Discussiones Mathematicae
General Algebra and Applications 37 (2017) 177–187
doi:10.7151/dmgaa.1273

A NOTE ON IDEAL BASED ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING

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Abstract

In this paper, we consider the ideal based zero divisor graph $\Gamma_I(R)$ of a commutative ring R. We discuss some graph theoretical properties of $\Gamma_I(R)$ in relation with zero divisor graph. We also relate certain parameters like vertex chromatic number, maximum degree and minimum degree for the graph $\Gamma_I(R)$ with that of $\Gamma(\frac{R}{I})$. Further we determine a necessary and sufficient condition for the graph to be Eulerian and regular.

Keywords: zero-divisor graph, chromatic number, ideal based zero divisor graph, clique number.

2010 Mathematics Subject Classification: 05C69, 05C45, 13A15.

1. Introduction

The study of algebraic graph theory is an interesting subject for mathematicians and goes back at least to 1973, when N. Biggs published his work. As he wrote in the preface of his book, his aim was "to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about graphs". Although Biggs spoke of algebraic methods and algebra in general, the kind of algebra he really used was linear algebra and some properties of polynomials. Later on, Beck [6], studied the graph of zero divisors of a commutative ring, where he was mainly interested in coloring. This investigation

of coloring of the zero divisor graph of a commutative ring was then continued by Anderson and Naseer [4]. Since then, the attention of algebraist's and graph theorists has focused on the graph of zero divisors. In this article we have the same aim as N. Biggs has had but we will study the generalized zero divisor graph of an associative ring. Let R be a commutative ring with identity 1 and $Z(R)$ be its zero divisors. The zero divisor graph of R denoted by $\Gamma(R)$ is an undirected graph whose vertices are the nonzero zero divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$. The zero-divisor graph has been extended to other algebraic structures in DeMeyer [8] *et al.* and Redmond [11].

Let R be a commutative ring and let I be an ideal of R. The ideal based zero divisor graph is an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I$ for some $y \in R - I\}$,where distinct vertices x and y are adjacent if and only if $xy \in I$. It was introduced by Redmond [12]. In [12], he found the values of parameters such as connectivity, clique, diameter, girth etc. in relation with zero divisor graph. Further various research work is going on here. In this paper, we find the values of parameters such as vertex chromatic number, clique number, maximum and minimum degree etc. In Section 2, we give the definition and theorem from [12] which are needed for subsequent sections. Section 3 discusses vertex chromatic number and the relation between clique number and chromatic number of the graph. In Section 4 discusses minimum and maximum degree of the graph. Further we find a necessary and sufficient condition for the graph $\Gamma_I(R)$ to be an Eulerian graph and a regular graph.

A ring R is said to be decomposable if R can be written as $R_1 \times R_2$, where R_1 and R_2 are rings; otherwise R is said to be indecomposable. If X is either an element or a subset of R, then $Ann(X)$ denotes the annihilator of X in R. For any subset X of R, we define $X^* = X - \{0\}$ and $|X|$ denote the number of elements in X.

For a graph G, the degree $\deg(v)$ of a vertex v in G is the number of edges incident with v. Denote the degree of the vertex v in $\Gamma_I(R)$ by $\deg(v)$ and that of $\Gamma(R)$ by $\deg_\Gamma(v)$. We denote the minimum and maximum degree of vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is regular if the degrees of all vertices of G are the same. A graph G is 1-factor if every vertex of G is of degree 1. We denote the complete graph with n vertices and complete bipartite graph with two parts of sizes m and n, by K_n and $K_{m,n}$, respectively. A subset X of the vertices of G is called a clique if the induced subgraph on X is a complete graph. The number of vertices in the set X is denoted by $|X|$.

An Eulerian trail is a closed trail which traverses each edge exactly once. A graph is Eulerian if it contains an Eulerian trail. A proper k-vertex coloring of a graph G is an assignment of k colors $\{1, \ldots, k\}$ to the vertices of G such that no two adjacent vertices have the same color. The vertex chromatic number $\chi(G)$ of a graph G, is the minimum k for which G has a proper k-vertex coloring.

2. Preliminaries

Definition 2.1 [12]**.** Let R be a commutative ring and let I be an ideal of R. The ideal based zero divisor graph is an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

Example 2.2. For $R \cong \mathbb{Z}_6 \times \mathbb{Z}_3$ and $I \cong 0 \times \mathbb{Z}_3$, $\Gamma_I(R)$ is shown in Figure 1.

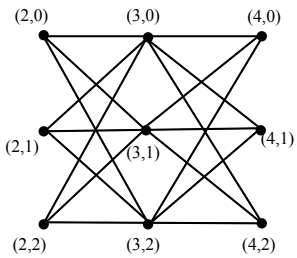


Figure 1.

Remark 2.3 [12]**.** Let I be an ideal of a ring R. Then $\Gamma_I(R)$ is a graph on a finite number of vertices if and only if either R is finite or I is a prime ideal. Moreover, if $\Gamma(\frac{R}{I})$ is a graph on N vertices, then $\Gamma_I(R)$ is a graph on $N\,|I|$ vertices.

Theorem 2.4 [12]**.** *Let I be an ideal of a ring R, and let $x, y \in R - I$. Then*

(a) *if $x + I$ is adjacent to $y + I$ in $\Gamma(\frac{R}{I})$, then x is adjacent to y in $\Gamma_I(R)$,*
(b) *if x is adjacent to y in $\Gamma_I(R)$ and $x + I \neq y + I$, then $x + I$ is adjacent to $y + I$ in $\Gamma(\frac{R}{I})$,*
(c) *if x is adjacent to y in $\Gamma_I(R)$ and $x + I = y + I$, then $x^2, y^2 \in I$.*

Corollary 1 [12]**.** *If x and y are (distinct) adjacent vertices in $\Gamma_I(R)$, then all (distinct) elements of $x + I$ and $y + I$ are adjacent in $\Gamma_I(R)$. If $x^2 \in I$, then all the distinct elements of $x + I$ are adjacent in $\Gamma_I(R)$.*

Remark 2.5 [12]**.** Clearly there is a strong relationship between $\Gamma(\frac{R}{I})$ and $\Gamma_I(R)$. Let I be an ideal of a ring R. One can verify that the following method can be used to construct a graph $\Gamma_I(R)$. Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq R$ be a set of coset representatives of the vertices of $\Gamma(\frac{R}{I})$. For each $i \in I$, define a graph G_i with vertices $\{a_\lambda + i : \lambda \in \Lambda\}$, where edges are defined by the relationship $a_\lambda + i$ is adjacent to $a_\beta + i$ in G_i if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(\frac{R}{I})$ (i.e., $a_\lambda a_\beta \in I$).

Define the graph G to have as its vertex set $V = \bigcup_{i\in I} G_i$. We define the edge set of G to be:

(1) all edges contained in G_i for each $i \in I$,

(2) for distinct $\lambda, \beta \in \Lambda$ and for any $i, j \in I$, $a_\lambda + i$ is adjacent to $a_\beta + j$ if and only if $a_\lambda + I$ is adjacent to $a_\beta + I$ in $\Gamma(\frac{R}{I})$ (i.e., $a_\lambda a_\beta \in I$),

(3) for $\lambda \in \Lambda$ and distinct $i, j \in I, a_\lambda + i$ is adjacent to $a_\lambda + j$ if and only if $a_\lambda^2 \in I$.

Definition 2.6 [12]**.** Using the notation as in the above construction, we call the subset $a_\lambda + I$ a column of $\Gamma_I(R)$. If $a_\lambda^2 \in I$, then we call $a_\lambda + I$ a connected column of $\Gamma_I(R)$.

Remark 2.7. Denote the vertices of $\Gamma(\frac{R}{I})$ by $V(\Gamma(\frac{R}{I})) = \{a_i + I : i \in \Lambda\}$. From the Remark 2.5, we can denote the vertex set of $\Gamma_I(R)$ as $V(\Gamma_I(R)) = \{a_i + h : i \in \Lambda, h \in I\}$.

Theorem 2.8 [2, Theorem 7]**.** *Let R be a finite ring. If $\Gamma(R)$ is a regular graph, then it is either a complete graph or a complete bipartite graph.*

Theorem 2.9 [12, Theorem 5.7]**.** *Let I be a nonzero ideal of a ring R. Then $\Gamma_I(R)$ is bipartite if and only if either* (a) *$gr(\Gamma_I(R)) = \infty$ or* (b) *$gr(\Gamma_I(R)) = 4$ and $\Gamma(\frac{R}{I}))$ is bipartite.*

Theorem 2.10 [3, Theorem 2.8]**.** *Let R be a commutative ring. Then $\Gamma(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for every $x, y \in Z(R)$. In particular, if R is a reduced commutative ring and not a field, then $\Gamma(R)$ is a complete graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

3. Chromatic number of $\Gamma_I(R)$

In this section we characterize when the chromatic number and clique number of $\Gamma_I(R)$ are equal and we also prove the following relationship $\omega(\Gamma_I(R)) \leq |I|\,\omega(\Gamma(\frac{R}{I}))$ and $\chi(\Gamma_I(R)) \leq |I|\,\chi(\Gamma(\frac{R}{I}))$.

Theorem 3.1. *Let R be a commutative ring and I be an ideal of R. If $\chi(\Gamma(\frac{R}{I})) = \omega(\Gamma(\frac{R}{I}))$, then $\chi(\Gamma_I(R))) = \omega(\Gamma_I(R))$.*

Proof. Assume $\chi(\Gamma(\frac{R}{I})) = \omega(\Gamma(\frac{R}{I})) = k$. Let $V(\Gamma(\frac{R}{I})) = \{a_i + I : i \in \Lambda\}$. Let $A_1, A_2, \ldots, A_k$ be the distinct color classes of $\Gamma(\frac{R}{I})$. Since $\omega(\Gamma(\frac{R}{I})) = k$, there exists elements $a_1 + I, a_2 + I, \ldots, a_k + I$ such that no two of them lie in same color class. Without loss of generality, let $a_i + I \in A_i$ for all i. Let $S = \{a_1 + I, a_2 + I, \ldots, a_k + I\}$. Then $\langle S \rangle$ is a maximal complete subgraph of $\Gamma(\frac{R}{I})$. Let $H = \{a_i : a_i + I \in S\} \cup \{a_i + m : a_i + I \in S, a_i^2 \in I, m \in I^*\}$. Since $\langle S \rangle$ is maximal, $\langle H \rangle$ is a maximal complete subgraph in $\Gamma_I(R)$. Hence $\omega(\Gamma_I(R)) \geq |H|$. So color the vertices of H with $|H|$ distinct colors. Clearly $a + I$ induces an independent set in $\Gamma_I(R)$ for $a^2 \notin I$ with $a + I \in S$ and so color that

vertices $a+m$ with the color of a, for all $m \in I^*$. Let $X = \{a : a+I \in S\}$. Then X have distinct colors. For each $y \notin X$, $y = a_s + m$, where $m \in I$ and $s \notin \{1, 2, \ldots, k\}$ and $y + I = a_s + I$. Since $a_s + I \in A_i$ and A_i's are independent, color the vertices $a_s + m$ with the color of $a_i + m$. Hence color the vertices which are not in X in this way and so this coloring is proper. Hence $\chi(\Gamma_I(R)) \leq |H|$. Since $\omega(\Gamma_I(R)) \leq \chi(\Gamma_I(R)), \chi(\Gamma_I(R)) = \omega(\Gamma_I(R))$. ■

Theorem 3.2. *Let R be a commutative ring and I be an ideal of R. Then $\omega(\Gamma_I(R)) \leq |I|\,\omega(\Gamma(\frac{R}{I}))$.*

Proof. Let $\omega(\Gamma(\frac{R}{I})) = m$. Choose $a_1 + I, a_2 + I, \ldots, a_m + I$ in $\Gamma(\frac{R}{I})$ such that $M = \bigcup_{1 \leq i \leq m}\{a_i + I\}$ induces a maximal complete subgraph. Let $S = \{a + h : a + I \in M, h \in I\}$. Assume $a^2 \in I$ for all $a + I \in M$. Then by Corollary 1, $\langle S \rangle$ is a complete subgraph in $\Gamma_I(R)$. If $S \cup \{p\}$ is complete subgraph in $\Gamma_I(R)$, then $p(a+h) \in I$ and so $(p+I)(a+I) = 0 + I$. So $M \cup \{p + I\}$ forms a clique of size $m+1$, which is a contradiction. Thus $\langle S \rangle$ is maximal and so $\omega(\Gamma_I(R)) = |S|$. Hence $\omega(\Gamma_I(R)) = |I|\,\omega(\Gamma(\frac{R}{I}))$. In all other cases, $\omega(\Gamma_I(R)) < |I|\,\omega(\Gamma(\frac{R}{I}))$ and so the result follows. ■

Theorem 3.3. *Let I be an ideal of a ring R. If $\Gamma_I(R)$ has no connected columns, then $\chi(\Gamma(\frac{R}{I})) = \chi(\Gamma_I(R))$.*

Proof. We have $\chi(\Gamma(\frac{R}{I})) \leq \chi(\Gamma_I(R))$. Assume $\chi(\Gamma(\frac{R}{I})) = k$. Let $A_1, A_2, \ldots, A_k$ be distinct color classes for $\Gamma(\frac{R}{I})$. Consider the set $B_i = \{a + h : a + I \in A_i, h \in I\}$. Since $a^2 \notin I$, B_i's are independent and $V(\Gamma_I(R)) = \bigcup_{i=1}^{k} B_i$. Hence $B_1, B_2, \ldots, B_k$ are distinct color classes of $\Gamma_I(R)$. $\Gamma_I(R)$ is a graph colored by k distinct colors and it is proper and so $\chi(\Gamma_I(R)) \leq k$. Thus $\chi(\Gamma(\frac{R}{I})) = \chi(\Gamma_I(R))$. ■

Theorem 3.4. *Let I be an ideal of a ring R. Then $2 \leq \chi(\Gamma(\frac{R}{I})) \leq \chi(\Gamma_I(R)) \leq |I|\,\chi(\Gamma(\frac{R}{I}))$.*

Proof. Since $\Gamma(\frac{R}{I})$ is connected, $\chi(\Gamma_I(R)) \geq 2$. Since $\Gamma(\frac{R}{I})$ is a subgraph of $\Gamma_I(R)$, we have $\chi(\Gamma(\frac{R}{I})) \leq \chi(\Gamma_I(R))$. Let $\chi(\Gamma(\frac{R}{I})) = k$ and $A_1, A_2, \ldots, A_k$ be distinct color classes of $\Gamma(\frac{R}{I})$. We have $V(\Gamma(\frac{R}{I})) = \{a_i + I : i \in \Lambda\}$. Assume $x^2 \in I$, for all $x + I \in \Gamma(\frac{R}{I})$. Then $x + I$ is a complete subgraph in $\Gamma_I(R)$. Now for each $1 \leq i \leq k$, and $h \in I$ define the set $B_{ih} = \{x + h : x + I \in A_i\}$. Since $A_i's$ are independent, so is B_{ih}. Also $\bigcup_{1 \leq i \leq k} \bigcup\_h \in IB_{ih} = V(\Gamma_I(R))$. Hence $\{B_{ih} : 1 \leq i \leq k \text{ and } h \in I\}$ are distinct color classes of $\Gamma_I(R)$. So it needs $|I|\,k$ colors. Hence this coloring is proper and $\chi(\Gamma(\frac{R}{I})) \leq k\,|I|$. In all other cases, $\chi(\Gamma(\frac{R}{I})) < k\,|I|$. Hence the result follows. ■

Example 3.5.

(1) Consider $R \cong \mathbb{Z}_{25} \times \mathbb{Z}_2$ and $I = \{0\} \times \mathbb{Z}_2$. Since $\Gamma(\frac{R}{I}) \cong K_4$ and every non zero element is nilpotent element of order 2, $\Gamma_I(R) \cong K_8$. So $\chi(\Gamma_I(R)) = 8 = |I|\,\chi(\Gamma(\frac{R}{I}))$.

(2) Consider $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = \mathbb{Z}_3 \times \{0\} \times \{0\}$. Then $\Gamma(\frac{R}{I}) \cong K_{2,2}$. By Theorem 2.9, $\Gamma_I(R)$ is complete bipartite graph and $\chi(\Gamma_I(R)) = 2 = \chi(\Gamma(\frac{R}{I}))$.

(3) Consider $R \cong \mathbb{Z}_{24}$ and $I = (8)$. Then $\Gamma(\frac{R}{I}) \cong K_{1,2}$. Also $\omega(\chi(\Gamma_I(R))) = 4$. so $\chi(\Gamma(\frac{R}{I})) < \chi(\Gamma_I(R))$.

Theorem 3.6. *Let $I \neq (0)$ be an ideal of R. If $\Gamma(\frac{R}{I})$ is a graph on a single vertex, then $\chi(\Gamma_I(R)) = |I|$.*

Proof. If $\Gamma(\frac{R}{I})$ has only one vertex, then $\Gamma_I(R)$ consists of a single connected column and, therefore, is the complete graph on $|I|$ vertices and so the result follows. ■

Theorem 3.7. *Let I be an ideal of a ring R. If $a + I$ is a connected column of $\Gamma_I(R)$, then $a + I$ is a complete subgraph of $\Gamma_I(R)$ and thus $\chi(\Gamma_I(R)) \geq |I|$.*

Proof. Since $a + I$ is a complete subgraph of $\Gamma_I(R)$, $\omega(\Gamma_I(R)) \geq |I|$. Hence the result follows. ■

Corollary 2. *If $\Gamma_I(R)$ has at least one connected column and I is infinite, then $\chi(\Gamma_I(R)) = \infty$.*

Corollary 3. *If $\Gamma_I(R)$ has a connected column and $\Gamma(\frac{R}{I})$ has at least two vertices, then $\chi(\Gamma_I(R)) \geq |I| + 1$.*

Proof. Let $a + I$ be a connected column of $\Gamma_I(R)$. By hypothesis, there exist $b \in R - I$ such that $a + I \neq b + I$ and $a + I$ is adjacent to $b + I$ in $\Gamma_I(R)$. Then each element of the connected column $a + I$ is adjacent to b and so $\{a + I\} \cup \{b\}$ forms a complete subgraph and it needs exactly $|I|$ colors. Hence $\chi(\Gamma_I(R)) \geq |I| + 1$. ■

4. Eulerian property of $\Gamma_I(R)$

In this section we discuss on maximum and minimum degree of $\Gamma_I(R)$. Using this result we prove the Eulerian property and regularity of $\Gamma_I(R)$.

Lemma 4.1. *Let I be an ideal of a ring R. Then in $\Gamma_I(R)$,*

$$\deg(a) = \begin{cases} |I| \deg_\Gamma(a + I) & \text{if} \quad a^2 \notin I, \\ |I| \deg_\Gamma(a + I) + |I| - 1 & \text{if} \quad a^2 \in I. \end{cases}$$

Proof. Let $a \in V(\Gamma_I(R))$. Then $\deg(a) \geq |I| \deg_\Gamma(a+I)$, since $a+h_1$ is adjacent to $b+h_2$ if and only if $a+I$ is adjacent to $b+I$. If $a^2 \in I$, then $a+I$ is a complete subgraph in $\Gamma_I(R)$. Hence $\deg(a) = |I| \deg_\Gamma(a+I) + |I| - 1$. If not, then $\deg(a) = |I| \deg_\Gamma(a+I)$ ■

Lemma 4.2. *Let $I \neq (0)$ be an ideal of a ring R. Then*

$$\delta(\Gamma_I(R)) = \begin{cases} |I| - 1 & \textit{if } \Gamma(\frac{R}{I}) \textit{ has a single vertex} \\ |I|\,\delta(\Gamma(\frac{R}{I})) + |I| - 1 & \textit{if } \Gamma_I(R) \textit{ has a connected column} \\ & a+I \textit{ with } \deg_\Gamma(a+I) = \delta(\Gamma(\frac{R}{I})) \\ |I|\,\delta(\Gamma(\frac{R}{I})) & \textit{otherwise.} \end{cases}$$

Proof. If $\Gamma(\frac{R}{I})$ is a graph on a single vertex, then $\Gamma_I(R)$ is a complete graph and $\delta(\Gamma_I(R)) = |I| - 1$. If $\Gamma_I(R)$ has a connected column, choose $a+I \in V(\Gamma_I(R))$ such that $\deg_\Gamma(a+I) = \delta(\Gamma(\frac{R}{I}))$. Clearly either $a^2 \in I$ or $a^2 \notin I$. If $a^2 \in I$, since $\deg_\Gamma(a+I) = \delta(\Gamma(\frac{R}{I}))$, $\deg(a) \leq \deg(b)$, for all $b \in V(\Gamma_I(R))$. By Lemma 4.1, $\deg(a) = |I| \deg_\Gamma(a+I) + |I| - 1$. So $\delta(\Gamma_I(R)) = |I|\,\delta(\Gamma(\frac{R}{I})) + |I| - 1$. If $a^2 \notin I$, Since $\Gamma_I(R)$ has a connected columns, $b^2 \in I$, for some $b \in V(\Gamma_I(R))$. Also $\deg_\Gamma(b+I) \geq \deg_\Gamma(a+I)$. This implies that $\deg(b) \geq \deg(a)$. By Lemma 4.1, $\deg(a) = |I| \deg(a+I)$. So $\delta(\Gamma_I(R)) = |I|\,\delta(\Gamma(\frac{R}{I}))$. If $\Gamma_I(R)$ has no connected column, then by Lemma 4.1 $\deg(a) = |I| \deg_\Gamma(a+I)$. So $\delta(\Gamma_I(R)) = |I|\,\delta(\Gamma(\frac{R}{I}))$. ■

Lemma 4.3. *Let $I \neq (0)$ be an ideal of a ring R which is not prime. Then*

$$\Delta(\Gamma_I(R)) = \begin{cases} |I| - 1 & \textit{if } \Gamma(\frac{R}{I}) \textit{ has a single vertex} \\ |I|\,\Delta(\Gamma(\frac{R}{I})) + |I| - 1 & \textit{if } \Gamma_I(R) \textit{ has a connected column} \\ & a+I \textit{ with } \deg_\Gamma(a+I) = \Delta(\Gamma(\frac{R}{I})) \\ |I|\,\Delta(\Gamma(\frac{R}{I})) & \textit{otherwise.} \end{cases}$$

Proof. If $\Gamma(\frac{R}{I})$ has a single vertex, then $\Gamma_I(R)$ is a complete graph and $\Delta(\Gamma_I(R)) = |I|-1$. If $\Gamma_I(R)$ has a connected column, then choose $a+I \in V(\Gamma(\frac{R}{I}))$ such that $\deg_\Gamma(a+I) = \Delta(\Gamma(\frac{R}{I}))$. Clearly either $a^2 \in I$ or $a^2 \notin I$. If $a^2 \in I$, then by Lemma 4.1 $\deg(a) = |I| \deg_\Gamma(a+I) + |I| - 1$ and so $\deg(a) \geq \deg(b)$. Thus $\Delta(\Gamma_I(R)) = |I|\,\Delta(\Gamma(\frac{R}{I})) + |I| - 1$. If $a^2 \notin I$, since $\Gamma_I(R)$ has a connected columns, $b^2 \in I$, for some $b \in V(\Gamma_I(R))$. We have $\deg_\Gamma(a+I) \geq \deg_\Gamma(b+I)$. This implies that $\deg(a) \geq \deg(b)$ and $\deg(a) = |I| \deg_\Gamma(a+I)$. In this case $\Delta(\Gamma_I(R)) = |I|\,\Delta(\Gamma(\frac{R}{I}))$. If $\Gamma_I(R)$ has no connected columns, then $\deg(a) = |I| \deg_\Gamma(a+I)$ and so the result follows. ■

Theorem 4.4. *Let I be an ideal of a ring R. If $\Gamma_I(R)$ has no connected column, then $\Gamma_I(R)$ is Eulerian if and only if either $|I|$ is even or $\Gamma(\frac{R}{I})$ is Eulerian.*

Proof. Since $\Gamma_I(R)$ has no connected column, $\deg(a) = |I|\deg(a+I)$, for all $a \in V(\Gamma_I(R))$. Assume $\Gamma(\frac{R}{I})$ is Eulerian. Then $\deg(a+I)$ is even, for all $a+I \in V(\Gamma(\frac{R}{I}))$. Therefore $\deg(a)$ is even, for all $a \in V(\Gamma_I(R))$ and so $\Gamma_I(R)$ is Eulerian. Similarly if $|I|$ is even, then $\deg(a)$ is even, for all $a \in V(\Gamma_I(R))$ and so $\Gamma_I(R)$ is Eulerian.

Conversely assume that $\Gamma_I(R)$ is Eulerian. Then $\deg(a)$ is even, for all $a \in V(\Gamma_I(R))$. So $|I|\deg_\Gamma(a+I)$, is even for all $a+I \in V(\Gamma(\frac{R}{I}))$. Hence either $|I|$ is even or $\deg_\Gamma(a+I)$ is even, for all $a+I \in V(\Gamma(\frac{R}{I}))$ and so the result follows. ■

Theorem 4.5. *Let I be an ideal of a ring R. If $\Gamma_I(R)$ has a connected column, then $\Gamma_I(R)$ is Eulerian if and only if $\Gamma(\frac{R}{I})$ is Eulerian and $|I|$ is odd.*

Proof. Assume $\Gamma_I(R)$ is Eulerian. Then $\deg(a)$ is even, where $a \in V(\Gamma_I(R))$. Since $\Gamma_I(R)$ has a connected column, there exist a vertex $a \in \Gamma_I(R)$ such that $a^2 \in I$. By Lemma 4.1 $\deg(a) = |I|\deg_\Gamma(a+I)+|I|-1$ is even. So $|I|\deg_\Gamma(a+I)$ is even and $|I|-1$ is even, for all $a+I \in V(\Gamma(\frac{R}{I}))$. This implies that $\deg_\Gamma(a+I)$ is even and $|I|$ is odd, for all $a+I \in V(\Gamma(\frac{R}{I}))$. Thus $\Gamma(\frac{R}{I})$ is Eulerian and $|I|$ is odd.

Conversely, assume that $\Gamma(\frac{R}{I})$ is Eulerian and $|I|$ is odd. Then $\deg_\Gamma(a+I)$ is even. By Lemma4.1, the result follows. ■

Example 4.6.

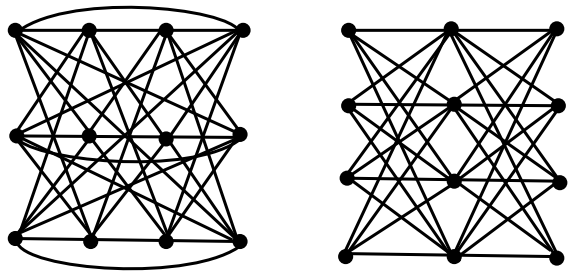
(1) Consider $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $I = \mathbb{Z}_3 \times \{0\} \times \{0\}$. Then $\Gamma(\frac{R}{I}) \cong K_{2,2}$ and $\Gamma_I(R)$ has no connected column. In this case $\Gamma(\frac{R}{I})$ is Eulerian. By Lemma 4.1, $\deg(a) = 2\,|I|$, for all $a \in V(\Gamma_I(R))$ and is even. So $\Gamma_I(R)$ is Eulerian (see Figure 2(a)).

(2) Consider $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $I = \mathbb{Z}_4 \times \{0\} \times \{0\}$. Then $\Gamma(\frac{R}{I})$ is not Eulerian and $\Gamma_I(R)$ has no connected column. By Lemma 4.1, $\deg(a) = 4\ \deg(a+I)$, for all $a \in V(\Gamma_I(R))$ and is even. So $\Gamma_I(R)$ is Eulerian (see Figure 2(b)).

(3) Consider $R \cong \mathbb{Z}_3 \times \frac{\mathbb{F}_4[x]}{(x^2)}$ and $I = \mathbb{Z}_3 \times \{0\}$. Then $\Gamma(\frac{R}{I}) \cong K_3$ is Eulerian and $|I|$ is odd. Also $\Gamma_I(R)$ has a connected column. By Lemma 4.1, $\Gamma_I(R) \cong K_9$ is Eulerian.

Theorem 4.7. *Let $I \neq (0)$ be an ideal of a ring R. If $\Gamma_I(R)$ has no connected columns, then $\Gamma_I(R)$ is a regular graph if and only if $\Gamma(\frac{R}{I})$ is a regular graph.*

Proof. Assume $\Gamma_I(R)$ is a regular graph. Then $\deg(a) = k$, for all $a \in V(\Gamma_I(R))$. Since $\Gamma_I(R)$ has no connected columns, $a^2 \notin I$, for all $a \in V(\Gamma_I(R))$. By Lemma 4.1, $|I|\deg_\Gamma(a+I) = k$, for all $a+I \in V(\Gamma(\frac{R}{I}))$. So $\deg_\Gamma(a+I) = \frac{k}{|I|}$,for all $a+I \in V(\Gamma(\frac{R}{I}))$. If k is prime, then $|I|\deg_\Gamma(a+I) = p$, for all $a+I \in V(\Gamma(\frac{R}{I}))$. Since $I \neq (0)$, $|I| = p$ and $\deg_\Gamma(a+I) = 1$, for all $a+I \in V(\Gamma(\frac{R}{I}))$. Then $\Gamma(\frac{R}{I})$ is a 1-factor graph. Since $diam(\Gamma(\frac{R}{I})) \leq 3$, $\Gamma(\frac{R}{I}) \cong K_2$ and is regular. If not, $\Gamma(\frac{R}{I})$ is a $\frac{k}{|I|}$-regular graph.



(a) (b)

Figure 2.

Conversely, assume that $\Gamma(\frac{R}{I})$ is a regular graph. Then $\deg_\Gamma(a+I)$ is k for all $a+I \in V(\Gamma(\frac{R}{I}))$. Since $\Gamma_I(R)$ has no connected columns, by Lemma 4.1, $\deg(a) = |I| \deg_\Gamma(a+I) = k\,|I|$, for all $a \in V(\Gamma_I(R))$. Thus $\Gamma_I(R)$ is a $k\,|I|$-regular graph. ■

Corollary 4. *Let $I \neq (0)$ be an ideal of a ring R and p be prime. Assume $\Gamma_I(R)$ has no connected columns. If $\Gamma_I(R)$ is a p-regular graph, then $\frac{R}{I}$ is isomorphic to one of the following rings:*
$\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9$, *or* $\frac{\mathbb{Z}_3[x]}{(x^2)}$.

Proof. If $\Gamma_I(R)$ is a p-regular graph, then as in the proof of Theorem 4.7 $\Gamma(\frac{R}{I}) \cong K_2$ and $\frac{R}{I}$ is isomorphic to the following rings $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_9$, or $\frac{\mathbb{Z}_3[x]}{(x^2)}$. ■

Converse of Corollary 4 is not true. For example consider $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $I = \{0\} \times \{0\} \times \mathbb{Z}_4$. Then $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\Gamma_I(R) \cong K_{4,4}$. So $\Gamma_I(R)$ is regular graph but not p-regular graph, for any prime p.

Theorem 4.8. *Let $I \neq (0)$ be an ideal of a ring R. Assume $a^2 \in I$, for all $a+I \in V(\Gamma(\frac{R}{I}))$. Then $\Gamma_I(R)$ is a k-regular graph, where $k \neq |I|-1$ if and only if $\Gamma(\frac{R}{I})$ is a regular graph.*

Proof. If $\Gamma_I(R)$ is a k-regular graph, then $\deg(a) = k$, for all $a \in V(\Gamma_I(R))$.

Since $a^2 \in I$, for all $a \in V(\Gamma(\frac{R}{I}))$. Then by Lemma 4.1, $\deg(a) = |I| \deg_\Gamma(a+I) + |I| - 1$ for all $a \in V(\Gamma_I(R))$ and so $\deg_\Gamma(a+I) = \frac{k-|I|+1}{|I|}$, for all $a+I \in V(\Gamma(\frac{R}{I}))$. Since $diam(\Gamma(\frac{R}{I})) \leq 3$, $\deg_\Gamma(a+I) \neq 0$. Thus $k \neq |I|-1$. For all remaining values of k, $\Gamma(\frac{R}{I})$ is a regular graph.

Conversely assume that $\Gamma(\frac{R}{I})$ is a regular graph. Then $\deg_\Gamma(a+I) = m$ and $a^2 \in I$, for all $a \in V(\Gamma(\frac{R}{I}))$. So $\deg(a) = |I|\,m + |I| - 1$, for all $a \in V(\Gamma_I(R))$. Thus $\Gamma_I(R)$ is a k-regular graph. ■

Theorem 4.9. *Let I be a non-zero ideal of a ring R. If $\Gamma(\frac{R}{I})$ is a graph on a single vertex, then $\Gamma_I(R)$ is $(|I|-1)$-regular graph.*

Proof. Since $\Gamma(\frac{R}{I})$ is a graph on single vertex, $\Gamma_I(R)$ is complete graph and so the result follows. ■

Theorem 4.10. *Let I be an ideal of a ring R such that $\frac{R}{I}$ is a finite ring. If $\Gamma_I(R)$ has no connected columns and is a regular graph, then $\Gamma(\frac{R}{I})$ is complete or a complete bipartite graph.*

Proof. The result follows from Theorems 4.7 and 2.8. ■

Acknowledgement

The authors are deeply grateful to the referee for careful reading of the paper and helpful suggestions. The work reported here is supported by the INSPIRE programme (IF 110684) of Department of Science and Technology, Government of India for the first author. The work is also supported by the UGC-Major Research Project (F. No. 42-8/2013(SR)) awarded to the third author by the University Grants Commission, Government of India.

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doi:10.1081/AGB-120022801

Received 18 November 2015
Revised 23 June 2017
Accepted 27 July 2017