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ON PERFECTNESS OF INTERSECTION GRAPH OF IDEALS OF $\mathbb{Z}_n$

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Abstract

In this short paper, we characterize the positive integers n for which intersection graph of ideals of $\mathbb{Z}_n$ is perfect.

Keywords: intersection graph, strong perfect graph theorem, weakly triangulated graph, induced odd cycle.

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1. Introduction

The idea of associating graphs to algebraic structures for characterizing the algebraic structures with graphs and vice versa dates back to Bosak [4]. Till then, a lot of research, e.g., [1, 2, 3, 5, 6, 8, 10, 11, 9, 14] has been done in connecting graph structures to various algebraic objects like groups, rings, vector spaces etc. However, the most prominent among them are the zero-divisor graphs [2] and intersection graph of ideals of rings [6]. Recently, authors in [15] proved that intersection graph of ideals of $\mathbb{Z}_n$ is weakly perfect for all $n > 0$. In this paper, we characterize the values of n for which the intersection graphs of ideals of $\mathbb{Z}_n$ is perfect. In particular, we prove the following theorem.

Main Theorem. *The intersection graph of ideals of $\mathbb{Z}_n$ is perfect if and only if $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$ where p_i's are distinct primes and $\alpha_i \in \mathbb{N} \cup \{0\}$, i.e., the number of distinct prime factors of n is less than or equal to 4.*

2. Definition, preliminaries and known results

In this section, for convenience of the reader and also for later use, we recall some definitions, notations and results concerning elementary graph theory and intersection graph of ideals of a ring. For undefined terms and concepts the reader is referred to [16].

By a graph $G = (V, E)$, we mean a non-empty set V and a symmetric binary relation (possibly empty) E on V. The set V is called the set of vertices and E is called the set of edges of G. Two elements u and v in V are said to be adjacent if $(u, v) \in E$. $H = (W, F)$ is called an *induced subgraph* of G if $\phi \neq W \subseteq V$ and F consists of all the edges between the vertices in W in G. A complete subgraph of a graph G is called a *clique*. A *maximal clique* is a clique which is maximal with respect to inclusion. The *clique number* of G, written as $\omega(G)$, is the maximum size of a clique in G. The *chromatic number* of G, denoted as $\chi(G)$, is the minimum number of colours needed to label the vertices so that the adjacent vertices receive different colours. It is easy to observe that $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$ and it is said to be *perfect* if $\omega(H) = \chi(H)$ for all induced subgraphs H of G. A graph G is said to be *weakly triangulated* if neither G nor its complement contains a chordless cycle of length more than 4. Hayward [13] proved that all weakly triangulated graphs are perfect. Chudnovsky *et al.* [7] in 2004 settled a long standing conjecture regarding perfect graphs and provided a characterization of perfect graphs.

Theorem 2.1 (Strong Perfect Graph Theorem) [7]**.** *A graph G is perfect if and only if neither G nor its complement contains an odd cycle of length at least* 5 *as an induced subgraph.*

Let R be a ring. The intersection graph of ideals of R (introduced in [6]), denoted by $G(R)$, consists of all non-trivial ideals as vertices and two ideals I and J are adjacent if and only if $I \cap J \neq \{0\}$. Throughout this paper, we take the ring R to be $\mathbb{Z}_n$, the ring of integers modulo n. We know that $\mathbb{Z}_n$ is a principal ideal ring and each of its ideals is generated by $\overline{m} \in \mathbb{Z}_n$ where m is a factor of n. For convenience, we denote this ideal by (m). Also without loss of generality, whenever we take an ideal (m) of $\mathbb{Z}_n$, we assume that m is a factor of n. It was proved in [15] that intersection graph of ideals of $\mathbb{Z}_n$ is weakly perfect, i.e., $\omega(G(\mathbb{Z}_n)) = \chi(G(\mathbb{Z}_n))$ for all $n > 0$.

3. Perfectness of intersection graph of ideals of $\mathbb{Z}_n$

In this section, we prove some preparatory results and subsequently use them to prove the main theorem of the paper.

Proposition 3.1. *Let $G(\mathbb{Z}_n)$ be the intersection graph of ideals of $\mathbb{Z}_n$ and (a) and (b) be two ideals in $\mathbb{Z}_n$ such that $a \mid n$ and $b \mid n$. Then (a) and (b) are adjacent in $G(\mathbb{Z}_n)$ if and only if $lcm(a, b)$ is a factor of n and $1 < lcm(a, b) < n$.*

Proof. Since $\mathbb{Z}_n$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ as ring via the correspondence $\overline{a} \leftrightarrow a + n\mathbb{Z}$, the ideal (a) in $\mathbb{Z}_n$ corresponds to the ideal $\langle a\rangle + n\mathbb{Z}$ in $\mathbb{Z}/n\mathbb{Z}$ where $\langle a\rangle$ denote the set of integer multiples of a. Now, let $(a) \sim (b)$ in $G(\mathbb{Z}_n)$, i.e., $(a) \cap (b) \neq \{\overline{0}\}$. Since $a \mid n$ and $b \mid n$, we have $lcm(a, b) \mid n$. On the other hand, using the correspondence described above, we have $\langle a\rangle + n\mathbb{Z} \cap \langle b\rangle + n\mathbb{Z} \neq \{n\mathbb{Z}\}$. But, we know that $\langle a\rangle + n\mathbb{Z} \cap \langle b\rangle + n\mathbb{Z} = \langle lcm(a, b)\rangle + n\mathbb{Z}$. Hence, we have $\langle lcm(a, b)\rangle + n\mathbb{Z} \neq \{n\mathbb{Z}\}$. This, together with the fact that $lcm(a, b) \mid n$, implies that $1 < lcm(a, b) < n$.

Conversely, let $lcm(a, b)$ be a factor of n and $1 < lcm(a, b) < n$. Clearly, $\overline{0} \neq \overline{lcm(a, b)} \in (a) \cap (b)$ in $\mathbb{Z}_n$ and hence $(a) \sim (b)$ in $G(\mathbb{Z}_n)$. ■

Theorem 3.1. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$. If $k \geq 5$, then $G(\mathbb{Z}_n)$ is not perfect.*

Proof. Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdots {p_5}^{\alpha_5}.s$ where $s = 1$ if $k = 5$ and $s = {p_6}^{\alpha_6} \cdots {p_k}^{\alpha_k}$ if $k > 5$. Consider the cycle C given by $({p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}s) \sim ({p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}s) \sim ({p_3}^{\alpha_3}{p_4}^{\alpha_4}{p_5}^{\alpha_5}s) \sim ({p_4}^{\alpha_4}{p_5}^{\alpha_5}{p_1}^{\alpha_1}s) \sim ({p_5}^{\alpha_5}{p_1}^{\alpha_1}{p_2}^{\alpha_2}s) \sim ({p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}s)$. Simple calculation using Proposition 3.1 shows that C is an induced 5-cycle in $G(\mathbb{Z}_n)$ and hence by Theorem 2.1, $G(\mathbb{Z}_n)$ is not perfect. ■

Theorem 3.2. *Let $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$. Then $G(\mathbb{Z}_n)$ does not contain any induced cycle of length greater than 4.*

Proof. Let us assume for contradiction that $G(\mathbb{Z}_n)$ contain an induced cycle C of length greater than 4, say $(a_1) \sim (a_2) \sim (a_3) \sim (a_4) \sim (a_5) \sim \cdots \sim (a_1)$. By Proposition 3.1, we have

$$lcm(a_1, a_3) = lcm(a_1, a_4) = lcm(a_2, a_4) = lcm(a_2, a_5) = lcm(a_3, a_5) = n.$$

Claim. $gcd(a_1, a_3) > 1$. If possible, let $gcd(a_1, a_3) = 1$. Since $gcd(a_1, a_3) \cdot lcm(a_1, a_3) = a_1a_3$, we have $lcm(a_1, a_3) = a_1a_3 = n$. Note that as $lcm(a_3, a_5) = n$, we have $n = \frac{a_1a_3}{gcd(a_1,a_3)} = \frac{a_3a_5}{gcd(a_3,a_5)}$, i.e., $a_1 \cdot gcd(a_3, a_5) = a_5 \cdot gcd(a_1, a_3)$, i.e., $a_5 = a_1 \cdot gcd(a_3, a_5)$, i.e., a_5 is a multiple of a_1. Now as a_1 and a_3 are coprime and their lcm is n, without loss of generality, two cases may arise: either $a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}; a_3 = {p_3}^{\alpha_3}{p_4}^{\alpha_4}$ or $a_1 = {p_1}^{\alpha_1}; a_3 = {p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$.

If $a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}; a_3 = {p_3}^{\alpha_3}{p_4}^{\alpha_4}$, we have $a_5 = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdot s$ for some natural number s such that $a_5 \mid n$. Also as $lcm(a_1, a_4) = n$, we have $a_4 = {p_3}^{\alpha_3}{p_4}^{\alpha_4} \cdot t$, for some natural number t such that $a_4 \mid n$. Thus $lcm(a_4, a_5) = n$ contradicting Proposition 3.1 and the fact that $a_4 \sim a_5$ in C.

If $a_1 = {p_1}^{\alpha_1}; a_3 = {p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$, similarly we have $a_5 = {p_1}^{\alpha_1} \cdot s$ and $a_4 = {p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4} \cdot t$ and hence $lcm(a_4, a_5) = n$ thereby leading to a contradiction. Thus by combining above two cases, we have $gcd(a_1, a_3) > 1$.

Thus we have $lcm(a_1, a_3) = n$ and $gcd(a_1, a_3) > 1$ with $a_1 \mid n$ and $a_3 \mid n$. Without loss of generality, let p_1 be a common factor of a_1 and a_3 and let $a_1 = {p_1}^x \cdot s$ and $a_3 = {p_1}^y \cdot t$ where p_1 is coprime with s and t. Now, if $max\{x, y\} < \alpha_1$, then $lcm(a_1, a_3) < n$, a contradiction. Thus either $x = \alpha_1$ or $y = \alpha_1$, i.e., for any common prime divisor p_i of a_1 and a_3, either ${p_i}^{\alpha_i} \mid a_1$ or ${p_i}^{\alpha_i} \mid a_3$ or both. Also as $lcm(a_1, a_3) = n$, all the ${p_i}^{\alpha_i}$ are factors of either a_1 or a_3 or both. Thus, without loss of generality, the forms of a_1 and a_3 are as follows: either

$$\text{Case 1: } a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\beta_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

or

$$\text{Case 2: } a_1 = {p_1}^{\alpha_1}{p_2}^{\beta_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

or

$$\text{Case 3: } a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

or

$$\text{Case 4: } a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

where $\beta_i < \alpha_i$. Note that in first two cases, a_1 and a_3 do not share any ${p_i}^{\alpha_i}$ as common factor. In the third case, they share only one ${p_i}^{\alpha_i}$ as common factor and in the fourth case, they share two ${p_i}^{\alpha_i}$'s as common factor.

Case 1. ($a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\beta_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$). Since $lcm(a_1, a_4) = n$, we have $a_4 = {p_1}^{\gamma_1}{p_2}^{\gamma_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$ where $\gamma_1 \leq \alpha_1, \gamma_2 \leq \alpha_2$ and $(\gamma_1, \gamma_2) \neq (\alpha_1, \alpha_2)$. Again, since $lcm(a_3, a_5) = n$, we have $a_5 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\delta_3}{p_4}^{\delta_4}$ where $\delta_3 \leq \alpha_3, \delta_4 \leq \alpha_4$ and $(\delta_3, \delta_4) \neq (\alpha_3, \alpha_4)$. Hence, we have $lcm(a_4, a_5) = n$, a contradiction to the fact that $a_4 \sim a_5$. Thus Case 1 is an impossibility.

Case 2. ($a_1 = {p_1}^{\alpha_1}{p_2}^{\beta_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$). Since $lcm(a_1, a_4) = n$, we have $a_4 = {p_1}^{\gamma_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$ where $\gamma_1 < \alpha_1$. Again, since $lcm(a_3, a_5) = n$, we have $a_5 = {p_1}^{\alpha_1}{p_2}^{\delta_2}{p_3}^{\delta_3}{p_4}^{\delta_4}$ where $\delta_i \leq \alpha_i$ and $(\delta_2, \delta_3, \delta_4) \neq (\alpha_2, \alpha_3, \alpha_4)$. Hence, we have $lcm(a_4, a_5) = n$, a contradiction to the fact that $a_4 \sim a_5$. Thus Case 2 is an impossibility.

Case 3. ($a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$). Since $lcm(a_1, a_4) = n$, we have ${p_3}^{\alpha_3}{p_4}^{\alpha_4} \mid a_4$. Again, since $lcm(a_3, a_5) = n$, we have ${p_1}^{\alpha_1} \mid a_5$. Now, as $lcm(a_2, a_5) = n$, we have either ${p_2}^{\alpha_2} \mid a_2$ or ${p_2}^{\alpha_2} \mid a_5$. But if ${p_2}^{\alpha_2} \mid a_5$, then we have $lcm(a_4, a_5) = n$, a contradiction. Thus, we have ${p_2}^{\alpha_2} \mid a_2$. Again, as $lcm(a_2, a_4) = n$, we have either ${p_1}^{\alpha_1} \mid a_2$ or ${p_1}^{\alpha_1} \mid a_4$. If ${p_1}^{\alpha_1} \mid a_2$, then $lcm(a_2, a_3) = n$, a contradiction. On the other hand, if ${p_1}^{\alpha_1} \mid a_4$, then $lcm(a_3, a_4) = n$, a contradiction. Thus Case 3 is an impossibility.

Case 4. ($a_1 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$). Since $lcm(a_1, a_4) = n$, we have ${p_4}^{\alpha_4} \mid a_4$. Now, as $lcm(a_2, a_4) = n$, we have either ${p_1}^{\alpha_1} \mid a_2$ or ${p_1}^{\alpha_1} \mid a_4$. If ${p_1}^{\alpha_1} \mid a_2$, then $lcm(a_2, a_3) = n$, a contradiction. On the other hand, if ${p_1}^{\alpha_1} \mid a_4$, then $lcm(a_3, a_4) = n$, a contradiction. Thus Case 4 is an impossibility.

Thus, combining all the cases we conclude that $G(\mathbb{Z}_n)$ does not contain any induced cycle of length greater than 4. ■

Theorem 3.3. *Let* $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$. *Then* $\overline{G(\mathbb{Z}_n)}$, *the complement of* $G(\mathbb{Z}_n)$, *does not contain any induced cycle of length greater than* 4.

Proof. Let us assume for contradiction that $\overline{G(\mathbb{Z}_n)}$ contain an induced cycle C of length greater than 4, say $(a_1) \sim (a_2) \sim (a_3) \sim (a_4) \sim \cdots \sim (a_t) \sim (a_1)$ with $t \geq 5$. Then, by Proposition 3.1, $lcm(a_1, a_2) = lcm(a_2, a_3) = lcm(a_3, a_4) = \cdots = lcm(a_t, a_1) = n$.

Claim. $gcd(a_2, a_3) > 1$. If possible, let $gcd(a_2, a_3) = 1$. Since $lcm(a_2, a_3) = n$, we have $n = a_2 a_3$. Thus without loss of generality, either

$$a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}; a_3 = {p_3}^{\alpha_3}{p_4}^{\alpha_4} \text{ or } a_2 = {p_1}^{\alpha_1}; a_3 = {p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}.$$

If $a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}; a_3 = {p_3}^{\alpha_3}{p_4}^{\alpha_4}$, as $lcm(a_3, a_4) = lcm(a_1, a_2) = n$, we have $a_1 = {p_3}^{\alpha_3}{p_4}^{\alpha_4} \cdot s$ and $a_4 = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \cdot t$ for some positive integer s, t. But this implies that $lcm(a_1, a_4) = n$, i.e., $a_1 \sim a_4$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction.

On the other hand, if $a_2 = {p_1}^{\alpha_1}; a_3 = {p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$, as $lcm(a_3, a_4) = lcm(a_1, a_2) = n$, we have $a_1 = {p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4} \cdot s$ and $a_4 = {p_1}^{\alpha_1} \cdot t$ for some positive integer s, t. But this implies that $lcm(a_1, a_4) = n$, i.e., $a_1 \sim a_4$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction. Hence the claim is true.

Now, we have $lcm(a_2, a_3) = n$ and $gcd(a_2, a_3) > 1$ with $a_2 \mid n$ and $a_3 \mid n$. Without loss of generality, let p_1 be a common factor of a_2 and a_3 and let $a_2 = {p_1}^x \cdot s$ and $a_3 = {p_1}^y \cdot t$ where p_1 is coprime with s and t. Now, if $max\{x, y\} < \alpha_1$, then $lcm(a_2, a_3) < n$, a contradiction. Thus either $x = \alpha_1$ or $y = \alpha_1$, i.e., for any common prime divisor p_i of a_2 or a_3, either ${p_i}^{\alpha_i} \mid a_2$ or ${p_i}^{\alpha_i} \mid a_3$ or both. Also as $lcm(a_2, a_3) = n$, all the ${p_i}^{\alpha_i}$ are factors of either a_2 or a_3. Thus, without loss of generality, the forms of a_2 and a_3 are as follows: either

$$\text{Case 1: } a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\beta_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

or

$$\text{Case 2: } a_2 = {p_1}^{\alpha_1}{p_2}^{\beta_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

or

$$\text{Case 3: } a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

or

$$\text{Case 4: } a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$$

where $\beta_i < \alpha_i$. Note that in first two cases, a_2 and a_3 do not share any ${p_i}^{\alpha_i}$ as common factor. In the third case, they share only one ${p_i}^{\alpha_i}$ as common factor and in the fourth case, they share two ${p_i}^{\alpha_i}$'s as common factor.

Case 1. $(a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\beta_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4})$. Since $lcm(a_1, a_2) = lcm(a_3, a_4) = n$, we have ${p_3}^{\alpha_3}{p_4}^{\alpha_4} \mid a_1$ and ${p_1}^{\alpha_1}{p_2}^{\alpha_2} \mid a_4$. But this implies $lcm(a_1, a_4) = n$, i.e., $a_1 \sim a_4$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction and hence Case 1 is an impossibility.

Case 2. $(a_2 = {p_1}^{\alpha_1}{p_2}^{\beta_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4})$. Since $lcm(a_1, a_2) = lcm(a_3, a_4) = n$, we have ${p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4} \mid a_1$ and ${p_1}^{\alpha_1} \mid a_4$. But this implies $lcm(a_1, a_4) = n$, a contradiction and hence Case 2 is an impossibility.

Case 3. $(a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\beta_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4})$. Since $lcm(a_1, a_2) = n$, we have ${p_3}^{\alpha_3}{p_4}^{\alpha_4} \mid a_1$. Also, since $lcm(a_t, a_1) = n$, either ${p_1}^{\alpha_1} \mid a_1$ or ${p_1}^{\alpha_1} \mid a_t$. If ${p_1}^{\alpha_1} \mid a_1$, then we have ${p_1}^{\alpha_1}{p_3}^{\alpha_3}{p_4}^{\alpha_4} \mid a_1$ which implies $lcm(a_1, a_3) = n$, i.e., $a_1 \sim a_3$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction. On the other hand, if ${p_1}^{\alpha_1} \mid a_t$, we have $lcm(a_t, a_3) = n$, i.e., $a_t \sim a_3$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction. Thus combining both the possibilities, Case 3 is an impossibility.

Case 4. $(a_2 = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\beta_4}; a_3 = {p_1}^{\beta_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4})$. Since $lcm(a_1, a_2) = n$, we have ${p_4}^{\alpha_4} \mid a_1$. Also, since $lcm(a_t, a_1) = n$, either ${p_1}^{\alpha_1} \mid a_1$ or ${p_1}^{\alpha_1} \mid a_t$. If ${p_1}^{\alpha_1} \mid a_1$, then we have ${p_1}^{\alpha_1}{p_4}^{\alpha_4} \mid a_1$ which implies $lcm(a_1, a_3) = n$, i.e., $a_1 \sim a_3$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction. On the other hand, if ${p_1}^{\alpha_1} \mid a_t$, we have $lcm(a_t, a_3) = n$, i.e., $a_t \sim a_3$ in $\overline{G(\mathbb{Z}_n)}$, a contradiction. Thus combining both the possibilities, Case 4 is an impossibility.

Thus, combining all the cases we conclude that $\overline{G(\mathbb{Z}_n)}$ does not contain any induced cycle of length greater than 4. ■

Finally, with Theorems 2.1, 3.1, 3.2 and 3.3 in hand, we are now in a position to prove the main result of this paper.

Main Theorem. *The intersection graph of ideals of $\mathbb{Z}_n$ is perfect if and only if $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}{p_3}^{\alpha_3}{p_4}^{\alpha_4}$ where p_i's are distinct primes and $\alpha_i \in \mathbb{N} \cup \{0\}$, i.e., the number of distinct prime factors of n is less than or equal to 4.*

Proof. Clearly, Theorem 3.1 shows that the condition is necessary. For the sufficiency part, first with the help of Theorems 3.2 and 3.3, we prove that the intersection graph of ideals of $\mathbb{Z}_n$ is weakly triangulated. Now, using the fact that weakly triangulated graphs are perfect [13], we conclude that the intersection graph of ideals of $\mathbb{Z}_n$ is perfect if n has exactly four distinct prime factors. The proofs for the cases when n has exactly three, two or one distinct prime factors follows similarly by suitably taking some of the α_i's to be zero. ■

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