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GENERALIZED DERIVATIONS WITH LEFT ANNIHILATOR CONDITIONS IN PRIME AND SEMIPRIME RINGS

BASUDEB DHARA

Basudeb Dhara, Department of Mathematics Belda College, Belda, Paschim Medinipur, 721424, W.B. India

e-mail: basu_dhara@yahoo.com

Abstract

Let R be a prime ring with its Utumi ring of quotients U, C = Z(U) be the extended centroid of R, H and G two generalized derivations of R, L a noncentral Lie ideal of R, I a nonzero ideal of R. The left annihilator of $S \subseteq$ R is denoted by $l_R(S)$ and defined by $l_R(S) = \{x \in R \mid xS = 0\}$. Suppose that $S = \{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$ and $T = \{H(x^n)x^n + x^nG(x^n) \mid x \in$ I}, where $n \geq 1$ is a fixed integer. In the paper, we investigate the cases when the sets $l_R(S)$ and $l_R(T)$ are nonzero.

Keywords: prime ring, derivation, Lie ideal, generalized derivation, extended centroid, Utumi quotient ring.

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1. INTRODUCTION

Let R be an associative ring with center Z(R). For $x, y \in R$, the commutator of x, y is denoted by [x, y] and defined by [x, y] = xy - yx. By d we mean a derivation of R. An additive mapping F from R to R is called a generalized derivation if there exists a derivation d from R to R such that F(xy) = F(x)y + xd(y) holds for all $x, y \in R$.

Throughout this paper, R will always present a prime ring with center Z(R), extended centroid C and U is its Utumi quotient ring. A well known result proved by Posner [20], states that if the commutators $[d(x), x] \in Z(R)$ for all $x \in R$, then either d = 0 or R is commutative. Then result of Posner was generalized in many

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directions by a number of authors. Posner's theorem was extended to Lie ideals in prime rings by Lee [17] and then by Lanski [12].

On the other hand, authors generalized Posner's theorem by considering two derivations. In [3], Brešar proved that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later Lee and Wong [18] consider the situation $d(x)x - x\delta(x) \in Z(R)$ for all x in some noncentral Lie ideal L of R and they proved that either $d = \delta = 0$ or R satisfies s_4 .

Recently in [22] Vukman proves that if d and δ are derivations on a 2mn(m+n-1)!-torsion free semiprime rings R such that $d(x^m)x^n + x^n\delta(x^m) = 0$ for all $x \in R$, where $m, n \ge 1$ are fixed integers, then both derivations d and δ map R into its center and $d = -\delta$.

In [23], Wei and Xiao studied the similar situation replacing derivations d and δ by generalized derivations G and H. More precisely they proved the following:

Let m, n be fixed positive integers, R be a noncommutative 2(m+n)!-torsion free prime ring and G, H be a pair of generalized Jordan derivations on R. If $G(x^m)x^n + x^nH(x^m) \in Z(R)$ for all $x \in R$, then G and H both are right (or left) multipliers.

In [14], Lee and Zhou studied the same situation of above result without considering torsion free restriction on R. In this paper, Lee and Zhou [14] proved the following:

Let R be a prime ring that is not commutative and such that $R \not\cong M_2(GF(2))$, let G, H be two generalized derivations of R, and let m, n be two fixed positive integers. Then $G(x^m)x^n - x^nH(x^m) = 0$ for all $x \in R$ iff the following two conditions hold:

- (1) There exists $w \in Q$ such that G(x) = xw and H(x) = wx for all $x \in R$;
- (2) either $w \in C$, or x^m and x^n are C-dependent for all $x \in R$.

There are many papers in the literature which studied the identities of generalized derivations with left annihilator conditions.

For any subset S of R, denote by $r_R(S)$ the right annihilator of S in R, that is, $r_R(S) = \{x \in R \mid Sx = 0\}$ and $l_R(S)$ the left annihilator of S in R that is, $l_R(S) = \{x \in R \mid xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $ann_R(S)$.

In [4], Carini *et al.* studied the left annihilator of the set $\{H(u)u-uG(u) \mid u \in L\}$, where L is a noncentral Lie ideal of R and H, G two non-zero generalized derivations of R. In case the annihilator is not zero, the conclusion is one of the following:

- (1) there exist $b', c' \in U$ such that H(x) = b'x + xc', G(x) = c'x with ab' = 0;
- (2) R satisfies s_4 and there exist $b', c', q' \in U$ such that H(x) = b'x + xc', G(x) = c'x + xq', with a(b' q') = 0.

Recently, Carini and De Filippis proved the following theorem:

Let R be a prime ring, U the Utumi quotient ring of R, C = Z(U) the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists an integer $n \ge 1$ such that $H(u^n)u^n + u^n G(u^n) \in C$, for all $u \in L$, then either there exists $a \in U$ such that H(x) = xa, G(x) = -ax, or R satisfies the standard identity s_4 . Moreover, in the last case the structures of the maps G, H are obtained.

In the present paper, we shall investigate the left annihilator of the sets $\{H(u^n)u^n + u^n G(u^n) | u \in L\}$ and $\{H(x^n)x^n + x^n G(x^n) | x \in I\}$, where L is a noncentral Lie ideal of R, I is a nonzero ideal of R, $n \ge 1$ is a fixed integer and H, G two non-zero generalized derivations of R. More precisely, we prove the following theorems:

Theorem 1.1. Let R be a prime ring with its Utumi ring of quotients U, C = Z(U) be the extended centroid of R, H and G two generalized derivations of R, L a noncentral Lie ideal of R and $S = \{H(u^n)u^n + u^nG(u^n) | u \in L\}$, where $n \ge 1$ is a fixed integer. If $l_R(S) \ne \{0\}$, then either there exist $b', p \in U$ such that H(x) = b'x - xp and G(x) = px for all $x \in R$ with ab' = 0 or R satisfies s_4 . Moreover, in the last case, if R satisfies s_4 , then one of the following holds:

- (1) char (R) = 2;
- (2) *n* is even, there exist $b, p \in U$ and derivations d, δ of R such that H(x) = bx + d(x) and $G(x) = px + \delta(x)$ for all $x \in R$, with a(b+p) = 0;
- (3) *n* is odd, there exist $b, p \in U$ and derivations d, δ of R such that H(x) = bx + d(x) and $G(x) = xp + \delta(x)$ for all $x \in R$, with a(b+p) = 0.

Theorem 1.2. Let R be a noncommutative prime ring with char $(R) \neq 2$, U its Utumi ring of quotients, C = Z(U) be the extended centroid of R, H and G two generalized derivations of R, I a nonzero ideal of R and $S = \{H(x^n)x^n + x^nG(x^n) | x \in I\}$, where $n \geq 1$ is a fixed integer. If $l_R(S) \neq \{0\}$, then there exist $b', p \in U$ such that H(x) = b'x - xp and G(x) = px for all $x \in R$ with ab' = 0.

As an immediate application of the Theorem 1.1, in particular when G = -H, then we have the following result which gives a particular result of Theorem 1.1 in [6].

Corollary 1.3. Let R be a prime ring with its Utumi ring of quotients U, C = Z(U) be the extended centroid of R, H a generalized derivation of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a[H(u^n), u^n] = 0$ for all $u \in L$, where $n \ge 1$ is a fixed integer. Then either there exists $\lambda \in C$ such that $H(x) = \lambda x$ for all $x \in R$ or R satisfies s_4 .

As an application of the Theorem 1.1, in particular when G = 0, then using

Theorem 2.2 in [8], we have the following result which gives a generalization of Theorem 1.1 in [21].

Corollary 1.4. Let R be a prime ring of char $(R) \neq 2$ with its Utumi ring of quotients U, C = Z(U) be the extended centroid of R, H a generalized derivation of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $aH(u^n)u^n = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exist b', $p \in U$ such that H(x) = b'x for all $x \in R$ with ab' = 0.

2. PROOF OF MAIN RESULTS IN PRIME RINGS

Let R be a prime ring with extended centroid C. Let H(x) = bx + xc and G(x) = px + xq for all $x \in R$ and for some $b, c, p, q \in U$, be two inner generalized derivations of R and L be a noncentral Lie ideal of R. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ implies $a(bx^{2n} + x^n(c+p)x^n + x^{2n}q) = 0$ for all $x \in L$. We know that if char $(R) \neq 2$, by [2, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char (R) = 2 and $dim_C RC > 4$ i.e., char (R) = 2 and R does not satisfy s_4 , then by [13, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. We assume that R does not satisfy s_4 . Then in any case of char (R) = 2 or char $(R) \neq 2$, we can conclude that there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. By hypothesis, we have

(1)
$$a(b[x_1, x_2]^{2n} + [x_1, x_2]^n (c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q) = 0$$

for all $x_1, x_2 \in I$. Then following lemmas are immediate consequences:

Lemma 2.1. R satisfies a nontrivial generalized polynomial identity (GPI) or $c, p, q \in C$ such that a(b + c + p + q) = 0.

Proof. Assume that R does not satisfy any nontrivial GPI. Then R must be noncommutative. Let $T = U *_C C\{x_1, x_2\}$, the free product of U and $C\{x_1, x_2\}$, the free C-algebra in noncommuting indeterminates x_1 and x_2 .

Then,

$$a\left(b[x_1, x_2]^{2n} + [x_1, x_2]^n (c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q\right)$$

is zero element in T. If $q \notin C$, then q and 1 are linearly independent over C. Then from above

$$a[x_1, x_2]^{2n}q = 0 \in T,$$

implying q = 0, since $a \neq 0$, a contradiction. Therefore, we conclude that $q \in C$.

Then by hypothesis

(2)
$$a((b+q)[x_1, x_2]^n + [x_1, x_2]^n (c+p))[x_1, x_2]^n = 0 \in T.$$

If $c + p \notin C$, then by (2)

$$a\left([x_1, x_2]^n (c+p)\right) [x_1, x_2]^n = 0 \in T,$$

implying c + p = 0, since $a \neq 0$, a contradiction. Therefore, we have $c + p \in C$ and hence

$$a(b+q+c+p)[x_1,x_2]^{2n} = 0 \in T.$$

This implies a(b+q+c+p) = 0.

Lemma 2.2. $c + p, q \in C$ with a(b + c + p + q) = 0, unless R satisfies s_4 .

Proof. By hypothesis, R satisfies GPI

(3)
$$f(x_1, x_2) = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n (c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q).$$

If R does not satisfy any nontrivial GPI, by Lemma 2.1, we obtain $c, p, q \in C$ with a(b+c+p+q) = 0 which gives the conclusion. So, we assume that R satisfies a nontrivial GPI. Since R and U satisfy the same generalized polynomial identities (see [5]), U satisfies $f(x_1, x_2)$. In case C is infinite, we have $f(x_1, x_2) = 0$ for all $x_1, x_2 \in U \otimes_C \overline{C}$, where \overline{C} is the algebraic closure of C. Moreover, both U and $U \otimes_C \overline{C}$ are prime and centrally closed algebras [9]. Hence, replacing R by U or $U \otimes_C \overline{C}$ according to C finite or infinite, without loss of generality we may assume that C = Z(R) and R is C-algebra centrally closed. By Martindale's theorem [19], R is then a primitive ring having nonzero socle soc(R) with C as the associated division ring. Hence, by Jacobson's theorem [10, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

If $\dim_C V = 2$, then $R \cong M_2(C)$, that is, R satisfies s_4 , a contradiction. So, let $\dim_C V \ge 3$.

We show that for any $v \in V$, v and qv are linearly C-dependent. Suppose that v and qv are linearly independent for some $v \in V$. Since $\dim_C V \ge 3$, there exists $u \in V$ such that v, qv, u are linearly C-independent set of vectors. By density, there exists $x_1, x_2 \in R$ such that

$$x_1v = v, x_1qv = 0, x_1u = qv; x_2v = 0, x_2qv = u, x_2u = 0.$$

Then $0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n (c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = aqv.$

This implies that if for some $v \in V$, $aqv \neq 0$, then by contradiction, v and qv are linearly C-dependent.

Now choose $v \in V$ such that v and qv are linearly C-independent. Then aqv = 0. Let us consider a subspace $W = \{\alpha v + \beta qv \mid \alpha, \beta \in C\}$ of V. Let $aq \neq 0$. Then, there exists $w \in V$ such that $aqw \neq 0$. Then $aq(v - w) = aqw \neq 0$. Then by the above argument, w, qw are linearly C-dependent and (v - w), q(v - w) too. Thus there exist $\alpha, \beta \in C$ such that $qw = \alpha w$ and $q(v - w) = \beta(v - w)$. Then $qv = \beta(v - w) + qw = \beta(v - w) + \alpha w$ i.e., $(\alpha - \beta)w = qv - \beta v \in W$. Now $\alpha = \beta$ implies that $qv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$.

Next assume that $u \in V$ such that aqu = 0. Then $aq(w+u) = aqw \neq 0$. By above argument, $aq(w+u) \neq 0$ implies $w+u \in W$. Since $w \in W$, we have $u \in W$. Thus it is observed that for any $v \in V$, $aqv \neq 0$ implies $v \in W$ and aqv = 0implies $v \in W$. This implies that V = W i.e., $\dim_C V = 2$, a contradiction.

Thus up to now we have proved that v and qv are linearly C-dependent for all $v \in V$, unless aq = 0. If $aq \neq 0$, by standard argument, it follows that $qv = \lambda v$ for all $v \in V$ and $\lambda \in C$ fixed. Then $(q - \lambda)V = 0$, implying $q = \lambda \in C$.

Now let aq = 0. Since $\dim_C V \ge 3$, there exists $w \in V$ such that v, qv, w are linearly *C*-independent set of vectors. By density, there exists $x_1, x_2 \in R$ such that

$$x_1v = v$$
, $x_1qv = 0$, $x_1w = v + qv$; $x_2v = 0$, $x_2qv = w$, $x_2w = 0$

Then $0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n (c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = av$. Then by above argument, since $a \neq 0, q \in C$.

Therefore, we have proved that in any case $q \in C$. Hence our identity reduces to

$$a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n) = 0,$$

where b' = b + q and c' = c + p.

Now we prove that v and c'v are linearly C-dependent. If possible let v and c'v be linearly independent for some $v \in V$. Then there exists $w \in V$ such that v, c'v and w are linearly independent over C. By density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0$$
, $x_1c'v = v$, $x_1w = 2c'v$; $x_2v = c'v$, $x_2c'v = w$, $x_2w = 0$.

Then $0 = a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n)v = a(b' + c')v$. As above, this implies either a(b' + c') = 0 or $c' \in C$. Let a(b' + c') = 0. Then we have that R satisfies $0 = a[c', [x_1, x_2]^n][x_1, x_2]^n$. By density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0$$
, $x_1c'v = v$, $x_1w = c'v$; $x_2v = c'v$, $x_2c'v = w$, $x_2w = 0$.

Thus $0 = a[c', [x_1, x_2]^n][x_1, x_2]^n v = ac'v$. This implies either ac' = 0 or $c' \in C$. Let ac' = 0. Then we have that R satisfies $0 = a[x_1, x_2]^n]c'[x_1, x_2]^n$. Again by density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0$$
, $x_1c'v = v$, $x_1w = v + c'v$; $x_2v = c'v$, $x_2c'v = w$, $x_2w = 0$.

Thus $0 = a[x_1, x_2]^n [c'[x_1, x_2]^n v = av$. Since $a \neq 0$, this implies $c' \in C$. Thus in any case, we have $c' \in C$. Hence R satisfies $0 = a(b' + c')[x_1, x_2]^{2n}$, which implies a(b' + c') = 0.

Proof of Theorem 1.1. Let $0 \neq a \in l_R(S)$. Then $a(H(u^n)u^n + u^nG(u^n)) = 0$ for all $u \in L$. If char (R) = 2 and R satisfies s_4 , then we obtain our conclusion (1). So we assume that either char $(R) \neq 2$ or R does not satisfy s_4 . Then by [2, Lemma 1] and [13, Theorem 13], since L is a noncentral Lie ideal of R, there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence, by our hypothesis, we have,

$$a(H([x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^n G([x_1, x_2]^n)) = 0$$

for all $x_1, x_2 \in I$. Since I, R and U satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [16]), they also satisfy the same generalized differential identities. Hence, by [15], U satisfies

$$a(H([x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^n G([x_1, x_2]^n)) = 0$$

for all $x_1, x_2 \in U$, where H(x) = bx + d(x) and $G(x) = px + \delta(x)$, for some $b, p \in U$ and derivations d and δ of U, that is, U satisfies

(4)
$$a(b[x_1, x_2]^{2n} + d([x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \delta([x_1, x_2]^n)) = 0.$$

Now we divide the proof into two cases:

Case I. Let d(x) = [c, x] for all $x \in U$ and $\delta(x) = [q, x]$ for all $x \in U$ i.e., d and δ be inner derivations of U. Then from (4), we obtain that U satisfies

(5)
$$a((b+c)[x_1,x_2]^{2n} + [x_1,x_2]^n(p-c+q)[x_1,x_2]^n - [x_1,x_2]^{2n}q) = 0.$$

By Lemma 2.2, when R does not satisfy s_4 , we have $q, p-c+q \in C$ with a(b+p) = 0. This implies $p-c \in C$. Hence H(x) = bx + [c, x] = bx + [p, x] = b'x - xp, G(x) = px for all $x \in U$ and so for all $x \in R$ with ab' = 0, where b' = b + p.

Moreover, when R satisfies s_4 (in this case by assumption char $(R) \neq 2$), then $R \subseteq M_2(F)$ and, R and $M_2(F)$ satisfy the same GPI, where $M_2(F)$ is a matrix ring over a field F. Hence $M_2(F)$ satisfies $a((b+c)[x_1, x_2]^{2n} + [x_1, x_2]^n (p <math>c+q)[x_1, x_2]^n - [x_1, x_2]^{2n}q) = 0$. Since $[x, y]^2 \in Z(M_2(F))$ for all $x, y \in M_2(F)$, $M_2(F)$ satisfies

(6)
$$a((b+c-q)[x_1,x_2]^{2n} + [x_1,x_2]^n(p-c+q)[x_1,x_2]^n) = 0.$$

If n is even, then by choosing $x_1 = e_{12}$, $x_2 = e_{21}$, we have 0 = a(b+p).

If n is odd, then $M_2(F)$ satisfies $a((b+c-q)[x_1,x_2] + [x_1,x_2](p-c+q))$ $[x_1,x_2]^{2n-1} = 0$. By Lemma 2.7 in [7], we conclude that $p-c+q \in Z(R)$ and a(b+p) = 0. Thus when R satisfies s_4 , one of the following holds:

- (i) n is even and a(b+p) = 0. In this case, H(x) = bx + [c, x] and G(x) = px + [q, x] for all $x \in R$, with a(b+p) = 0. This is our conclusion (2).
- (ii) n is odd and $p c + q \in C$ and a(b + p) = 0. Hence H(x) = bx + [c, x]and G(x) = px + [q, x] = px - [p - c, x] = xp + [c, x] for all $x \in R$, with a(b + p) = 0. This is our conclusion (3).

Case II. Next assume that d and δ are not both inner derivations of U, but they are C-dependent modulo inner derivations of U. Suppose $d = \lambda \delta + ad_c$, that is, $d(x) = \lambda \delta(x) + [c, x]$ for all $x \in U$, where $\lambda \in C$, $c \in U$. Then d can not be inner derivation of U. From (4), we have that U satisfies

$$a\left(b[x_1, x_2]^{2n} + \lambda\delta([x_1, x_2]^n)[x_1, x_2]^n + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n\delta([x_1, x_2]^n)\right) = 0.$$

This gives

$$\begin{aligned} & a \bigg(b[x_1, x_2]^{2n} + \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i \delta([x_1, x_2]) [x_1, x_2]^{n-1-i} [x_1, x_2]^n + [c, [x_1, x_2]^n] [x_1, x_2]^n \\ & + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i \delta([x_1, x_2]) [x_1, x_2]^{n-1-i} \bigg) = 0. \end{aligned}$$

Then by Kharchenko's theorem [11], we have that U satisfies

$$a\left(b[x_1, x_2]^{2n} + \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-1-i} [x_1, x_2]^n + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-1-i}\right) = 0.$$

In particular U satisfies blended component

(8)
$$a\left(b[x_1, x_2]^{2n} + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n\right) = 0$$

and

(9)
$$a\left(\lambda\sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2]) [x_1, x_2]^{n-1-i} [x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2]) [x_1, x_2]^{n-1-i}\right) = 0.$$

For $y_1 = [q, x_1]$ and $y_2 = [q, x_2]$, where $q \notin C$ we have that U satisfies

(10)
$$a([\lambda q, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n[q, [x_1, x_2]^n]) = 0.$$

By Lemma 2.2, if R does not satisfy s_4 , then $q \in C$, a contradiction. Hence we conclude that R satisfies s_4 . Now the relations (8) and (10) are similar to the relation (5). Thus by same argument as given in Case I, when R satisfies s_4 (in this case char (R) must be not equal to 2), one of the following holds:

(i) Let n be even. Then by (8), a(b+p) = 0. Thus H(x) = bx + d(x) and $G(x) = px + \delta(x)$ for all $x \in R$, with a(b+p) = 0. This is our conclusion (2).

(ii) Let *n* be odd. Then by (8), $p - c \in C$ and a(b + p) = 0. Again by (10), $q - \lambda q = q(1 - \lambda) \in C$. Since $q \notin C$, we have $\lambda = 1$. Then replacing $y_1 = x_1$ and $y_2 = 0$, (9) gives $na(\lambda + 1)[x_1, x_2]^{2n} = 0$, implying 2na = 0. Since char $(R) \neq 2$, na = 0. Hence $H(x) = bx + \lambda\delta(x) + [c, x] = bx + \delta(x) + [c, x]$ and $G(x) = px + \delta(x) = (p - c)x + cx + \delta(x) = x(p - c) + cx + \delta(x) = xp + \delta(x) + [c, x]$ for all $x \in R$. This is our conclusion (3).

The situation when $\delta = \lambda d + a d_c$ is similar.

Next assume that d and δ are C-independent modulo inner derivations of U. Since neither d nor δ is inner, by Kharchenko's Theorem [11], we have from (4) that U satisfies

$$(11) \quad a\left(b[x_1, x_2]^{2n} + \sum_{i=0}^{n-1} [x_1, x_2]^i ([u_1, x_2] + [x_1, u_2])[x_1, x_2]^{n-1-i} [x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\right) = 0$$

Then U satisfies blended component

(12)
$$a\left(b[x_1, x_2]^{2n} + [x_1, x_2]^n p[x_1, x_2]^n\right) = 0$$

and

(13)
$$a\left([x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([v_1, x_2] + [x_1, v_2]) [x_1, x_2]^{n-1-i}\right) = 0.$$

Replacing v_1 with $[q, x_1]$ and v_2 with $[q, x_2]$ for some $q \notin C$ in (13), we obtain that U satisfies

(14)
$$a([x_1, x_2]^n [q, [x_1, x_2]^n]) = 0.$$

By Lemma 2.2, we have $q \in C$, a contradiction, unless R satisfies s_4 . So we consider the case when R satisfies s_4 . In this case by same argument of Case I, (12) and (14) together implies that n is even and a(b + p) = 0. This gives our conclusion (2). Hence the theorem is proved.

Corollary 2.3. Let R be a prime ring with its Utumi ring of quotients U, C = Z(U) be the extended centroid of R, H and G two generalized derivations of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(H(u^2)u^2 + u^2G(u^2)) = 0$ for all $u \in L$. Then either there exist b', $p \in U$ such that H(x) = b'x - xp and G(x) = px for all $x \in R$ with ab' = 0 or R satisfies s_4 . Moreover, if R satisfies s_4 , then one of the following holds:

- (1) char (R) = 2;
- (2) there exist $b, p \in U$ and derivations d, δ of R such that H(x) = bx + d(x)and $G(x) = px + \delta(x)$ for all $x \in R$, with a(b+p) = 0.

Proof of Theorem 1.2. Let $0 \neq a \in l_R(S)$. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ for all $x \in I$. By Theorem 1.1, we have only to consider the case when R satisfies s_4 . In this case R is a PI-ring, and so there exists a field K such that $R \subseteq M_2(K)$ and, R and $M_2(K)$ satisfy the same GPI. First we assume that H and G are inner generalized derivations of R, that is, H(x) = bx + xc for all $x \in R$ and G(x) = px + xq for all $x \in R$, for some $b, c, p, q \in R$. Since $M_2(F)$ is a simple ring, by our hypothesis, $M_2(F)$ satisfies

(15)
$$a(bx^{2n} + x^n(c+p)x^n + x^{2n}q) = 0.$$

Moreover, R is a dense ring of K-linear transformations over a vector space V. Let $aq \neq 0$. Assume there exists $v \neq 0$, such that $\{v, qv\}$ is linear K-independent. By the density of R, there exists $r \in R$ such that

$$rv = 0; \quad r(qv) = qv.$$

Hence

$$0 = a(br^{2n} + r^n(c+p)r^n + r^{2n}q)v = aqv$$

Of course for any $w \in V$ such that $\{w, v\}$ are linearly K-dependent implies aqw = 0. Since $aq \neq 0$, there exists $w \in V$ such that $aqw \neq 0$. Then $\{w, v\}$ must be linearly K-independent. By the above argument it follows that w and qw are linearly K-dependent, as are $\{w + v, q(w + v)\}$ and $\{w - v, q(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in K$ such that

$$qw = \alpha_w w, \quad q(w+v) = \alpha_{w+v}(w+v), \quad q(w-v) = \alpha_{w-v}(w-v).$$

In other words we have

(16)
$$\alpha_w w + qv = \alpha_{w+v} w + \alpha_{w+v} v$$

and

(17)
$$\alpha_w w - qv = \alpha_{w-v} w - \alpha_{w-v} v.$$

By comparing (16) with (17) we get both

(18)
$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0$$

and

(19)
$$2qv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.$$

By (18) and since $\{w, v\}$ is K-independent and $char(K) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (19) it follows $2qv = 2\alpha_w v$. Since $\{qv, v\}$ is K-independent, the conclusion $\alpha_w = \alpha_{w+v} = 0$ follows, that is qw = 0 and q(w+v) = 0, which implies the contradiction qv = 0.

Hence we conclude that for any $v \in V$, $\{v, qv\}$ is linearly K-dependent. Thus there exists a suitable $\alpha_v \in K$ such that $qv = \alpha_v v$, and standard argument shows that there is $\alpha \in K$ such that $qv = \alpha v$ for all $v \in V$. Now let $r \in R, v \in V$. Since $qv = \alpha v$,

(20)
$$[q,r]v = (qr)v - (rq)v = q(rv) - r(qv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus [q, r]v = 0 for all $v \in V$ i.e., [q, r]V = 0. Since [q, r] acts faithfully as a linear transformation on the vector space V, [q, r] = 0 for all $r \in R$. Therefore, $q \in C$.

Thus up to now, we have proved that either aq = 0 or $q \in C$.

Let aq = 0. In this case, assume that there exists $v \neq 0$, such that $\{v, qv\}$ is linear K-independent. By the density of R, there exists $r \in R$ such that

$$rv = 0;$$
 $r(qv) = v + qv.$

Hence

$$0 = a(br^{2n} + r^n(c+p)r^n + r^{2n}q)v = av.$$

Thus by the same argument as above, this implies either a = 0 or $q \in C$. Since $a \neq 0, q \in C$.

Thus in any case we conclude that $q \in C$. Then (15) reduces to

(21)
$$a((b+q)x^{n} + x^{n}(c+p))x^{n} = 0.$$

Let there exists $v \neq 0$, such that $\{v, (c+p)v\}$ is linear K-independent. By the density of R, there exists $r \in R$ such that

$$rv = 0; \quad r((c+p)v) = (c+p)v.$$

Hence

$$0 = a((b+q)r^{n} + r^{n}(c+p))r^{n}v = a(c+p)v.$$

Then again by same argument, $c + p \in C$. Then (21) reduces to

(22)
$$a(b+c+p+q)x^{2n} = 0$$

for all $x \in R$. This implies a(b+c+p+q) = 0, where $q, c+p \in C$. Hence H(x) = bx + xc = bx + x(c+p) - xp = (b+c+p)x - xp = (b+c+p+q)x - x(p+q) for all $x \in R$ and G(x) = (p+q)x for all $x \in R$. This gives our conclusion.

Next assume that H(x) = bx + d(x) and $G(x) = px + \delta(x)$, where d, δ are not both inner derivations of R. In this case by our hypothesis, R satisfies

(23)
$$a\left(bx^{2n} + d(x^n)x^n + x^n px^n + x^n \delta(x^n)\right) = 0.$$

If d and δ are C-dependent modulo inner derivations of R, then $d = \lambda \delta + ad_c$ for some $\lambda \in C$. In this case (23) reduces to

(24)
$$a\left(bx^{2n} + \lambda\delta(x^{n})x^{n} + [c, x^{n}]x^{n} + x^{n}px^{n} + x^{n}\delta(x^{n})\right) = 0.$$

By Kharchenko's Theorem [11], R satisfies

(25)
$$a\left(bx^{2n} + \lambda \sum_{i} x^{i}yx^{n-i-1}x^{n} + [c, x^{n}]x^{n} + x^{n}px^{n} + x^{n}\sum_{i} x^{i}yx^{n-i-1}\right) = 0.$$

Replacing y with [p, x] for some $p \notin C$, we have from (25) that

(26)
$$a\left(bx^{2n} + \lambda[p, x^n]x^n + [c, x^n]x^n + x^npx^n + x^n[p, x^n]\right) = 0.$$

Then this implies as above (for inner derivation case) that $p \in C$, a contradiction.

The case when $\delta = \lambda d + ad_{c'}$ for some $\lambda \in C$, is similar.

Next assume that d and δ are C-independent modulo inner derivations of R. Then by Kharchenko's Theorem [11], R satisfies

(27)
$$a\left(bx^{2n} + \sum_{i} x^{i}yx^{n-i-1}x^{n} + x^{n}px^{n} + x^{n}\sum_{i} x^{i}zx^{n-i-1}\right) = 0.$$

Replacing y with [p, x] and z with [p', x] for some $p, p' \notin C$, we have

(28)
$$a\left(bx^{2n} + [p, x^n]x^n + x^n px^n + x^n[p', x^n]\right) = 0.$$

Then by same argument as above, it yields that $p' \in C$, a contradiction.

In particular, when H and G are two derivations of R, we have the following:

Corollary 2.4. Let R be a noncommutative prime ring with char $(R) \neq 2$ and C the extended centroid of R. Let d and δ be two derivations of R. If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $d = \delta = 0$.

3. **Results on semiprime rings**

In this section we extend the Corollary 2.4 to semiprime rings. Let R be a semiprime ring and U the left Utumi ring of quotients of R. Then C = Z(U), center of U, is called extended centroid of R. It is well known that C is a Von Neumann regular ring. It is known that C is a field if and only if R is a prime ring. The set of all idempotents of C is denoted by E. The elements of E are called central idempotents.

We know that any derivation of R can be uniquely extended to a derivation of U (see [16, Lemma 2]).

By using the standard theory of orthogonal completions for semiprime rings, we prove the following:

Theorem 3.1. Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and d, δ be two derivations of R. If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then there exist orthogonal central idempotents e_1 , e_2 , $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $(d + \delta)(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.

Proof. Since any derivation d can be uniquely extended to a derivation in U, and U and R satisfy the same differential identities (see [16]), $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in U$.

Let *B* be the complete Boolean algebra of *E*. We choose a maximal ideal *P* of *B* such that U/PU is 2-torsion free. Then *PU* is a prime ideal of *U*, which is *d*-invariant. Denote $\overline{U} = U/PU$ and $\overline{d}, \overline{\delta}$ be the canonical pair of derivations on \overline{U} induced by *d* and δ respectively. Then by hypothesis, $\overline{a}(\overline{d}(\overline{x}^n)\overline{x}^n + \overline{x}^n\overline{\delta}(\overline{x}^n)) = 0$ for all $\overline{x} \in \overline{U}$. Since \overline{U} is a prime ring, by Corollary 2.4, either $\overline{d} = \overline{\delta} = 0$ or $[\overline{U}, \overline{U}] = 0$ or $\overline{a} = 0$. In any case, we have $ad(U)[U, U] \subseteq PU$ and $a\delta(U)[U, U] \subseteq PU$ for all *P*, that is, $aD(U)[U, U] \subseteq PU$ for all *P*, where $D = d + \delta$. Since $\bigcap \{PU : P \text{ is any maximal ideal in$ *B*with <math>U/PU 2-torsion free} = 0, we have aD(U)[U, U] = 0.

By using the theory of orthogonal completion for semiprime rings (see, [1, Chapter 3]), it follows that there exist orthogonal central idempotents e_1 , e_2 , $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $D(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.

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