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ON L-FUZZY MULTIPLICATION MODULES

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Abstract

Let L be a complete lattice. In a manner analogous to a commutative ring, we introduce and investigate the L-fuzzy multiplication modules over a commutative ring with non-zero identity. The basic properties of the prime L-fuzzy submodules of L-fuzzy multiplication modules are characterized.

Keywords: L-fuzzy multiplication modules, L-fuzzy Noetherian modules, L-fuzzy radical, generalized maximal L-fuzzy submodules.

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1. Introduction

The idea of investigating a mathematical structure via its representation in simpler structure is commonly used and often successful. The representation theory of multiplication modules over a commutative ring has developed greatly in the recent years. Among the most interesting modules are multiplication modules because, for example, they are top module (an R-module M equipped with Zariski topology is called top module, see [13]). Let R be a commutative ring and M an R-module. Then M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M) = \{r \in R : rm \subseteq N\}$. The literature on multiplication ideals and modules is quite extensive, for example, see [1, 2, 4, 6, 7] and [16]. In particular [2, 7], and [16] contain a number of characterizations of multiplication modules.

Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, neural networks and planning. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy module theory is of recent interest. In the last few years a considerable amount of work has been done on fuzzy modules. Zadeh in [17] introduced the notion of a fuzzy subset μ of a non-empty set X as a function from X to $[0, 1]$. Goguen in [8] generalized the notion of fuzzy sets of X to a lattice L. In [14], Rosenfeld considered the fuzzification of algebraic structures. Liu [10] introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on L-fuzzy ideals of a ring and L-fuzzy modules (see [3, 4, 5, 9, 11] and [14]). See also [12] for a comprehensive survey of the literature of these developments. Hence the study of the L-fuzzy multiplication modules theory is worthy of study.

In the present paper, we introduce and study the L-fuzzy multiplication modules over a commutative ring with non-zero identity. There are many basic open questions concerning the L-fuzzy module theory. The most essential one among them is to know whether or not an L-fuzzy $\mathcal{P}$-module is a $\mathcal{P}$-module and vice versa. We give a condition giving an affirmative answer to these questions. Our main purpose is to establish a connection between the L-fuzzy multiplication modules (resp. the L-fuzzy Noetherian modules) and the multiplication modules (resp. the Noetherian modules) over a commutative ring. In Section 3, we introduce the L-fuzzy multiplication modules and make an intensive study of this notion. It is shown that, in Theorem 10, an R-module M is a multiplication module if and only if M is an L-fuzzy multiplication module (so it is top module). Also, we introduce L-fuzzy Noetherian modules and show that every L-fuzzy Noetherian module is a Noetherian module (Theorem 8), but the converse is not true. In Section 4, we introduce the notion of L-fuzzy radical of an L-fuzzy submodule of an L-fuzzy module over a commutative ring. Finally, in Theorem 21, we formulate the L-fuzzy radical of L-fuzzy submodules of an L-fuzzy multiplication module.

2. Preliminaries

Throughout this paper R is a commutative ring with non-zero identity, M is an unitary R-module, and L stands for a complete lattice with least element 0 and greatest element 1. Let 0_M denote the zero element of M. In order to make this paper easier to follow, we recall in this section various notions from fuzzy commutative algebra theory which will be used in the sequel.

An element α, $\alpha \neq 1$, is called a prime element in L if for all $a, b \in L$ such that $a \wedge b \leq \alpha$, then either $a \leq \alpha$ or $b \leq \alpha$. Given a nonempty set X, an L-fuzzy subset μ is a function from X to L. We denote by $F(X)$ the set of all L-fuzzy subsets of X. For $\mu, \nu \in F(X)$ we write $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$. Also, $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$. Let $\mu \in F(X)$ and $t \in L$. Then the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called the level subset of X with respect to μ. By an L-fuzzy point x_r of X, $x \in X$, $r \in L \setminus \{0\}$, we mean $x_r \in F(X)$ is defined by

$$x_r(y) = \begin{cases} r & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

If x_r is an L-fuzzy point of X and $x_r \subseteq \mu \in F(X)$, we write $x_r \in \mu$. We let χ_A denote the characteristic function of a subset A of X. The following are two very basic definition given in [12].

Definition. (a) Let $\gamma \in F(R)$. Then γ is called an *L-fuzzy ideal* of R if for all $x, y \in R$,

(1) $\gamma(x - y) \geq \gamma(x) \wedge \gamma(y)$,

(2) $\gamma(xy) \geq \gamma(x) \vee \gamma(y)$

(b) Let $\mu \in F(M)$. Then μ is called an *L-fuzzy R-module* of M if for all $x, y \in M$ and for all $r \in R$,

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,

(2) $\mu(rx) \geq \mu(x)$,

(3) $\mu(0_M) = 1$.

Let $L(M)$ denote the set of all L-fuzzy R-modules of M and $LI(R)$ denote the set of all L-fuzzy ideals of R.

Theorem 1 [3, Theorem 2.4]**.** *Let $\mu \in F(M)$. Then μ is an L-fuzzy module if and only if for all $t \in L$ such that $\mu_t \neq \emptyset$, then μ_t is an R-submodule of M.*

Definition [3, Definition 2.5]**.** For a non-constant $\gamma \in LI(R)$, γ is called an L-fuzzy prime ideal of R if for any L-fuzzy points $x_r, y_s \in F(R)$, $x_r y_s \in \gamma$ implies that either $x_r \in \gamma$ or $y_s \in \gamma$.

Definition [3, Definition 3.1]**.** For $\mu, \nu \in L(M)$, ν is called an *L-fuzzy submodule* of μ if and only if $\nu \subset \mu$. In particular, if $\mu = \chi_M$, then we say ν is an L-fuzzy submodule of M.

Definition [3, Definition 3.2]**.** Let ν be an L-fuzzy submodule of μ. Then ν is called an *L-fuzzy prime submodule* of μ if for $r_t \in F(R)$, $x_s \in F(M)$ ($r \in R$, $x \in M$ and $t, s \in L$), $r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $r_t\mu \subseteq \nu$.

In particular, taking $\mu = \chi_M$, if $r_t \in F(R)$, $x_s \in F(M)$ we have $r_t x_s \in \nu$ implies that either $x_s \in \nu$ or $r_t\chi_M \subseteq \nu$, then ν is called an L-fuzzy prime submodule of M.

Theorem 2 [3, Theorem 3.4]**.** *Let ν be an L-fuzzy prime submodule of μ. If $v_t \neq \mu_t$ for some $t \in L$, then ν_t is a prime submodule of μ_t.*

Lemma 3 [3, Corollary 3.5]**.** *Let ν be an L-fuzzy prime submodule of M. Then $\nu_* = \{x \in M : \nu(x) = \nu(0_M)\}$ is a prime submodule of M.*

Definition [12, Definition 4.1.6]**.** Let $\zeta \in LI(R)$ and $\mu \in L(M)$. Define:

$$(\zeta\mu)(x) = \vee\{\zeta(r) \wedge \mu(y) : r \in R, y \in M \text{ and } ry = x\}.$$

If $\zeta(0_R) = 1$, then $\zeta\mu \in L(M)$ by [12, Theorem 4.1.16]. It is clear that, if $\zeta' \subseteq \zeta$ and $\mu \subseteq \nu$ for some $\zeta, \zeta' \in LI(R)$ and $\mu, \nu \in L(M)$, then $\zeta'\mu \subseteq \zeta\nu$.

Definition [12, Definition 4.5.1]**.** For $\mu, \nu \in L(M)$ and $\zeta \in LI(R)$, define the residual quotients as

$$(\mu : \nu) = \bigcup\{\eta : \eta \in LI(R), \eta\nu \subseteq \mu\}.$$

By [12, Theorem 4.5.6], $(\mu : \nu)$ is an L-fuzzy ideal of R and it is easy to see that $(\mu : \nu)(0_R) = 1$.

Theorem 4 [12, Theorem 4.5.3]**.** *Let $\mu, \nu \in F(M)$ and $\zeta \in F(R)$. Then*

(1) $(\mu : \nu)\nu \subseteq \mu$.

(2) $\zeta\nu \subseteq \mu$ *if and only if* $\zeta \subseteq (\mu : \nu)$.

Definition [12, Theorem 4.3.1]**.** Let $\mu \in L(M)$. The L-fuzzy submodule generated by the L-fuzzy subset μ is denoted by $< \mu >$ and defined by $< \mu >= \bigcap\{\nu : \mu \subseteq \nu, \nu \in L(M)\}$.

3. L-fuzzy multiplication modules

In this section we list some basic properties concerning L-fuzzy multiplication modules over a commutative ring. We begin with the key definition of this paper.

Definition. Let M be a module over a commutative ring R. M is called an *L-fuzzy multiplication module* provided for each L-fuzzy submodule μ of M, there exists $\zeta \in LI(R)$ with $\zeta(0_R) = 1$ such that $\mu = \zeta\chi_M$. One can easily show that if $\mu = \zeta\chi_M$ for some $\zeta \in LI(R)$ with $\zeta(0_R) = 1$, then $\mu = (\mu : \chi_M)\chi_M$.

In [7] it was proved that an R-module M is a multiplication module if and only if for each m in M there exists an ideal I of R such that $Rm = IM$. Now, we have the following proposition for L-fuzzy multiplication R-modules.

Proposition 5. *An R-module M is an L-fuzzy multiplication module if and only if for each $x \in M$ and $a \in L$, there exists an L-fuzzy ideal $\zeta \in LI(R)$ with $\zeta(0_R) = 1$ such that $< x_a >= \zeta\chi_M$.*

Proof. The necessity is clear. Conversely, suppose that for each $x \in M$ and $a \in L$, there exists $\zeta \in LI(R)$ with $\zeta(0_R) = 1$ such that $< x_a >= \zeta\chi_M$. Let $\mu \in L(M)$ and $x \in M$. There exists $a \in L$ such that $\mu(x) = a$, so $x_a \in \mu$; hence $< x_a >\subseteq \mu$ by definition. By assumption, there exists $\zeta_x \in LI(R)$ with $\zeta(0_R) = 1$ such that $< x_a >= \zeta_x\chi_M$. Then $\zeta_x\chi_M \subseteq \mu$; thus $\zeta_x \subseteq (\mu : \chi_M)$ by Theorem 4. It follows that $\zeta_x(r) \leq (\mu : \chi_M)(r)$ for each $r \in R$. Set $\zeta = \bigcup\{\zeta_x : x \in M\}$. So $\zeta(r) = \vee\{\zeta_x(r) : x \in M\}$ for each $r \in R$ and $\zeta(0_R) = 1$; hence $\zeta(r) \leq (\mu : \chi_M)(r)$ for every $r \in R$. Thus $\zeta\chi_M \subseteq \mu$ by Theorem 4. For the other containment, assume that $m \in M$. There exists $a \in L$ such that $\mu(m) = a$, and hence $m_a \in \mu$. It follows that $< m_a >= \zeta_{m_a}\chi_M \subseteq \zeta\chi_M$. On the other hand, $< m_a > (m) = a$ by [12, Theorem 4.3.4]. Therefore, $\mu(m) =< m_a > (m) \leq \zeta\chi_M(m)$. Thus $\mu \subseteq \zeta\chi_M$, and so we have the equality. ∎

Lemma 6. *Let M be an R-module and $\mu \in L(M)$. Then $(\mu : \chi_M) = \bigcup\{r_a : a \in L, r \in R \cap (\mu_a : M)\}$.*

Proof. By [12, Theorem 4.5.2], we have

$$(\mu : \chi_M) = \bigcup\{r_a : a \in L, r \in R, r_a\chi_M \subseteq \mu\}.$$

On the other hand, for each $y \in M$, by definition we have

$$r_a\chi_M(y) = \begin{cases} a & \text{if } y = rm \text{ for some } m \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $(\mu : \chi_M) = \bigcup\{r_a : a \in L, r \in R, a \leq \mu(rm) \text{ for each } m \in M\} =$

$$\bigcup\{r_a : a \in L, r \in R, \text{ and } rm \in \mu_a \text{ for each } m \in M\}$$

$$= \bigcup\{r_a : a \in L, r \in R, rM \subseteq \mu_a\} = \bigcup\{r_a : a \in L, r \in R, r \in (\mu_a : \chi_M)\}. \quad ∎$$

Let us now define a basic concept and new properties of them over commutative rings.

Definition. An R-module M is called an *L-fuzzy Noetherian module*, if every ascending chain of L-fuzzy submodules in M is stationary.

Theorem 7. *Let R be an L-fuzzy Noetherian ring and M be an L-fuzzy multiplication module. Then M is an L-fuzzy Noetherian module.*

Proof. Let $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq \cdots$ be an ascending chain of L-fuzzy submodules of M. Then $(\mu_1 : \chi_M) \subseteq (\mu_2 : \chi_M) \subseteq (\mu_3 : \chi_M) \subseteq \cdots$ is an ascending chain of L-fuzzy ideals of R. By assumption, there is a positive integer t such that $(\mu_t : \chi_M) = (\mu_{t+s} : \chi_M)$ for every positive integer s; hence $\mu_t = (\mu_t : \chi_M)\chi_M = (\mu_{t+s} : \chi_M)\chi_M = \mu_{t+s}$ for every s, and so the chain is stationary. ■

In the next theorem, we show that the notion of L-fuzzy Noetherian R-module is a generalization of the notion of Noetherian R-module.

Theorem 8. *If M is an L-fuzzy Noetherian module, then M is a Noetherian R-module.*

Proof. Assume that $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ be an ascending chain of submodules of M. For each positive integer i, we define the mapping $\mu_i : M \to L$ by

$$\mu_i(x) = \begin{cases} 1 & \text{if } x \in N_i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\mu_i = \chi_{N_i}$ and $\mu_i \in L(M)$. Then $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq \cdots$ is an ascending chain of L-fuzzy submodules of M; so there exists a positive integer n such that $\mu_n = \mu_{n+k}$ for every positive integer k. Now we show that $N_n = N_{n+k}$ for all k. Let k be a positive integer and $x \in N_{n+k}$. So $\mu_n(x) = \mu_{n+k}(x) = 1$. Hence $x \in N_n$. Thus $N_n \subseteq N_{n+k} \subseteq N_n$, and so we have the equality. Therefore M is a Noetherian R-module. ■

Example 9. Let $M = R$ denote the field of real numbers with the usual addition and multiplication, and let $L = [0, 1]$. So M is a Noetherian R-module. We define the mappings $\mu_n : M \to L$ by

$$\mu_n(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1 - 1/n & \text{otherwise.} \end{cases}$$

for all positive integer n. An inspection will show that μ_n is an L-fuzzy submodule of M for each n. Then $\mu_1 \subset \mu_2 \subset \cdots$ is an infinite strictly ascending chain of L-fuzzy submodules of M, and M is not an L-fuzzy Noetherian R-module.

The Example 9 shows that the converse of Theorem 8 is not true. So a Noetherian module need not be an L-fuzzy Noetherian module, but we have the following theorem for multiplication modules.

Theorem 10. *Let M be an R-module. Then M is a multiplication module if and only M is an L-fuzzy multiplication module.*

Proof. Let M be a multiplication module, and let $\mu \in L(M)$. Since the inclusion $(\mu : \chi_M)\chi_M \subseteq \mu$ is clear, we will prove the reverse inclusion. Let $x \in M$ and $\mu(x) = a$ for some $a \in L$. It suffices to show that $a \leq (\mu : \chi_M)\chi_M(x)$. By assumption, $\mu_a = (\mu_a : \chi_M)\chi_M$, since $\mu(x) = a$, we must have $x \in \mu_a$. Then $x = \sum_{i=1}^n r_i x_i$, where $x_i \in M$ and $r_i \in (\mu_a : M)$ $(i = 1, 2, \ldots, n)$. It then follows from Lemma 6 that $(r_i)_a \in (\mu : \chi_M)$ for each i. On the other hand, for each i, $(x_i)_a \in \chi_M$. Hence $(r_i x_i)_a = (r_i)_a (x_i)_a \leq (\mu : \chi_M)\chi_M$; thus $(r_i x_i)_a \in (\mu : \chi_M)\chi_M$ $(1 \leq i \leq n)$. It follows that $((\mu : \chi_M)\chi_M)(r_i x_i) \geq a$ for each i $(1 \leq i \leq n)$. Then

$$((\mu : \chi_M)\chi_M)(x) \geq ((\mu : \chi_M)\chi_M)(r_1 x_1) \wedge \cdots \wedge ((\mu : \chi_M)\chi_M)(r_n x_n) \geq a.$$

Therefore, $\mu(x) = a \leq ((\mu : \chi_M)\chi_M)(x)$, so $\mu(m) \leq ((\mu : \chi_M)\chi_M)(m)$ for each $m \in M$. Hence $\mu \subseteq (\mu : \chi_M)\chi_M$, and we have the equality.

Conversely, assume that M is an L-fuzzy multiplication module and let N be a proper submodule of M. It suffices to show that $N \subseteq (N : M)M$. We define the mapping $\mu : M \to L$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N, \\ 0 & \text{otherwise.} \end{cases}$$

One can easily to see that $\mu = \chi_N$, $\mu \in L(M)$ and $\mu_a = N$ for each $0 \neq a \in L$. So $\mu = (\mu : \chi_M)\chi_M$ since M is an L-fuzzy multiplication module. Let $m \in N$. Then $\mu(m) = ((\mu : \chi_M)\chi_M)(m) = 1$. So by definition,

$$\begin{aligned} ((\mu : \chi_M)\chi_M)(m) &= \vee \{(\mu : \chi_M)(s) \wedge \chi_M(x) : sx = m \text{ for some } s \in R, x \in M\} \\ &= \vee \{(\mu : \chi_M)(s) : m \in sM \text{ for some } s \in R\}. \end{aligned}$$

On the other hand, $(\mu : \chi_M)(s) = \vee\{r_a(s) : r \in (N : M)\}$ by [12, Theorem 4.5.2] and the fact that $\mu_a = N$ for each non-zero element a of L. If there is no $1 \neq r \in R$ such that $m \in rM$, then

$$(\mu : \chi_M)(1) = \vee\{r_a(1) : r \in (N : M)\} = 0$$

since $N \neq M$ and $1 \notin (N : M)$. Therefore,

$$((\mu : \chi_M)\chi_M)(m) = \vee\{(\mu : \chi_M)(s) : m \in sM \text{ for some } s \in R\} = (\mu : \chi_M)(1) = 0,$$

which is a contradiction. So there exists $r' \in R$ such that $m \in r'M$. Set $A = \{r \in (N : M) : m \in rM\}$. If $A = \emptyset$, then for each $t \in R$ with $m \in tM$, we have $t \notin (N : M)$. So $(\mu : \chi_M)(t) = \vee\{r_a(t) : r \in (N : M)\} = 0$. Therefore,

$$((\mu : \chi_M)\chi_M)(m) = \vee\{(\mu : \chi_M)(s) : m \in sM \text{ for some } s \in R\} = 0,$$

a contradiction. So we may assume that $A \neq \emptyset$. Then there exists $r \in R$ such that $m \in rM$ and $r \in (N : M)$; so $N \subseteq (N : M)M$. Thus M is a multiplication R-module. ■

Let M be an R-module. A submodule N of M is called prime if $N \neq M$ and whenever $r \in R$ and $m \in M$ satisfy $rm \in N$ then $rM \subseteq N$ or $m \in N$. Let $\mathrm{Spec}(M)$ denote the collection of prime submodule of M. We define $V(N)$ to be the set of all prime submodules of M containing N (so $V(M) = \emptyset$ and $V(0) = \mathrm{Spec}(M)$). If $\zeta(M)$ denotes the collection of all subset $V(N)$ of $\mathrm{Spec}(M)$, then $\zeta(M)$ contains the empty set and $\mathrm{Spec}(M)$, and $\zeta(M)$ is closed under arbitrary intersections. We shall say that M is a module with a Zariski topology, or a top module for short, if $\zeta(M)$ is closed under finite unions, i.e. for any submodules N and L of M there exists a submodule J of M such that $V(N) \cup V(L) = V(J)$, for in this case $\zeta(M)$ satisfies the axioms for the closed subsets of a topological space [13]. We recall that by [13, Theorem 3.5] every multiplication module is a top module. Now by [4, Theorem 4.5] and Theorem 10 we obtain the following theorem.

Theorem 11. *Every L-fuzzy multiplication R-module is L-top module.*

4. Radical of an L-fuzzy submodule

In [12], the notion of L-fuzzy radical of an L-fuzzy ideal and its properties are given. We generalized their definition to any L-fuzzy submodule of an L-fuzzy module over a commutative ring.

Definition. Let M be an R-module and $\mu \in L(M)$. Let P_μ be the family of all L-fuzzy prime submodules of M containing μ. The L-radical of μ, denoted by $\mathrm{rad}_M(\mu)$, is defined by

$$(rad_M(\mu))(x) = \begin{cases} \bigwedge_{\nu \in P_\mu} \nu(x) & \text{if } P_\mu \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

First, we have the following lemma.

Lemma 12. *Let M be an R-module and $\mu \in L(M)$. If μ is an L-fuzzy submodule of M, then* $\mathrm{rad}_M(\mu_*) \subseteq (\mathrm{rad}_M(\mu))_*$.

Proof. If $P_\mu = \emptyset$, then $\mathrm{rad}_M(\mu) = \chi_M$, and hence $(\mathrm{rad}_M(\mu))_* = M$. So we may assume that $P_\mu \neq \emptyset$. Let $\nu \in P_\mu$. Then ν_* is a prime submodule of M by Lemma 3 and $\mu_* \subseteq \nu_*$; hence $\mathrm{rad}_M(\mu_*) \subseteq \bigcap \{\nu_* : \nu \in P_\mu\}$. Let $m \in \bigcap \{\nu_* : \nu \in P_\mu\}$. So $m \in \nu_*$; thus $\nu(m) = 1$ for every $\nu \in P_\mu$; hence $(\mathrm{rad}_M(\mu))(m) = 1$ by definition. It follows that $m \in (\mathrm{rad}_M(\mu))_*$, so $\mathrm{rad}_M(\mu_*) \subseteq \bigcap \{\nu_* : \nu \in P_\mu\} \subseteq (\mathrm{rad}_M(\mu))_*$. ■

In view of Lemma 12 and [3, Theorem 3.6], we have the following theorem.

Theorem 13. *Let μ be a nonconstant L-fuzzy submodule of M, and let $\bigwedge\{\mu(x) : x \notin \mathrm{rad}_M(\mu_*)\} = a$. If $a < 1$ and a is a prime element of L, then* $\mathrm{rad}_M(\mu_*) = (\mathrm{rad}_M(\mu))_*$.

Proof. By Lemma 12, it suffices to show that $(\mathrm{rad}_M(\mu))_* \subseteq \mathrm{rad}_M(\mu_*)$. Let P be a prime submodule of M containing μ_*. Then the mapping $\nu : M \to L$ is defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in P, \\ a & \text{otherwise.} \end{cases}$$

is an L-fuzzy prime submodule of M by [3, Theorem 3.6]. Now we show that $\mu \subseteq \nu$. If $x \in P$, then $\mu(x) \leq \nu(x) = 1$. If $x \notin P$, then $\nu(x) = a$, so $x \notin \mathrm{rad}_M(\mu_*)$. Since by hypothesis $\mu(x) \leq a$, we must have $\mu(x) \leq a = \nu(x)$. Hence $\mu \subseteq \nu$. It follows that ν is an L-fuzzy prime submodule of M containing μ and $\nu_* = P$. Now, let $x \in (\mathrm{rad}_M(\mu))_*$. Then $(\mathrm{rad}_M(\mu))(x) = 1$, so $\nu(x) = 1$ and $x \in \nu_* = P$; hence $x \in \mathrm{rad}_M(\mu_*)$, and so we have the equality. ∎

Definition. Let M be an R-module and $\mu \in L(M)$. Then μ is called *a generalized maximal L-fuzzy submodule* of M, if for any L-fuzzy submodule ν of M, if $\mu \subseteq \nu$, then either $\mu_* = \nu_*$ or $\nu = \chi_M$.

In [7, Theorem 2.5], it was proved that every proper submodule of a non-zero multiplication R-module is contained in a maximal submodule of M. Now we have the following theorem for L-fuzzy multiplication R-modules.

Theorem 14. *Let M be a non-zero L-fuzzy multiplication R-module. Then every L-fuzzy submodule $\mu \neq \chi_M$ of M is contained in a generalized maximal L-fuzzy submodule of M.*

Proof. Let μ be a non-constant L-fuzzy submodule of M. So $\mu_* \neq M$ and there exists a maximal submodule N of M such that $\mu_* \subseteq N$ by [7, Theorem 2.5] and Theorem 10. Let $a = \vee\{\mu(x) : x \in M\}$. We define the mapping $\nu : M \to L$ by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in N, \\ a & \text{otherwise.} \end{cases}$$

One can easily to see that ν is an L-fuzzy submodule of M and $\mu \subseteq \nu$. Now we show that ν is a generalized maximal L-fuzzy submodule of M. Let $\nu \subseteq \beta$ and $\beta \in L(M)$. Therefore $N = \nu_* \subseteq \beta_*$. So either $N = \beta_*$ or $\beta_* = M$ since N is a maximal submodule of M. Hence $\nu_* = N = \beta_*$ or $\beta = \chi_M$. ∎

Definition. Let M be an R-module. We define the *L-fuzzy radical of M*, denoted by $\mathrm{Jac}(\chi_M)$, to be the intersection of all the generalized maximal L-fuzzy submodules of M if such exist, and χ_M otherwise.

Proposition 15. *Let μ be a non-constant L-fuzzy submodule of an L-fuzzy multiplication module M such that $\chi_M = \mu + \mathrm{Jac}(\chi_M)$. Then $\chi_M = \mu$.*

Proof. If $\chi_M \neq \mu$, then μ is contained in a generalized maximal L-fuzzy submodule ν of M. So $\chi_M = \mu + \mathrm{Jac}(\chi_M) \subseteq \nu$, which is a contradiction. ∎

Here we have the following proposition that is a generalization of [7, Lemma 2.10] for L-fuzzy multiplication R-modules.

Proposition 16. *Assume that M is a faithful L-fuzzy multiplication R-module and let ζ be a prime L-fuzzy ideal of R, $a, b \in L$ and $r_a x_b \in \zeta\chi_M$ for some $r \in R$ and $x \in M$. Then $r_a \in \zeta$ or $x_b \in \zeta\chi_M$.*

Proof. Let ζ be a prime L-fuzzy ideal of R. Then by [12, Theorem 3.5.5], for each $r \in R$, there exist a prime ideal P of R and a prime element $c \in L$ such that

$$\zeta(r) = \begin{cases} 1 & \text{if } r \in P, \\ c & \text{otherwise.} \end{cases}$$

By assumption, $\zeta\chi_M(rx) \geq a \wedge b$. On the other hand, by definition, we have

$$\begin{aligned} \zeta\chi_M(rx) &= \vee\{\zeta(s) \wedge \chi_M(y) : s \in R, y \in M, rx = sy\} \\ &= \vee\{\zeta(s) : s \in R, rx \in sM\}. \end{aligned}$$

Set $K = \{s \in P : rx \in sM\}$. First suppose that $K = \emptyset$. Then there is no $s \in P$ such that $rx \in sM$. Hence $\zeta\chi_M(rx) = c \geq a \wedge b$. It follows that either $c \geq a$ or $c \geq b$ since c is a prime element of L. We split the proof into two cases:

Case 1. $c \geq a$. Since $\zeta(r) \in \{1, c\}$, we must have $\zeta(r) \geq a$, and so $r_a \in \zeta$.

Case 2. $c \geq b$. Similarly, $\zeta\chi_M(x) = \vee\{\zeta(s') : s' \in R, x \in s'M\}$. So $\zeta\chi_M(x) \in \{1, c\}$. Therefore, $\zeta\chi_M(x) \geq b$, and so $x_b \in \zeta\chi_M$. So we may assume that $K \neq \emptyset$. Then there exists $s' \in P$ such that $rx \in s'M$. Therefore we have $\zeta\chi_M(rx) = \vee\{\zeta(s) : s \in R,\, rx \in sM\} = 1$ and $rx \in s'M \subseteq PM$. It then follows from [7, Lemma 2.10] and Theorem 10 that either $r \in P$ or $x \in PM$. If $r \in P$, then $\zeta(r) = 1 \geq a$, and so $r_a \in \zeta$. If $x \in PM$, then $x = r_1x_1 + \cdots + r_nx_n$ for some $r_i \in P$ and $x_i \in M$ such that $i = 1, 2, \ldots, n$; hence

$$\zeta\chi_M(x) = \zeta\chi_M\left(\sum_{i=1}^{n} r_i x_i\right) \geq \zeta\chi_M(r_1x_1) \wedge \cdots \wedge \zeta\chi_M(r_nx_n) \geq 1 \wedge \cdots \wedge 1 = 1.$$

Thus $\zeta\chi_M(x) = 1 \geq b$, so $x_b \in \zeta\chi_M$, and the proof is complete. ∎

In view of Proposition 16 and [4, Theorem 3.6] we have the following theorem.

Theorem 17. *The following statements are equivalent for a non-constant L-fuzzy submodule μ of an L-fuzzy multiplication module M.*

(1) *μ is an L-fuzzy prime submodule of M.*

(2) *$\mu = \zeta\chi_M$ for some L-fuzzy prime ideal ζ of R.*

Proposition 18. *Assume that M is a faithful L-fuzzy multiplication R-module and let ζ be a prime L-fuzzy ideal of R. If η is an L-fuzzy ideal of R such that $\eta\chi_M \subseteq \zeta\chi_M$ and $\zeta\chi_M \neq \chi_M$, then $\eta \subseteq \zeta$. In particular, $(\zeta\chi_M : \chi_M) = \zeta$.*

Proof. Let $r \in R$ and $\eta(r) = a$ for some $a \in L$. Then $r_a \in \eta$ and there exists $m \in M$ such that $\zeta\chi_M(m) < 1$, since $\zeta\chi_M \neq \chi_M$. So $m_{\{1\}} \notin \zeta\chi_M$; hence $r_a m_{\{1\}} \in \eta\chi_M \subseteq \zeta\chi_M$. Therefore $r_a \in \zeta$ by Proposition 16. Hence $\zeta(r) \geq a = \eta(r)$, and so $\eta \subseteq \zeta$. The particular statement is clear. ■

Proposition 19. *Let M be an R-module. If μ is an L-fuzzy submodule of M, then $\sqrt{(\mu : \chi_M)}\chi_M \subseteq \mathrm{rad}_M(\mu)$.*

Proof. If $\mathrm{rad}_M(\mu) = \chi_M$, the result is clear. Otherwise, if $\nu \in P_\mu$, then we have $(\mu : \chi_M) \subseteq (\nu : \chi_M)$. So $\sqrt{(\mu : \chi_M)} \subseteq (\nu : \chi_M)$ by [4, Theorem 3.6]; hence $\sqrt{(\mu : \chi_M)}\chi_M \subseteq (\nu : \chi_M)\chi_M \subseteq \nu$. Since ν is an arbitrary prime L-fuzzy submodule of M containing μ, we have $\sqrt{(\mu : \chi_M)}\chi_M \subseteq \mathrm{rad}_M(\mu)$. ■

Lemma 20. *Let μ, ν be L-fuzzy submodules of an R-module M. If η is an L-fuzzy ideal of R, then the following hold:*

(1) $(\mu_* : M) \subseteq (\mu : \chi_M)_*$.

(2) $\eta_*\mu_* \subseteq (\eta\mu)_*$.

Proof. (1) Let $r \in (\mu_* : M)$. Then $rM \subseteq \mu_* \subseteq \mu_a$ for each $a \in L$. Therefore we have

$$(\mu : \chi_M)(r) = \vee\{s_a(r) : s \in R, a \in L, s \in (\mu_a : M)\} \geq \vee\{r_a(r) = a : a \in L\} = 1.$$

Hence $r \in (\mu : \chi_M)_*$, and the proof is complete.

(2) Let $m \in \eta_*\mu_*$. Then $m = \sum_{i=1}^n r_i m_i$ for some $r_i \in \eta_*$ and $m_i \in \mu_*$ such that $i = 1, 2, \ldots, n$. Therefore,

$$\eta\mu(m) \geq \eta\mu(r_1 m_1) \wedge \cdots \wedge \eta\mu(r_n m_n) = 1 \wedge \cdots \wedge 1 = 1.$$

Thus $m \in (\eta\mu)_*$, as required. ■

Now, we have the following theorem that is a generalization of [7, Theorem 2.12].

Theorem 21. *Let M be a faithful L-fuzzy multiplication R-module. If μ is an L-fuzzy submodule of M, then $\mathrm{rad}_M(\mu) = \sqrt{(\mu : \chi_M)}\chi_M$.*

Proof. By Proposition 19, it suffices to show that $\mathrm{rad}_M(\mu) \subseteq \sqrt{(\mu : \chi_M)}\chi_M$. Since M is an L-fuzzy multiplication module, we must have

$$\mathrm{rad}_M(\mu) = (\mathrm{rad}_M(\mu) : \chi_M)\chi_M.$$

Now it is enough to show that $(\mathrm{rad}_M(\mu) : \chi_M) \subseteq \sqrt{(\mu : \chi_M)}$. Let η be a prime L-fuzzy ideal of R containing $(\mu : \chi_M)$. Then η_* is a prime ideal of R by [12, Theorem 3.5.5], and we have

$$(0 : M) \subseteq (\mu_* : M) \subseteq (\mu : \chi_M)_* \subseteq \eta_*$$

by Proposition 20. Hence $\eta_* M$ is a prime submodule of M by [7, Corollary 2.11]. Therefore, $\eta_* M \neq M$, and Proposition 20 gives $\eta\chi_M \neq \chi_M$. Then by Theorem 17, $\eta\chi_M$ is a prime L-fuzzy submodule of M; so $\mu = (\mu : \chi_M)\chi_M \subseteq \eta\chi_M$ and $\mathrm{rad}_M(\mu) \subseteq \eta\chi_M$. Hence $(\mathrm{rad}_M(\mu) : \chi_M)\chi_M \subseteq \eta\chi_M$. It then follows from Proposition 18 that $(\mathrm{rad}_M(\mu) : \chi_M) \subseteq \eta$, and so we have equality. ∎

5. Conclusion

Letting $L - \zeta(M) = \{V(\eta\chi_M) \mid \eta \in LI(R)\}$. R. Ameri and R. Mahjoob showed that $L - \zeta(M)$ induces a topology which is called Zariski topology if and only if M is an L-top module. By following them we define L-fuzzy multiplication R-modules and we show that every L-fuzzy multiplication R-module is an L-top module. Also we find a connection between the L-fuzzy multiplication R-modules and the multiplication R-modules.

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