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DEVELOPED ZARISKI TOPOLOGY-GRAPH

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Abstract

In this paper, we introduce the developed Zariski topology-graph associated to an R-module M with respect to a subset X of the set of all quasi-prime submodules of M and investigate the relationship between the algebraic properties of M and the properties of its associated developed Zariski topology-graph.

Keywords: developed Zariski topology-graph, annihilating-submodule graph, quasi-prime submodules.

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1. Introduction

For the last few decades several mathematicians studied graphs on the various algebraic structures (groups, rings, modules, ...). These interdisciplinary studies allow us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa. Various constructions of graphs related to the algebraic structures are found in [3, 6, 7, 8, 11, 13, 14, 18, 19].

Throughout this paper all rings are commutative with nonzero identity and all modules are unitary. A proper ideal I of a ring R is said to be *quasi-prime* if for each pair of ideals A and B of R, $A \cap B \subseteq I$ yields either $A \subseteq I$ or $B \subseteq I$ (see [9, 12] and [17]). It is easy to see that every prime ideal is a quasi-prime ideal. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal

$\{r \in R \mid rM \subseteq N\}$ and annihilator of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. M is called *faithful* if $\mathrm{Ann}(M) = (0)$. If there is no ambiguity we write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$). A proper submodule N of an R-module M is called *quasi-prime* if $(N :_R M)$ is a quasi-prime ideal of R (see [2]). We define the *quasi-prime spectrum* of an R-module M to be the set of all quasi-prime submodules of M and denote it by $q\mathrm{Spec}^R(M)$. If there is no ambiguity we write only $q\mathrm{Spec}(M)$ instead of $q\mathrm{Spec}^R(M)$. The notion of quasi-prime submodule as a generalization of quasi-prime ideal of rings was introduced and investigated in [2]. For a submodule N of M we define $D(N) = \{L \in q\mathrm{Spec}(M) \mid (L : M) \supseteq (N : M)\}$. For any R-module M there exists a topology, τ say, on $q\mathrm{Spec}(M)$ having $\zeta(M) = \{D(N) \,|\, N \leq M\}$ as the family of all closed sets. The topology τ is called the *developed Zariski topology on* $q\mathrm{Spec}(M)$ (see [2]).

In this paper, we employ sets $D(N)$ and define a new graph called the *developed Zariski topology-graph*, and by using this graph, we study algebraic (resp. topological) properties of M (resp. $q\mathrm{Spec}(M)$).

2. The developed Zariski topology-graph

For the reminder of this paper, we will use the letter X to denote an arbitrary subset of $q\mathrm{Spec}(M)$. We will denote the intersection of all elements in X by $\Im(X)$ and the closure of X in $q\mathrm{Spec}(M)$ with respect to the developed Zariski topology by $Cl(X)$.

Definition 2.1. Let M be an R-module. Then we define the *developed Zariski topology-graph* $G_X(M)$ as an undirected graph in which the set of vertices $V\big(G_X(M)\big)$ is defined by

$$\big\{N < M \,|\, \exists\, 0 \neq L < M \text{ such that } D(N) \cup D(L) = X \text{ and } D(N), D(L) \neq X\big\}$$

and distinct vertices N and L are adjacent if and only if $D(N) \cup D(L) = X$.

Recall that a topological space T is irreducible if for any decomposition $T = A_1 \cup A_2$ with closed subsets A_i of T with $i = 1, 2$, we have $A_1 = T$ or $A_2 = T$. A subset Y of T is irreducible if it is irreducible as a subspace of T.

Lemma 2.2. *Let M be an R-module. Then $G_X(M) \neq \emptyset$ if and only if X is closed and reducible subset of* $q\mathrm{Spec}(M)$.

Proof. Suppose $G_X(M) \neq \emptyset$. Then there exists a submodule L of M such that $L \in V(G_X(M))$. Hence, there exists a nonzero proper submodule N of M where $D(N \cap L) = X$. This shows that X is a closed subset of $q\mathrm{Spec}(M)$. Moreover, $D(N) \cup D(L) = X$, $D(L) \neq X$ and $D(N) \neq X$ by Definition 2.1. It follows that $G_X(M)$ is a reducible subset of $q\mathrm{Spec}(M)$. Conversely, let X be a closed and reducible subset of $q\mathrm{Spec}(M)$. Then, there are submodules L_1 and L_2 of M, with $X = D(L_1) \cup D(L_1)$ and $D(L_1) \neq X$, $D(L_2) \neq X$. Therefore, $L_1, L_2 \in V\big(G_X(M)\big)$ and the proof is completed. ∎

Lemma 2.3. *Let M be an R-module, $Y \subseteq q\mathrm{Spec}(M)$ and let $L \in q\mathrm{Spec}(M)$. Then the following statements hold:*

(1) $D\big(\Im(Y)\big) = Cl(Y)$. *In particular,* $Cl(\{L\}) = D(L)$*;*

(2) $\Im(Y)$ *is a quasi-prime submodule of M if and only if Y is an irreducible space.*

Proof. See [2, Proposition 3.4(1) and Proposition 3.10]. ∎

Remark 2.4. By Lemma 2.3, X is closed if and only if $X = D\big(\Im(X)\big)$. Therefore, by Lemma 2.2, $G_X(M) \neq \emptyset$ if and only if $X = D\big(\Im(X)\big)$ and X is a reducible subset of $q\mathrm{Spec}(M)$. Moreover, we infer from Lemma 2.3 that $G_X(M) \neq \emptyset$ if and only if $X = D\big(\Im(X)\big)$ and $\Im(X)$ is not a quasi-prime submodule of M.

Recall that a graph is said to be *connected* if for each pair of distinct vertices v and w, there is a finite sequence

$$\boxed{v = v_1} \text{———} \boxed{v_2} \text{———} \cdots \text{———} \boxed{v_{n-1}} \text{———} \boxed{v_n = w}$$

of distinct vertices where each pair $\{v_i, v_{i+1}\}$ is an edge of graph. Such a sequence is said to be a *path* and the *distance*, $d(v, w)$, between connected vertices v and w is the length of the shortest path connecting them ($d(v, v) = 0$ and $d(v, w) = \infty$ if there is no such path). The *diameter* of a connected graph G is the supremum of the distances between vertices and is denoted by $diam(G)$. The *girth* of a graph G denoted by $gr(G)$ is the length of a shortest cycle in G (for more details see [16]). Let N and L be submodules of an R-module M. Then the *product* of N and L is defined by $(N : M)(L : M)M$ and denoted by NL (see [5]).

Let M be an R-module. For a submodule N of M we define $\Omega(N) = \{L \in q\mathrm{Spec}(M) \mid L \supseteq N\}$. Then we define $\sqrt[D]{N} := \Im(D(N))$. If $D(N) = \emptyset$, then $\sqrt[D]{N} = M$. Also, we define $\sqrt[\Omega]{N} := \Im(\Omega(N))$. If $\Omega(N) = \emptyset$, then $\sqrt[\Omega]{N} = M$. Consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/24\mathbb{Z}) \oplus (\mathbb{Z}/28\mathbb{Z})$ and let $N = (4\mathbb{Z}/24\mathbb{Z}) \oplus (2\mathbb{Z}/28\mathbb{Z})$, $P_1 = (4\mathbb{Z}/24\mathbb{Z}) \oplus (\mathbb{Z}/28\mathbb{Z})$ and $P_2 = (4\mathbb{Z}/24\mathbb{Z}) \oplus (4\mathbb{Z}/28\mathbb{Z})$. Note that P_1 and P_2 are quasi-prime submodules of M. Then we have $\sqrt[\Omega]{N} = P_1 \cap (2)M = N$ and $\sqrt[D]{N} \subseteq P_1 \cap P_2 = P_2 \subsetneq N$. Therefore, this example shows $\sqrt[\Omega]{N}$ and $\sqrt[D]{N}$ are different in general.

Lemma 2.5. *Let M be an R-module and let I be an ideal of R. For arbitrary submodules N and L of an R-module M we have*

(1) $D(N) = D\big((N : M)M\big) = \Omega\big((N : M)M\big)$*;*

(2) $D(N) \cup D(L) = D(NL) = \Omega(NL)$;
(3) $D(IM) = D(\sqrt[D]{I}M) = \Omega(IM) = \Omega(\sqrt[D]{I}M)$.

Proof. (1) This is easy. (2) By definition and part (1) we have

$$\begin{aligned} D(NL) &= D\left((N:M)(L:M)M\right) \\ &= D\left((N:M)M\right) \cup D\left((L:M)M\right) \\ &= D(N) \cup D(L). \end{aligned}$$

(3) This follows from definitions. ∎

Theorem 2.6. *Let M be an R-module. Then the developed Zariski topology-graph $G_X(M)$ is connected and $diam\big(G_X(M)\big) \le 3$. Moreover, if $G_X(M)$ contains a cycle, then $gr\big(G_X(M)\big) \le 4$.*

Proof. Suppose $N, L \in V\big(G_X(M)\big)$ and $D(N) \cup D(L) \neq X$. Then there exist nonzero proper submodules V and W of M such that $D(N) \cup D(V) = X$ and $D(L) \cup D(W) = X$. If $V = W$, then

$$\begin{tikzcd} \boxed{N} \arrow[r, dash] & \boxed{V} \arrow[r, dash] & \boxed{L} \end{tikzcd}$$

is a path of length 2. Suppose $V \neq W$ and $D(V \cap W) = X$. Then

$$\begin{tikzcd} \boxed{N} \arrow[r, dash] & \boxed{V} \arrow[r, dash] & \boxed{W} \arrow[r, dash] & \boxed{L} \end{tikzcd}$$

is a path of length 3. Let $V \neq W$ and $D(V \cap W) \neq X$. Then

$$\begin{tikzcd} \boxed{N} \arrow[r, dash] & \boxed{V \cap W} \arrow[r, dash] & \boxed{L} \end{tikzcd}$$

is a path of length 2. Note that if $N = V \cap W$, then $N \cap L = V \cap W \cap L$, and so $D(N \cap L) = D(V \cap W \cap L) = X$, a contradiction. Similarly, $W \cap V \neq L$. Therefore, $G_X(M)$ is connected and $diam\big(G_X(M)\big) \le 3$.

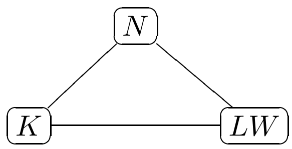
Now, let $G_X(M)$ contains a cycle. Suppose the assertion of the theorem is false. Without loss of generality, we assume that $gr\big(G_X(M)\big) = 5$. Hence, we have a cycle

$$\begin{tikzcd} \boxed{N} \arrow[r, dash] \arrow[d, dash] & \boxed{L} \arrow[r, dash] & \boxed{V} \arrow[d, dash] \\ \boxed{K} \arrow[rr, dash] & & \boxed{W} \end{tikzcd}$$

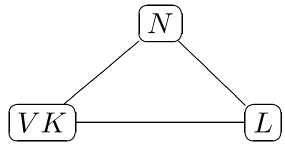
of length 5. Clearly $D(L \cap W) \neq X$ and $D(V \cap K) \neq X$. Note that

$$D(N) \cup D(LW) = D(N) \cup D(L) \cup D(W) = X.$$

Therefore



and



are two cycles with length 3, a contradiction. So $gr\big(G_X(M)\big) \leq 4$. ■

Let M be an R-module. When $q\mathrm{Spec}(M) \neq \emptyset$, the map $\psi : q\mathrm{Spec}(M) \to q\mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(L) = (L : M)/\mathrm{Ann}(M)$ for every $L \in q\mathrm{Spec}(M)$, will be called the *natural map of* $q\mathrm{Spec}(M)$. An R-module M is called *quasi-primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and has a surjective natural map (see [1]). Let $\Sigma := q\mathrm{Spec}(\mathbb{Z}) \setminus \{(0)\}$. Consider the $\mathbb{Z}$-module $M = \bigoplus_{I \in \Sigma} \mathbb{Z}/I$. Then M is a quasi-primeful $\mathbb{Z}$-module (see [1, Example 2.11]).

Lemma 2.7. *Let M be an R-module.*

(1) *ψ is continuous with respect to the developed Zariski topology;*

(2) *ψ is bijective if and only if it is a homeomorphism;*

(3) *if M is quasi-primeful, then ψ is both closed and open; more precisely, for any submodule N of M, $\psi\big(D(N)\big) = D\big(\overline{(N : M)}\big)$ and*

$$\psi\big(q\mathrm{Spec}(M) \setminus D(N)\big) = q\mathrm{Spec}(\bar{R}) \setminus D\big(\overline{(N : M)}\big).$$

Proof. See [2, Proposition 3.2]. ■

An R-module M is called a *multiplication* module if every submodule N of M is of the form IM for some ideal I of R (see [10, 15]). It is easy to see that if M is a multiplication R-module, then the natural map is bijective (see [1, Corollary 2.23]), and so by Lemma 2.7, it is homeomorphism.

Theorem 2.8. *Suppose M is an R-module such that $q\mathrm{Spec}(M)$ is homeomorphic by $q\mathrm{Spec}\big(\frac{R}{\mathrm{Ann}(M)}\big)$ under the natural map ψ (e.g. multiplication module). If N and L are adjacent in $G_X(M)$ and $X' = \{\overline{(K : M)} \mid K \in X\}$, then $\overline{(N : M)}$ and $\overline{(L : M)}$ are adjacent in $G_{X'}(\bar{R})$. Conversely, if $\bar{I}$ and $\bar{J}$ are adjacent in $G_{X'}(\bar{R})$, then IM and JM are adjacent in $G_X(M)$.*

Proof. Let $J \in X'$. Then there exists a quasi-prime submodule $K \in X$ such that $\overline{(K:M)} = J$. Hence, $\psi(K) = J$. By Lemma 2.7(2), ψ is injective, therefore $\psi^{-1}(J) = K$. This implies that

$$\psi^{-1}(X') \subseteq X. \tag{2.1}$$

Suppose $K \in X$. Then $\psi(K) = \overline{(K:M)}$ and so

$$K = \psi^{-1}\overline{(K:M)} \in \psi^{-1}(X').$$

This implies that

$$X \subseteq \psi^{-1}(X'). \tag{2.2}$$

From (2.1) and (2.2) it follows that $X = \psi^{-1}(X')$. By assumption $D(N) \cup D(L) = X$, so

$$X' = \psi(X) = \psi\big(D(N)\big) \cup \psi\big(D(L)\big).$$

From Lemma 2.7(3) it follows that $X' = D\overline{(N:M)} \cup D\big(\overline{(L:M)}\big)$. Therefore, $\overline{(N:M)}$ and $\overline{(L:M)}$ are adjacent in $G_{X'}(\bar{R})$.

Conversely, let $D(\bar{I}) \cup D(\bar{J}) = X'$. Then we have

$$X = \psi^{-1}\big(D(\bar{I})\big) \cup \psi^{-1}\big(D(\bar{J})\big) = D(IM) \cup D(JM).$$

Therefore, IM and JM are adjacent in $G_X(M)$. ∎

Lemma 2.9. *Let M be an R-module. Then $D(L)$ is an irreducible closed subset of $q\mathrm{Spec}(M)$ for every quasi-prime submodule L of M.*

Proof. See [2, Corollary 3.9]. ∎

Lemma 2.10. *Suppose $G_X(M) \neq \emptyset$ and $K \in X$. Then K is a vertex of $G_X(M)$ if either of the following statements holds:*

(1) *There exists a subset X' of X such that $K \in X'$, $D(\bigcap_{Q\in X'} Q) = X$ and $D(\bigcap_{Q\in X', Q\neq K} Q) \neq X$.*

(2) *For a submodule N of M, $N \in V\big(G_X(M)\big)$ and $N \cap K \notin V\big(G_X(M)\big)$.*

Proof. (1) First we recall that K is a vertex if there exists a submodule L of M such that $D(K) \cup D(L) = X$ and $D(K), D(L) \neq X$. Suppose $L := (\bigcap_{Q\in X', Q\neq K} Q)$. Then by assumption $D(\bigcap_{Q\in X', Q\neq K} Q) \neq X$ and

$$D\left(\left(\bigcap_{Q\in X', Q\neq K} Q\right) \cap K\right) = D\left(\bigcap_{Q\in X', Q\neq K} Q\right) \cup D(K) = X.$$

From Lemma 2.9, it follows that $D(K) \neq X$. Hence K is a vertex.

(2) Since $N \in V\big(G_X(M)\big)$, then there exists a nonzero proper submodule L of M such that $D(N) \cup D(L) = X$, $D(N) \neq X$ and $D(L) \neq X$. Moreover, since $N \cap K \notin V\big(G_X(M)\big)$, we infer that, $D(N \cap K) \cup D(L) \neq X$ or $D(N \cap K) = X$ or $D(L) = X$. By assumption, only the second case is true. Hence K is a vertex. ■

Theorem 2.11. *Let X be a finite set and $G_X(M) \neq \emptyset$. Then $X \cap V(G_X(M)) \neq \emptyset$.*

Proof. Suppose $K \in X$. Then $D(K) \cup D(\bigcap_{Q \in X, Q \neq K} Q) = X$. If $D(\bigcap_{Q \in X, Q \neq K} Q) \neq X$, then K is a vertex of $G_X(M)$. Otherwise, if $D(\bigcap_{Q \in X, Q \neq K} Q) = X$, since X is reducible there exists a subset X' of X and $K' \in X$ such that $D(\bigcap_{K \in (X \setminus X')} K) \neq X$ and $D(\bigcap_{K \in (X \setminus X') \cup \{K'\}} K) = X$. Hence, $K' \in X \cap V$ $(G_X(M))$. Therefore $X \cap V\big(G_X(M)\big) \neq \emptyset$. ■

3. Relation between $G_X(M)$ and $AG(M)$

Recall that the *annihilating-submodule graph* $AG(M)$ is a graph with vertices $V(AG(M)) = \{N \leq M \mid \exists\, 0 \neq L < M \text{ with } NL = 0\}$, where distinct vertices N and L are adjacent if and only if $NL = 0$ (see [6]).

Lemma 3.1. *Suppose that M is an R-module such that $M \notin V(AG(M))$. If $AG(M)$ is empty, then the submodule (0) of M is a quasi-prime submodule of M.*

Proof. This follows from [6, Proposition 3.2]. ■

Theorem 3.2. *Suppose M is an R-module and $M/\Im(X)$ is not a vertex in $AG\big(M/\Im(X)\big)$. Then we have the following statements:*

(1) *$AG\big(M/\Im(X)\big)$ is isomorphic to a subgraph of $G_X(M)$.*

(2) *$AG\big(M/\Im(X)\big) = \emptyset$ if and only if $G_X(M) = \emptyset$.*

Proof. (1) Let $N/\Im(X)$ be a vertex in $AG\big(M/\Im(X)\big)$. Then there exists a nonzero proper submodule $L/\Im(X)$ adjacent to $N/\Im(X)$. Hence,

$$\begin{aligned} \frac{N}{\Im(X)} \cdot \frac{L}{\Im(X)} &= \left(\frac{N}{\Im(X)} : \frac{M}{\Im(X)}\right)\left(\frac{L}{\Im(X)} : \frac{M}{\Im(X)}\right)\frac{M}{\Im(X)} \\ &= \frac{(N:M)(L:M)M + \Im(X)}{\Im(X)} = (\bar{0}). \end{aligned}$$

Therefore, $(N : M)(L : M)M = NL \subseteq \Im(X)$. This yields that $(NL : M) \subseteq (\Im(X) : M)$. Consequently, by Lemma 2.5 we have $D(NL) = D(N) \cup D(L) = X$.

If $D(N) = X$, then $(NL : M) = (\Im(X) : M)$. Hence, $M/\Im(X)$ is a vertex, a contradiction. Similarly $D(L) \neq X$.

(2) Suppose $AG(M/\Im(X)) = \emptyset$. By Lemma 3.1, $\Im(X)$ is a quasi-prime submodule of M. Therefore, by Remark 2.4 we have $G_X(M) = \emptyset$. Conversely, let $AG\big(M/\Im(X)\big) \neq \emptyset$. Thus, from part (1) of this theorem, we have $G_X(M) \neq \emptyset$. ■

Remark 3.3. In the annihilating-submodule graph $AG(M)$, M itself can be a vertex. M is a vertex if and only if there exists a nonzero proper submodule N of M such that $(N : M) = \mathrm{Ann}(M)$ (see page 3289 of [6]).

Lemma 3.4. *Let M be an R-module. Then $M/\Im(X)$ is a vertex in $AG\big(M/\Im(X)\big)$ if and only if there exists a proper submodule N of M such that $\Im(X) < N$ and $D(N) = X$.*

Proof. We first show that if $\Im(X) < N$ and $D(N) = X$, then $\overline{M} := M/\Im(X)$ is a vertex in $AG\big(M/\Im(X)\big)$. By definition, it suffices to show that there exists a nonzero proper submodule $\overline{L}$ of $\overline{M}$ such that $\overline{LM} = (\overline{0})$. Therefore,

$$\begin{aligned}\frac{M}{\Im(X)} \cdot \frac{L}{\Im(X)} &= \left(\frac{M}{\Im(X)} : \frac{M}{\Im(X)}\right)\left(\frac{L}{\Im(X)} : \frac{M}{\Im(X)}\right)\frac{M}{\Im(X)} \\ &= \frac{(L : M)M + \Im(X)}{\Im(X)}.\end{aligned}$$

Thus, suffices to show $(L : M)M \subseteq \Im(X)$. By assumption, $D(N) = X$. Therefore,

$$(N : M) \subseteq \bigcap_{Q \in X} (Q : M) = \left(\bigcap_{Q \in X} Q : M\right) = \big(\Im(X) : M\big) \Rightarrow (N : M)M \subseteq \Im(X).$$

Hence, $\frac{M}{\Im(X)}$ is a vertex because adjacent vertex is $\frac{N}{\Im(X)}$.

Conversely, if $M/\Im(X)$ is a vertex in $AG\big(M/\Im(X)\big)$, then by Remark 3.3, there exists a nonzero proper submodule $\overline{N} := N/\Im(X)$ of $\overline{M}$ such that $(\overline{N} : \overline{M}) = \mathrm{Ann}(\overline{M})$. Therefore, $N \neq M$, $\Im(X) < N$ and $(\Im(X) : M) = (N : M)$. Thus, $D(\Im(X)) = D(N)$. Since $AG(M/\Im(X)) \neq \emptyset$ by Theorem 3.2(2), it follows that $G_X(M) \neq \emptyset$ and from Remark 2.4, it follows that $D(\Im(X)) = X$. Consequently, $D(N) = X$. ■

Theorem 3.5. *Suppose M is a quasi-primeful R-module and N, L are adjacent in $G_X(M)$. Then $\frac{\sqrt[D]{(N:M)M}}{\Im(X)}$ and $\frac{\sqrt[D]{(L:M)M}}{\Im(X)}$ are adjacent in $AG(\frac{M}{\Im(X)})$.*

Proof. By assumption $D(N) \cup D(L) = X$. By Lemma 2.5 we have

$$\begin{aligned}X &= D(N) \cup D(L) = D(NL) = \Omega(NL) = \Omega\big((NL : M)M\big) \\ &= \Omega((N : M)(L : M)M).\end{aligned}$$

Therefore,

$$\begin{aligned}\Im(X) &= \sqrt[D]{(N:M)(L:M)M}\\ &\subseteq \sqrt[D]{(N:M)M\cap(L:M)M}\\ &\subseteq \sqrt[D]{(N:M)M}\cap\sqrt[D]{(L:M)M}.\end{aligned}$$

Now, we show that $\frac{\sqrt[D]{(N:M)M}}{\Im(X)}\frac{\sqrt[D]{(L:M)M}}{\Im(X)}=(\bar{0})$. We have

$$\begin{aligned}&\frac{\sqrt[D]{(N:M)M}}{\Im(X)}\cdot\frac{\sqrt[D]{(L:M)M}}{\Im(X)}\\ &=\left(\frac{\sqrt[D]{(N:M)M}}{\Im(X)}:\frac{M}{\Im(X)}\right)\left(\frac{\sqrt[D]{(L:M)M}}{\Im(X)}:\frac{M}{\Im(X)}\right)\frac{M}{\Im(X)}\\ &=\frac{\big(\sqrt[D]{(N:M)M}:M\big)\big(\sqrt[D]{(L:M)M}:M\big)M+\Im(X)}{\Im(X)}.\end{aligned}$$

It is sufficient for us to prove that $\big(\sqrt[D]{(N:M)M}:M\big)\big(\sqrt[D]{(L:M)M}:M\big)$ $M\subseteq\Im(X)$. Since M is quasi-primeful, we have

$$\left(\sqrt[D]{(N:M)M}:M\right)=\left(\sqrt[D]{N}:M\right)=\sqrt[D]{N:M} \tag{3.1}$$

and

$$\left(\sqrt[D]{(L:M)M}:M\right)=\left(\sqrt[D]{L}:M\right)=\sqrt[D]{L:M}. \tag{3.2}$$

Thus by (3.1) and (3.2) we infer that

$$\begin{aligned}\left(\sqrt[D]{(N:M)M}:M\big)\big(\sqrt[D]{(L:M)M}:M\right)M &= \sqrt[D]{N:M}\,\sqrt[D]{L:M}M\\ &\subseteq \sqrt[D]{(N:M)(L:M)}M\\ &\subseteq \sqrt[\Omega]{(N:M)(L:M)M}\\ &= \sqrt[\Omega]{NL}=\Im(X).\end{aligned}$$

If we show that $\frac{\sqrt[D]{(N:M)M}}{\Im(X)}$ is a nonzero proper submodule of $\frac{M}{\Im(X)}$, then we can infer that $\frac{\sqrt[D]{(N:M)M}}{\Im(X)}$ is a vertex in $AG(\frac{M}{\Im(X)})$. Let $\frac{\sqrt[D]{(N:M)M}}{\Im(X)}=0$. Then $(N:M)M\subseteq\sqrt[D]{(N:M)M}=\Im(X)$, and so $(N:M)\subseteq(\Im(X):M)$. Consequently, $D(N)=X$ a contradiction. Therefore $\sqrt[D]{(N:M)M}\neq\Im(X)$ and similarly $\sqrt[D]{(L:M)M}\neq\Im(X)$.

To show that $\frac{\sqrt[D]{(N:M)M}}{\Im(X)}$ and $\frac{\sqrt[D]{(L:M)M}}{\Im(X)}$ are adjacent in $AG(\frac{M}{\Im(X)})$. We must prove that these are distinct. Let $\sqrt[D]{(N:M)M}/\Im(X) = \sqrt[D]{(L:M)M}/\Im(X)$. Then we have

$$\begin{aligned}
(\bar{0}) &= \frac{\sqrt[D]{(N:M)M}}{\Im(X)} \cdot \frac{\sqrt[D]{(L:M)M}}{\Im(X)} \\
&= \left(\frac{\sqrt[D]{(L:M)M}}{\Im(X)} : \frac{M}{\Im(X)}\right)\left(\frac{\sqrt[D]{(L:M)M}}{\Im(X)} : \frac{M}{\Im(X)}\right)\frac{M}{\Im(X)} \\
&= \left(\sqrt[D]{(L:M)M}\right)^2 \frac{M}{\Im(X)} = \frac{\left(\sqrt[D]{L:M}\right)^2 M + \Im(X)}{\Im(X)}.
\end{aligned}$$

So $(\sqrt[D]{L:M})^2 M \subseteq \Im(X)$. Then $(\sqrt[D]{L:M})^2 \subseteq (\Im(X):M)$, Hence, $(L:M) \subseteq \sqrt[D]{L:M} \subseteq \sqrt[D]{(\Im(X):M)} = (\Im(X):M)$. Thus $D(L) = X$, a contradiction. Hence $\sqrt[D]{(N:M)M} \neq \sqrt[D]{(L:M)M}$ and the proof is completed. ∎

Corollary 3.6. *If the conditions of Theorem 3.5 hold, then* $\frac{\sqrt[D]{N}}{\Im(X)}$ *and* $\frac{\sqrt[D]{L}}{\Im(X)}$ *are adjacent in* $AG(\frac{M}{\Im(X)})$.

Proof. Apply the same technique in the proof of Theorem 3.5. ∎

A submodule N of an R-module M is said *quasi-semiprime* if it is an intersection of quasi-prime submodules (see [2]). We recall that an R-module M is *co-semisimple* in case every submodule of M is the intersection of maximal submodules (see [4, p. 122]). Every proper submodule of a co-semisimple module is a quasi-semiprime submodule.

Lemma 3.7. *If A is a quasi-primeful R-module and $B \leq A$, then for every $\mathfrak{p} \in q\mathrm{Spec}(R)$ where $(B:A) = \mathfrak{p}$, there exists $P \in q\mathrm{Spec}(A)$ such that $B \leq P$ and $(P:A) = \mathfrak{p}$.*

Proof. The proof is easy and we omit it. ∎

Theorem 3.8. *Let M be an R-module. Suppose that $\frac{N}{\Im(X)}$ and $\frac{L}{\Im(X)}$ are adjacent in $AG(\frac{M}{\Im(X)})$. Then N and L are adjacent in $G_X(M)$ if one of the following conditions holds:*

(1) *$\frac{M}{\Im(X)}$ is not a vertex in $AG(\frac{M}{\Im(X)})$. Particularly, this holds when $\frac{M}{\Im(X)}$ is a quasi-primeful module and contains no quasi-semiprime $S \neq \Im(X)$ with $D(S) \neq X$.*

(2) *$\frac{M}{N}$ and $\frac{M}{L}$ are quasi-primeful and contains no quasi-semiprime $S \neq \Im(X)$ with $D(S) \neq X$.*

Proof. (1) If $\frac{M}{\Im(X)}$ is not a vertex in $AG(\frac{M}{\Im(X)})$, then by the proof of Theorem 3.2, N and L are adjacent in $G_X(M)$. To see the second assertion, let $\frac{M}{\Im(X)}$ be a vertex in $AG(\frac{M}{\Im(X)})$. By Lemma 3.4, there exists a nonzero proper submodule N' of M such that $\Im(X) < N'$. Clearly, $\frac{M}{\Im(X)}$ has structure of $\frac{R}{(\Im(X):M)}$-module. Suppose that Q is an arbitrary element of X. Then we have $(N' : M) \subseteq (Q : M)$ and therefore

$$\left(\frac{N'}{\Im(X)} : \frac{M}{\Im(X)}\right) \subseteq \frac{(Q : M)}{(\Im(X) : M)}.$$

Now, by Lemma 3.7 there exists a quasi-prime submodule $\frac{K}{\Im(X)}$ such that $N' < K$ and $(K : M) = (Q : M)$. Thus we have $\Im(X) \leq K$ and so $D(K) \subseteq D(\Im(X)) = X$. Since $(K : M) = (Q : M)$, we have $Q \in D(K)$. Hence $X \subseteq D(K)$. It follows that $D(K) = D(N') = X$ and this means that there exists a quasi-semiprime submodule $N' \neq \Im(X)$ such that $D(N') = X$, a contradiction.

(2) Since $\frac{N}{\Im(X)}$ and $\frac{L}{\Im(X)}$ are adjacent in $AG(\frac{M}{\Im(X)})$ we have

$$\begin{aligned}
\frac{N}{\Im(X)} \cdot \frac{L}{\Im(X)} = 0 &\Rightarrow \left(\frac{N}{\Im(X)} : \frac{M}{\Im(X)}\right)\left(\frac{L}{\Im(X)} : \frac{M}{\Im(X)}\right)\frac{M}{\Im(X)} = 0\\
&\Rightarrow \frac{(N : M)(L : M)M + \Im(X)}{\Im(X)} = 0\\
&\Rightarrow NL \subseteq \Im(X)\\
&\Rightarrow D(NL) = D(N) \cup D(L) = X.
\end{aligned}$$

We now claim that $D(N) \neq X$ and $D(L) \neq X$. If $D(N) = X$, then $(N : M) \subseteq (Q : M)$, for every $Q \in X$. Since $\frac{M}{N}$ is quasi-primeful, there exists a quasi-prime submodule K of M such that $(K : M) = (Q : M)$. Thus, $N \subseteq \Im(X)$ a contradiction. Therefore, $D(N) \neq X$. Similarly $D(L) \neq X$ and the proof is completed. ■

Lemma 3.9. *Let M be an R-module.*

(1) *If the zero submodule of M is not quasi-prime, then $AG(M)$ has ACC (resp. DCC) on vertices if and only if M is a Noetherian (resp. an Artinian) module.*

(2) *Suppose that R is an Artinian ring and M is a finitely generated R-module. Then every nonzero proper submodule N of M is a vertex in $AG(M)$.*

Proof. See [6, Proposition 3.5 and Theorem 3.6]. ■

Theorem 3.10. *Suppose M is an R-module such that $\frac{M}{\Im(X)}$ is a faithful module which is not a vertex in $AG(\frac{M}{\Im(X)})$. Then the following statements are equivalent:*

(1) *$G_X(M)$ is a finite graph;*

(2) $AG(\frac{M}{\Im(X)})$ *is a finite graph;*

(3) $\frac{M}{\Im(X)}$ *has finite number of submodules. Moreover,* $G_X(M)$ *has* $n\,(n \geq 1)$ *vertices if and only if* $\frac{M}{\Im(X)}$ *has* n *nonzero proper submodules.*

Proof. (1)⇒(2) Suppose $G_X(M)$ is a finite graph. Since $\frac{M}{\Im(X)}$ is not a vertex in $AG(\frac{M}{\Im(X)})$, by Theorem 3.2(1) it follows that $AG(\frac{M}{\Im(X)})$ is a finite graph.

(2)⇒(3) If $AG(\frac{M}{\Im(X)})$ is a finite graph with $n\,(n \geq 1)$ vertices, then by Lemma 3.1, $\Im(X)$ is not a quasi-prime submodule. By Lemma 3.9, $\frac{M}{\Im(X)}$ has finite length. Therefore, $\frac{M}{\Im(X)}$ is a Noetherian, Artinian and finitely generated R-module. Since $M/\Im(X)$ is a faithful R-module we have

$$\frac{R}{\operatorname{Ann}(\frac{M}{\Im(X)})} = \frac{R}{(\Im(X) : M)} = R.$$

Thus, R is an Artinian ring. By Lemma 3.9 every nonzero proper submodule $\frac{N}{\Im(X)}$ of $\frac{M}{\Im(X)}$ is a vertex. Consequently, $\frac{M}{\Im(X)}$ has finite number of submodules. If N is a vertex in $G_X(M)$, then there exists a nonzero proper submodule L of M such that $D(N) \cup D(L) = X$. By Theorem 3.2(1), $\frac{N}{\Im(X)}$ is a vertex in $AG(\frac{M}{\Im(X)})$.

(3)⇒(1) If $\frac{N}{\Im(X)}$ is a nonzero proper submodule of $\frac{M}{\Im(X)}$. Then by Lemma 3.9, $\frac{N}{\Im(X)}$ is a vertex in $AG(\frac{M}{\Im(X)})$ and it follows from Theorem 3.2(1) that N is a vertex in $G_X(M)$. Thus, $G_X(M)$ is a finite graph. ■

Lemma 3.11. *Suppose* M *is an* R*-module such that* $\frac{M}{\Im(X)}$ *is not a vertex in* $AG(\frac{M}{\Im(X)})$ *and for every* $P \in q\mathrm{Spec}(M) \cap V\big(G_X(M)\big)$ *there exists a quasi-semiprime submodule of* M *adjacent with* P*. Then*

$$q\mathrm{Spec}\left(\frac{M}{\Im(X)}\right) \cap V\left(AG\left(\frac{M}{\Im(X)}\right)\right) \neq \emptyset \Leftrightarrow q\mathrm{Spec}(M) \cap V\big(G_X(M)\big) \neq \emptyset.$$

Proof. Suppose that $P \in q\mathrm{Spec}(M) \cap V\big(G_X(M)\big)$. By assumption, $D(P) \cup D(\bigcap_{P' \in D'} P') = X$ where D' is an open subset of X. Now, we show that $\frac{P}{\Im(X)}$ and $\frac{\bigcap_{P' \in D'} P'}{\Im(X)}$ are adjacent. We must prove that $\frac{P}{\Im(X)} \cdot \frac{\bigcap_{P' \in D'} P'}{\Im(X)} = 0$. By definition we have

$$\begin{aligned}\frac{P}{\Im(X)} \cdot \frac{\bigcap_{P' \in D'} P'}{\Im(X)} &= \left(\frac{P}{\Im(X)} : \frac{M}{\Im(X)}\right)\left(\frac{\bigcap_{P' \in D'} P'}{\Im(X)} : \frac{M}{\Im(X)}\right)\frac{M}{\Im(X)} \\ &= \frac{(P : M)\left(\bigcap_{P' \in D'} (P' : M)\right) M + \Im(X)}{\Im(X)}.\end{aligned}$$

It is enough for us to show that $(P:M)(\bigcap_{P'\in D'}(P':M))M \subseteq \Im(X)$. By assumption, we have

$$X = D(P) \cup D\left(\bigcap_{P'\in D'} P'\right) = \Omega\left(P \cdot \left(\bigcap_{P'\in D'} P'\right)\right).$$

Therefore,

$$\Im(X) = \sqrt[\Omega]{P \cdot \left(\bigcap_{P'\in D'} P'\right)} \supseteq P \cdot \left(\bigcap_{P'\in D'} P'\right).$$

This implies that

$$(P:M)\left(\bigcap_{P'\in D'} (P':M)\right)M = P \cdot \bigcap_{P'\in D'} P' \subseteq \Im(X).$$

Thus, $q\mathrm{Spec}(\frac{M}{\Im(X)}) \cap V\big(AG(\frac{M}{\Im(X)})\big) \neq \emptyset$.

Now suppose that $\frac{P}{\Im(X)} \in q\mathrm{Spec}(\frac{M}{\Im(X)}) \cap V\big(AG(\frac{M}{\Im(X)})\big) \neq \emptyset$. Since $\frac{P}{\Im(X)} \in V\big(AG(\frac{M}{\Im(X)})\big)$, there exists a proper submodule $\frac{L}{\Im(X)}$ such that

$$\begin{aligned} 0 &= \frac{P}{\Im(X)} \cdot \frac{L}{\Im(X)} \\ &= \left(\frac{P}{\Im(X)} : \frac{M}{\Im(X)}\right)\left(\frac{L}{\Im(X)} : \frac{M}{\Im(X)}\right)\frac{M}{\Im(X)} \\ &= \frac{(P:M)(L:M)M + \Im(X)}{\Im(X)}. \end{aligned}$$

This implies that $(P:M)(L:M)M \subseteq \Im(X)$. Thus $PL \subseteq \Im(X)$ and so $(PL:M) \subseteq (\Im(X):M)$. Hence, we have $D(P) \cup D(L) = D(PL) = X$ and this follows that $P \in q\mathrm{Spec}(M) \cap V\big(G_X(M)\big)$. Therefore, $q\mathrm{Spec}(M) \cap V\big(G_X(M)\big) \neq \emptyset$. ■

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