeroid and a right zeroid, then every element of M is a zeroid.*

Theorem 3.16. *If e is a α-idempotent left zeroid of a Γ-semiring M then $M\Gamma e$ is a regular Γ-semiring.*

Proof. Obviously e is a right identity of $M\Gamma e$. Suppose $z\gamma e \in M\Gamma e$. Then there exist $g \in M, \beta \in \Gamma$ such that $g\beta z\gamma e = e$

$$\text{consider} \quad \begin{aligned} e &= e\alpha e \\ &= e\alpha(g\beta z\gamma e) \\ &= (e\alpha g)\beta(z\gamma e). \end{aligned}$$

Therefore e is a left zeroid of $M\Gamma e$. Suppose $x \in M\Gamma e$ and $\beta \in \Gamma$, then there exists $y \in M\Gamma e$ such that $y\beta x = e$. Then by Theorem [3.2], Corollary [3.4] $x\alpha y\beta x = x\alpha e = x$. Thus $M\Gamma e$ is a regular Γ-semiring. ■

Theorem 3.17. *Let M be a Γ-semiring. If e is the only idempotent of M, which is a left zeroid of M, then e is a zeroid of M.*

Proof. Let e be the only idempotent of Γ-semiring M, which is a left zeroid of M. Then by Theorem [3.16], $M\Gamma e$ is a regular Γ-semiring. Suppose $b \in M\Gamma e$, then there exists $\alpha, \beta \in \Gamma$, $x \in M\Gamma e$, such that $b = b\alpha x\beta b$. Therefore $b\alpha x$ is an idempotent of M. Hence $b\alpha x = e$. Each element of $M\Gamma e$ has right inverse and e is a right identity of $M\Gamma e$. Therefore $M\Gamma e$ is a Γ division semiring. Let $c \in M, \gamma \in \Gamma$. Then $c\gamma e \in M\Gamma e$. There exist $d\delta e \in M\Gamma e$, $\beta \in \Gamma$ such that

$$(c\gamma e)\beta(d\delta e) = e$$
$$\Rightarrow c\gamma(e\beta d\delta e) = e.$$

Therefore e is a right zeroid of M. Thus e is a zeroid of M. ■

We define a relation " $\leq$ " on the non-empty set of idempotents of a Γ-semiring M as follows:

$$e \leq f \Leftrightarrow e\alpha f = e, \quad \text{for some} \quad \alpha \in \Gamma.$$

Theorem 3.18. *Let M be a Γ-semiring. If e is a unique least idempotent and the left (right) zeroid of M, then e is a zeroid of M.*

Proof. Suppose e is the least unique idempotent and the left zeroid of M. Let M contains an idempotent f, which is a left zeroid of M and $\alpha \in \Gamma$. By Theorem [3.1] $f\alpha e = f$. Then $f \leq e$. Since e is the unique least idempotent, we have $f = e$. Therefore by Theorem [3.17], e is a zeroid of M. Suppose that e is a right zeroid of M. Let M contains an idempotent f which is a right zeroid of M, $\alpha \in \Gamma$. By Corollary [3.2], we have $f\alpha e = e$. Therefore $f \leq e$. Hence $e = f$. Thus e is the only idempotent of M which is a right zeroid of M. By Theorem [3.17], e is a zeroid of M. Hence the Theorem. ■

Theorem 3.19. *A Γ-semiring M with a left zeroid μ contains a left zeroid idempotent if and only if μ is a regular of M.*

Proof. Suppose left zeroid μ is a regular element of M. Then there exist $\alpha, \beta \in \Gamma$ and $x \in M$ such that $\mu = \mu\alpha x\beta\mu$. Then $x\beta\mu = x\beta\mu\alpha x\beta\mu$. Hence $x\beta\mu$ is a left zeroid idempotent. Conversely suppose that e is a left zeroid idempotent of M. We can prove e is a left zeroid of $M\alpha\mu$. By Theorem [3.16], $M\alpha\mu\alpha e$ is regular. $M\alpha\mu\alpha e = M\alpha(\mu\alpha e) = M\alpha\mu$. Hence $M\alpha\mu$ is regular. Therefore $\mu \in M\gamma\mu$, since μ is left zeroid. Thus μ is regular. ■

Theorem 3.20. *Let M be a simple Γ-semiring. Then a left zeroid μ of a simple Γ-semiring M is regular if and only if M is a regular Γ-semiring.*

Proof. Suppose M is a simple Γ-semiring with a regular left zeroid μ of M. Since μ is regular, there exist $\gamma, \beta \in \Gamma$, $x \in M$ such that $\mu = \mu\gamma x\beta\mu$. Then $x\beta\mu$ is an idempotent of M. Suppose $b \in M$ and $\gamma \in \Gamma$. Then there exists $c \in M$ such that $c\gamma b = \mu$ and there esists $d \in M$ such that $d\gamma c = \mu$. $\mu\gamma b = d\gamma c\gamma b = d\gamma\mu$. Therefore $M\gamma\mu\gamma b = M\gamma(d\gamma\mu) = (M\gamma d)\gamma\mu \subseteq M\gamma\mu$. Thus $M\gamma\mu$ is a right ideal of M. Obviously $M\gamma\mu$ is a left ideal of M. Hence $M\gamma\mu = M$, since M is simple. Every element of M is a left zeroid of M. Thus $x\beta\mu$ is a left zeroid idempotent of M. By Theorem [3.16], $M\gamma x\beta\mu$ is regular.

$$\begin{aligned} M\gamma x\beta\mu &= M\gamma\mu\gamma x\beta\mu \\ &= M\gamma\mu \\ &= M. \end{aligned}$$

Converse is obvious. Hence the Theorem. ■

Acknowledgement

The authors are highly grateful to referee for his valuable suggestions for improving this paper.

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Received 4 March 2017
Revised 13 April 20177
Accepted 24 April 2017