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ON QI-ALGEBRAS

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Abstract

In this paper, the notion of a QI-algebra is introduced which is a generalization of a BI-algebra and there are studied its properties. We considered ideals, congruence kernels in a QI-algebra and characterized congruence kernels whenever a QI-algebra is right distributive.

Keywords: BI-algebra, QI-algebra, right distributive, ideal, congruence kernel.

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1. Introduction

BCK-algebras and BCI-algebras were introduced by Imai and Iséki [4, 5]. Since their introduction, several generalizations of BCK-algebras were introduced and extensively studied by many researchers. Abbott [2] introduced a concept of an implication algebra in the sake to formalize the logical connective implication in the classical propositional logic. Recently, Saeid *et al.* introduced the concept of a BI-algebra [1] as a generalization of (dual) implication algebra and studied its properties.

In this paper, we introduce the concept of a QI-algebra which is a generalization of a BI-algebra and study its properties. We consider the concept of ideals, congruences in a QI-algebra and give connection between ideals and congruence kernels whenever a QI-algebra is right distributive.

2. Preliminaries

First, we recall certain definitions from [1, 2, 4] and [5] that are required in the paper.

Definition 2.1 ([5]). A BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(1) $(x * y) * (x * z) \leq (z * y)$,

(2) $x * (x * y) \leq y$,

(3) $x \leq x$,

(4) $x \leq y$ and $y \leq x$ imply $x = y$,

(5) $x \leq 0$ implies $x = 0$,

where $x \leq y$ is defined by $x * y = 0$.

If (5) is replaced by (6) $0 \leq x$, then the algebra is called a BCK-algebra [3]. It is known that every BCK-algebra is a BCI-algebra but not conversely. A BCK-algebra satisfying the property $x * (y * x) = x$ for all $x, y \in X$ is called an implicative BCK-algebra.

Several generalizations of a BCK-algebra, in the form of definitions, one can see in the paper [1].

Definition 2.2 ([2]). A groupoid $(X, *)$ is called an implication algebra if it satisfies the following identities:

(a) $(x * y) * x = x$,

(b) $(x * y) * y = (y * x) * x$,

(c) $x * (y * z) = y * (x * z)$,

for all $x, y, z \in X$.

Definition 2.3 ([2]). Let $(X, *)$ be an implication algebra and binary operation "$\circ$" on X be defined by

$$x * y = y \circ x.$$

Then $(X, \circ)$ is said to be a dual implication algebra. In fact, the axioms of that are as follows:

(a) $x \circ (y \circ x) = x$,

(b) $x \circ (x \circ y) = y \circ (y \circ x)$,

(c) $(x \circ y) \circ z = (x \circ z) \circ y$,

for all $x, y, z \in X$.

Chen and Oliveira [6] proved that in any implication algebra $(X, *)$ the identity $x * x = y * y$ holds for all $x, y \in X$. We denote the identity $x * x = y * y$ by the constant 0. The notion of BI-algebras comes from the (dual) implication algebra.

Definition 2.4 ([1]). An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BI-algebra if

(BI1) $x * x = 0$,

(BI2) $x * (y * x) = x$,

for all $x, y, \in X$.

It can be observed that every dual implication algebra is a BI-algebra but converse need not be true.

3. QI-algebras

In this section, we define the notion of a QI-algebra which is a generalization of a BI-algebra and study its properties.

Definition 3.1. A QI-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms:

(QI1) $x * x = 0$,

(QI2) $x * 0 = x$,

(QI3) $x * (y * (x * y)) = x * y$,

for all $x, y \in X$.

Let $(X, *, 0)$ be a QI-algebra. We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0$. A relation $\leq$ is not a partially order, but it is only reflexive.

Note that every BI-algebra is a QI-algebra but converse need not be true.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the following table.

$*$	0	1	2	3
0	0	2	1	0
1	1	0	1	0
2	2	2	0	2
3	3	2	1	0

Then $(X, *, 0)$ is a QI-algebra but not a BI-algebra because

$$3 * (2 * 3) = 3 * 2 = 1 \neq 3.$$

Also, every implicative BCK-algebra is a QI-algebra but converse need not be true.

Example 3.3. Let $X = \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	b
c	c	0	a	0

Then $(X, *, 0)$ is a QI-algebra but not an implicative BCK-algebra because

$$a * c = 0 \ \& \ c * a = 0 \text{ but } a \neq c \text{ and } c * (b * c) = c * b = a \neq c.$$

Example 3.4. Let $X = \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	b	a	0
a	a	0	a	0
b	b	b	0	b
c	c	b	a	0

Then $(X, *, 0)$ is a QI-algebra but not a BCI/BCK-algebra because

$$[(a * a) * (a * c)] * (c * a) = (0 * 0) * b = a \neq 0 \text{ and } 0 * a \neq 0.$$

Proposition 3.5. *Let $(X, *, 0)$ be a QI-algebra. Then*

(i) $x * (0 * x) = x$,

(ii) *if* $x \leq 0$, *then* $x = 0$,

(iii) *if* $x * y = y$, *then* $x = y$,

(iv) *if* $x * y = x$, *then* $x * (y * x) = x$,

(v) *if* $(x * y) * (z * u) = (x * z) * (u * y)$, *then* $X = \{0\}$,

for all $x, y, z, u \in X$.

Proof. (i) Using $(QI2)$ and $(QI3)$ we have $x*(0*x) = x*(0*(x*0)) = x*0 = x$.
(ii) Let $x \leq 0$. Then $x * 0 = 0$ and hence $x = 0$.
(iii) Let $x * y = y$. Then, by $(QI3), (QI1)$ and $(QI2)$, we have

$$y = x * y = x * (y * (x * y)) = x * (y * y) = x * 0 = x.$$

(iv) Let $x * y = x$. Then

$$x = x * y = x * (y * (x * y)) = x * (y * x).$$

(v) If $x \in X$, then we have

$$x = x * (0 * x) = (x * 0) * (0 * x) = (x * 0) * (x * 0) = x * x = 0.$$

Hence $X = \{0\}$. ■

Definition 3.6. A QI-algebra X is said to be right distributive (or left distributive, resp.) if

$$(QI4) \quad (x * y) * z = (x * z) * (y * z), (z * (x * y) = (z * x) * (z * y), \text{ resp.})$$

for all $x, y, z \in X$.

Example 3.7. (i) Example 3.3 is a right distributive QI-algebra.
(ii) Example 3.2 is not a right distributive QI-algebra, since

$$(3 * 1) * 3 = 2 * 3 = 2 \neq 0 = 0 * 0 = (3 * 3) * (1 * 3).$$

Proposition 3.8. *If X is a left distributive* QI-*algebra, then $X = \{0\}$.*

Proof. Let X be a left distributive QI-algebra and $x \in X$. Then by (QI2) and (QI1), we have

$$x = x * 0 = x * (x * x) = (x * x) * (x * x) = 0 * 0 = 0.$$

■

Proposition 3.9. *If X is a right distributive QI-algebra, then*

(QI5) $0 * x = 0$,

(QI6) $(x * y) * y = x * y$,

for any $x, y \in X$.

Proof. Let $x, y \in X$. Then

$$(QI5) : 0 * x = (x * x) * x = (x * x) * (x * x) = 0 * 0 = 0.$$

$$(QI6) : (x * y) * y = (x * y) * (y * y) = (x * y) * 0 = x * y.$$

■

Proposition 3.10. *In a right distributive QI-algebra X, for all $x, y, z \in X$, the following conditions hold:*

(1) $y * x \leq y$,

(2) $(y * x) * x \leq y$,

(3) $(x * z) * (y * z) \leq x * y$,

(4) $x \leq y$ *implies* $x * z \leq y * z$,

(5) $(x * y) * z \leq x * (y * z)$.

(6) *If* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$,

(7) $x \leq y$ *implies* $z * y \leq z * x$,

(8) $(x * y) * z \leq (x * z) * y$,

(9) $(z * x) * (z * y) \leq (y * x)$.

(10) *If* $x * y = z * y$, *then* $(x * z) * y = 0$.

Proof. We can easily prove (1) to (6) by the application of $(QI1), (QI2), (QI4)$ and $(QI5)$. Let $x \leq y$. Then $x * y = 0$ and hence $[(z * y) * (z * x)] * (x * y) = [(z * y) * (x * y)] * [(z * x) * (x * y)] = [(z * x) * y] * (z * x) = 0$. Therefore (7) follows. By (1), $z * y \leq z$. Then, by (7), $(x * y) * z \leq (x * y) * (z * y)$ and hence $(x * y) * z \leq (x * z) * y$ which proves (8). Now $[(z * x) * (z * y)] * (y * x) \leq [(z * x) * (y * x)] * (z * y) = [(z * y) * x] * (z * y) = 0$. Hence $(z * x) * (z * y) \leq y * x$ which proves (9). Let $x * y = z * y$. Then $(x * z) * y = (x * y) * (z * y) = 0$ which proves (10). ■

4. Ideals in QI-algebras

In this section, we introduce the concept of an ideal in a QI-algebra and study its properties.

Definition 4.1. Let $(X, *, 0)$ be a QI-algebra and $I \subseteq X$. Then I is called an ideal of X if it satisfies the following:

(I1) $0 \in I$,

(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

Clearly, $\{0\}$ and X are ideals of X and we call them as zero ideal and trivial ideal respectively. An ideal I is said to be proper if $I \neq X$.

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with the following table.

*	0	a	b	c
0	0	b	a	0
a	a	0	a	0
b	b	b	0	b
c	c	b	a	0

Then $(X, *, 0)$ is a QI-algebra. Clearly, $I_1 = \{0, a\}$ and $I_2 = \{0, a, c\}$ are ideals of X. But $I_3 = \{0, a, b\}$ is not an ideal of X.

Lemma 4.3. *Let* X *be a QI-algebra and* I *a non-empty subset of* X *satisfying the following conditions:*

(I3) $x \in X$ *and* $y \in I$ *imply* $y * x \in I$,

(I4) $x \in X, a, b \in I$ *imply* $x * ((x * a) * b) \in I$.

Then I *is an ideal of* X.

Proof. Let I be a non-empty subset of X satisfying $(I3)$ and $(I4)$. Then $0 \in I$. Let $y \in I$ and $x * y \in I$. Then, by $(I4)$, we have $x * (x * y) = x * ((x * y) * 0) \in I$. Put $a = x * y$, $b = x * (x * y)$. Then $a, b \in I$ and $x = x * 0 = x * ((x * (x * y)) * (x * (x * y))) \in I$. Hence I is an ideal of X. ∎

The converse of the above lemma does not hold in general.

Example 4.4. Let $X = \{0, a, b, c\}$ be a set with the following table.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then $(X, *, 0)$ is a QI-algebra. Clearly, $I = \{0, a\}$ is an ideal of X but it doesn't satisfy $(I3)$ and $(I4)$.

However, for right distributive QI-algebras we have

Theorem 4.5. *If* X *is a right distributive QI-algebra and* I *is an ideal of* X*. Then* I *satisfies* $(I3)$ *and* $(I4)$.

Proof. Let I be an ideal of X and $a \in I, x \in X$. Then $(a * x) * a = 0 \in I$ and, applying (I2), we conclude $a * x \in I$, i.e., I satisfies $(I3)$. Now, suppose $a, b \in I$ and $x \in X$. Then $(x * ((x * a) * b)) * b = (x * b) * [((x * a) * b) * b] = (x * b) * [((x * a) * b) * (b * b)] = (x * b) * ((x * a) * b) = (x * (x * a)) * b \leq (x * (x * a)) \leq a$ and hence $[(x * ((x * a) * b)) * b] * a = 0 \in I$ and applying $(I2)$ twice, we get $x * ((x * a) * b) \in I$ proving $(I4)$. ∎

Theorem 4.6. If X is a right distributive QI-algebra and I a non-empty subset of X. Then I is an ideal of X if and only if I satisfies $(I3)$ and $(I4)$.

5. Congruence kernels

In this section, we give a characterization of congruence kernels in a right distributive QI-algebra. Let θ be a binary relation on a QI-algebra $(X, *, 0)$. We denote $\{x \in X \mid (x, 0) \in \theta\}$ by $[0]_\theta$. If θ is a congruence relation on X then $[0]_\theta$ is called a congruence kernel.

Lemma 5.1. *Let $(X, *, 0)$ be a QI-algebra and θ a congruence relation on X. Then the kernel $[0]_\theta$ is an ideal of X.*

Proof. Clearly $0 \in [0]_\theta$. Suppose $y \in [0]_\theta$ and $x*y \in [0]_\theta$. Then $(y, 0), (x*y, 0) \in \theta$ and hence $(x*y, x) = (x*y, x*0) \in \theta$. By symmetry of θ, $(x, x*y) \in \theta$. Therefore, by transitivity of θ, we obtain $(x, 0) \in \theta$ proving $x \in [0]_\theta$. ■

Lemma 5.2. *Let $(X, *, 0)$ be a QI-algebra and θ a congruence relation on X. Then the kernel $[0]_\theta$ satisfies $(I3)$ and $(I4)$.*

Proof. Clearly $0 \in [0]_\theta$. Suppose $x \in X$ and $y \in [0]_\theta$. Then $(y, 0) \in \theta$ and hence $(y*x, 0) = (y*x, 0*x) \in \theta$. Therefore $y*x \in [0]_\theta$ proving $(I3)$. Suppose $x \in X$ and $a, b \in [0]_\theta$. Then $(x*((x*a)*b), 0) = (x*((x*a)*b), x*((x*0)*0)) \in \theta$ and hence $x*((x*a)*b) \in [0]_\theta$ proving $(I4)$. ■

Theorem 5.3. *Let $(X, *, 0)$ be a right distributive QI-algebra. Then every ideal I of X is a kernel of a congruence θ_I given by*

$$(x, y) \in \theta_I \textit{ if and only if } x*y \in I \textit{ and } y*x \in I.$$

Moreover, θ_I is the greatest congruence on X having the kernel I.

Proof. Let I be an ideal of X. Since $0 \in I$, we have θ_I is reflexive. Clearly θ_I is symmetric. We prove transitivity of θ_I. Let $(x, y) \in \theta_I$ and $(y, z) \in \theta_I$. Then $x*y, y*x, y*z, z*y \in I$ and, by Theorem 4.6, $(x*y)*z \in I$. Hence $(x*z)*(y*z) \in I$ so that $x*z \in I$. Similarly we can prove that $z*x \in I$. Thus $(x, z) \in \theta_I$. Now, we prove the substitution property of θ_I. Let $(x, y) \in \theta_I$ and $(u, v) \in \theta_I$. Then $x*y, y*x, u*v, v*u \in I$ and hence, by Theorem 4.6, $(x*u)*(y*u) = (x*y)*u \in I$ and $(y*u)*(x*u) = (y*x)*u \in I$. Therefore, $(x*u, y*u) \in \theta_I$. Further, by Proposition 2.10(9), we have $(y*u)*(y*v) \leq v*u$ and $(y*v)*(y*u) \leq u*v$. Since I is an ideal of X, we have $(y*u)*(y*v) \in I$ and $(y*v)*(y*u) \in I$. Hence $(y*u, y*v) \in \theta_I$. By transitivity of θ_I, we conclude $(x*u, y*v) \in \theta_I$. Thus θ_I is a congruence relation on X.

If $x \in I$ then $x*0 = x \in I$ and $0*x = 0 \in I$. Therefore $(x, 0) \in \theta_I$, i.e., $x \in [0]_{\theta_I}$. Conversely, let $x \in [0]_{\theta_I}$. Then $(x, 0) \in \theta_I$ and hence $x = x*0 \in I$ which shows that $I = [0]_{\theta_I}$. Thus I is the kernel of congruence θ_I.

Finally, if ψ is a congruence relation on X such that $[0]_\psi = I$, then for $(x, y) \in \psi$ we have $(x*y, 0) = (x*y, y*y) \in \psi$ and $(y*x, 0) = (y*x, y*y) \in \psi$ thus $x*y \in I$ and $y*x \in I$ which gives $(x, y) \in \theta_I$. Hence $\psi \subseteq \theta_I$ i.e., θ_I is the greatest congruence relation of X having the kernel I. ■

We have observed that, in Example 4.4, for general QI-algebras ideals can not coincide with (I3) and (I4), they can satisfy or not these properties. The following example shows that also ideals need not be congruence kernels.

Example 5.4. In Example 4.4, $I = \{0, a\}$ is an ideal of X. Let $(0, a) \in \theta$ for some congruence relation θ on X. Then $(c, b) \in \theta$ and hence $(0, c) \in \theta$ which shows that $c \in [0]_\theta \neq \{0, a\}$. Hence I is not a congruence kernel.

Finally, we conclude this section with the following theorem.

Theorem 5.5. *Let $(X, *, 0)$ be a right distributive QI-algebra and I a non-empty subset of X. Then the following are equivalent:*

(1) *I is an ideal of X.*
(2) *I satisfies $(I3)$ and $(I4)$.*
(3) *I is a congruence kernel.*

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