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TRACE INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES

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Abstract

Certain trace inequalities for positive definite matrices are generalized for positive semidefinite matrices using the notion of the group generalized inverse.

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1. Introduction and preliminaries

We study certain trace inequalities for positive semidefinite matrices. We show how some recently obtained results on trace inequalities for positive definite matrices can be extended to positive semidefinite matrices. We provide a framework where our results apply.

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex numbers. A Hermitian matrix $A \in \mathbb{C}^{n\times n}$ is positive semidefinite if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ and positive definite if $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$. The trace of $A \in \mathbb{C}^{n\times n}$ is the sum of its main diagonal entries, or, equivalently, the sum of its eigenvalues.

Obtaining solutions of many algebraic Riccati equations and Lyapunov equations is very difficult in general and impossible in some particular instances. In such cases, one looks for bounds for the trace of products of positive definite or positive semidefinite matrices. For more details, we refer to [6] (and the references cited therein). These inequalities are also important in problems in communication systems with multiple input and multiple output (see [8]).

Let us start with the following result. If A and B are positive definite matrices then

$$Tr\{(A-B)(B^{-1}-A^{-1})\} \geq 0.$$

This was generalized to the result, viz., if A, B are positive definite and C, D are positive semidefinite then

$$Tr[(A-B)(B^{-1}-A^{-1}) + (C-D)\{(B+D)^{-1} - (A+C)^{-1}\}] \geq 0.$$

This was further improved to the following inequality

$$Tr[(A-B)(B^{-1}-A^{-1}) + 4(C-D)\{(B+D)^{-1} - (A+C)^{-1}\}] \geq 0.$$

Another interesting result for a positive definite block matrix is that such a matrix could be written as a product PP^* and Q^*Q, where P and Q are lower triangular block matrices whose subblocks are expressed in terms of the Schur complement and the complementary Schur complement of the matrix that one started with.

We generalize the results mentioned above in Theorem 7 and Theorem 13. In order to state the next result, we need some terminology.

For $A \in \mathbb{C}^{n\times n}$, let $R(A)$, $N(A)$, $Tr(A)$, and $rk(A)$ denote the range space of A, the null space of A, the trace of A and the rank of A. The Moore-Penrose (generalized) inverse of $A \in \mathbb{C}^{m\times n}$ is the unique $X \in \mathbb{C}^{n\times m}$ satisfying $A = AXA, X = XAX, (AX)^* = AX$ and $(XA)^* = XA$ and is denoted by $A^\dagger$. The group (generalized) inverse of $A \in \mathbb{C}^{n\times n}$, if it exists, is the unique $X \in \mathbb{C}^{n\times n}$ satisfying $A = AXA, X = XAX$ and $AX = XA$ and is denoted by $A^\#$. If A is nonsingular, then $A^{-1} = A^\dagger = A^\#$. A necessary and sufficient condition for a matrix to have a group inverse is $rk(A^2) = rk(A)$. Another characterization is that the subspaces $R(A)$ and $N(A)$ are complementary. Let us recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called range-Hermitian if $R(A^*) = R(A)$. It is known that $A^\dagger = A^\#$ if and only if A is range-Hermitian. In particular, if A is positive semidefinite then it is Hermitian and so it follows that $A^\#$ exists. It may be verified that (if $A^\#$ exists, then) one has $R(A) = R(A^\#)$. Let us also include a result that will be used quite frequently in our proofs. If $A, B \in \mathbb{C}^{n\times n}$ are such that $A^\#$ exists and $R(B) \subseteq R(A)$, then $AA^\#B = B$. One also has the following: If $R(B) = R(A)$, then $AA^\# = BB^\#$. For proofs and other details, we refer to [4].

Let $U = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix} \in \mathbb{C}^{n\times n}$ be positive semidefinite. Since A_U and C_U are Hermitian, $A_U^\#$ and $C_U^\#$ exist. The *pseudo Schur complement* of A_U, denoted by F_U, is $F_U = C_U - B_U^* A_U^\# B_U$ and the *complementary pseudo Schur complement* denoted by G_U is $G_U = A_U - B_U C_U^\# B_U^*$. Extensions of the formulae for the usual Schur complements have been discussed in detail for the case of the Moore-Penrose inverse and several properties have been obtained [12, section 1.6].

In Theorem 14, we prove that a positive semidefinite matrix can be written as a product of a block upper triangular matrix and its conjugate transpose in two different ways. As an application of this result, we derive a trace inequality for positive semidefinite block matrices in Theorem 15.

The following three results give trace inequalities for products of matrices.

Theorem 1 [11, Theorem 3.1]. *Let $A, B \in \mathbb{C}^{n\times n}$. Then for any positive integer m,*

$$|Tr(AB)^{2m}| \leq Tr(A^*ABB^*)^m \leq Tr\{(A^*A)^m(BB^*)^m\}.$$

Theorem 2 [1, Exercise 12.14]. *Let $A, B \in \mathbb{C}^{n\times n}$ be positive semidefinite. Then*

$$Tr(AB) \geq 0.$$

Theorem 3 [5, Lemma 3.3]. *For Hermitian matrices X_1, X_2 and positive semidefinite S_1, S_2, set $\alpha = Tr(X_1S_1X_1S_2)$, $\beta = Tr(X_2S_1X_2S_2)$ and $\gamma = Tr(X_1S_1X_2S_2)$. Then, for positive real numbers a and b one has*

$$a\alpha + b\beta \geq 2\sqrt{ab}|\gamma|.$$

The next result concerns the reverse monotonicity of the group inverse. Let us note that the original result is proved for the Moore-Penrose inverse. However, since the matrix under consideration is Hermitian, the Moore-Penrose inverse coincides with the group inverse.

Theorem 4 [10, Theorem 1]. *Let $A, B \in \mathbb{C}^{n\times n}$ be positive semidefinite. Then any two of the following conditions imply the third:*

(i) *$A - B$ is positive semidefinite*

(ii) *$rk(A) = rk(B)$*

(iii) *$B^{\#} - A^{\#}$ is positive semidefinite.*

2. Main results

In this section, we present the main results. The first result gives certain relationships between the range spaces of the subblocks of a positive semidefinite block matrix.

Theorem 5 [13, Exercise 34, p. 226]. *Let $U = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix}$ be positive semidefinite. Then*

(i) $R(B_U) \subseteq R(A_U) = R(A_U^{1/2})$.

(ii) $R(B_U^*) \subseteq R(C_U) = R(C_U^{1/2})$.

If A is a positive definite matrix, then A^{-1} (exists and) is positive definite. We have an analogue for the group inverse, too.

Lemma 6. *$A \in \mathbb{C}^{n\times n}$ be positive semidefinite. Then $A^{\#}$ is positive semidefinite.*

Proof. Note that since A is Hermitian, $A^{\#}$ exists. Also, $(A^{\#})^* = (A^*)^{\#} = A^{\#}$. Now for any $x \in \mathbb{C}^n$, one has $x^*A^{\#}x = x^*A^{\#}AA^{\#}x = (A^{\#}x)^*A(A^{\#}x) \geq 0$. Thus $A^{\#}$ is positive semidefinite. ■

If A and B are positive definite matrices, then $Tr\{(A-B)(B^{-1}-A^{-1})\} \geq 0$ [1, Exercise 12.28 (c)]. We generalize this result to the case of positive semidefinite matrices.

Theorem 7. *Let $A, B \in \mathbb{C}^{n\times n}$ be positive semidefinite with $R(A) = R(B)$. Then*

$$Tr\{(A-B)(B^{\#}-A^{\#})\} \geq 0.$$

Proof. Since $R(A) = R(B)$, so there exists a unitary matrix U such that

$$A = U\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}U^* \text{ and } B = U\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}U^*,$$

where E and F are positive definite matrices of the same size. Now

$$A-B = U\begin{pmatrix} E-F & 0 \\ 0 & 0 \end{pmatrix}U^* \text{ and } B^{\#}-A^{\#} = U\begin{pmatrix} F^{-1}-E^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

Thus $Tr\{(A-B)(B^{\#}-A^{\#})\} = Tr\{(E-F)(F^{-1}-E^{-1})\} \geq 0$. ■

In the preceding result, the condition $R(A) = R(B)$ is indispensable.

Example 8. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then A, B are positive semidefinite and $R(A) \neq R(B)$. Now $(A-B)(B^{\#}-A^{\#}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has negative trace.

To motivate the next result, let us recall Lemma 2.3 of [3]. Let A, B be positive definite and C, D be positive semidefinite matrices of the same size. Let X be any Hermitian matrix. Then $Tr(XA^{-1}XB^{-1}) - Tr\{X(A+C)^{-1}X(B+D)^{-1}\} \geq 0$. We extend this inequality to the case when both A and B are just positive semidefinite. Let us note that, for a positive semidefinite matrix E, since $E = S^*S$, it follows that $x^*Ex = 0$ if and only if $Ex = 0$.

Theorem 9. *Let $A, B, C, D \in \mathbb{C}^{n\times n}$ be positive semidefinite and let $X \in \mathbb{C}^{n\times n}$ be Hermitian. If $R(C) \subseteq R(A)$ and $R(D) \subseteq R(B)$, then*

$$Tr(XA^{\#}XB^{\#}) - Tr\{X(A+C)^{\#}X(B+D)^{\#}\} \geq 0.$$

Proof. First, we shall show that $R(A+C) = R(A)$. Let $(A+C)x = 0$. Then $x^*(A+C)x = 0$ so that $x^*Ax + x^*Cx = 0$. As both the terms are nonnegative (since A and C are positive semidefinite), we get $x^*Ax = 0 = x^*Cx$. In particular, $Ax = 0$. So $N(A+C) \subseteq N(A)$. Hence $R(A) \subseteq R(A+C) \subseteq R(A)$ using $R(C) \subseteq R(A)$. Thus $R(A+C) = R(A)$ and so $rk(A+C) = rk(A)$. Now $(A+C)-A$ is positive semidefinite. Hence by Theorem 4, $A^{\#}-(A+C)^{\#}$ is positive semidefinite. Similarly, it follows that $B^{\#} - (B+D)^{\#}$ is positive semidefinite. Since $XA^{\#}X$ is positive semidefinite (as X is Hermitian) by Theorem 2, we have

$$Tr\{(XA^{\#}X)(B^{\#} - (B+D)^{\#})\} \geq 0,$$

so that

$$Tr(XA^{\#}XB^{\#}) \geq Tr\{XA^{\#}X(B+D)^{\#}\}.$$

Again, by Theorem 2,

$$Tr[\{X(A^{\#} - (A+C)^{\#})X\}\{(B+D)^{\#}\}] \geq 0,$$

so that

$$Tr\{XA^{\#}X(B+D)^{\#}\} \geq Tr\{X(A+C)^{\#}X(B+D)^{\#}\}.$$

Combining these inequalities we have

$$Tr(XA^{\#}XB^{\#}) \geq Tr\{X(A+C)^{\#}X(B+D)^{\#}\}.$$

■

Remark 10. The inclusion conditions $R(C) \subseteq R(A)$ and $R(D) \subseteq R(B)$ are indispensable. Let $X = I, A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = C = D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then A, B, C, D are positive semidefinite, $R(C) \nsubseteq R(A), XA^{\#}XB^{\#} = 0$ and $X(A+C)^{\#}X(B+D)^{\#} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ so that $Tr(XA^{\#}XB^{\#}) - Tr\{X(A+C)^{\#}X(B+D)^{\#}\} < 0$. Similarly, it may be shown that the other inclusion $R(D) \subseteq R(B)$ is indispensable.

To place the next result in a proper perspective, let us recall Lemma 2.2 of [2]. Let A, B be positive definite and C, D be positive semidefinite matrices of the same size. Set

$$M = (A-B)(B+D)^{-1}(C-D)(A+C)^{-1}$$

and

$$N = (C - D)(B + D)^{-1}(A - B)(A + C)^{-1},$$

i.e., N is obtained from M by interchanging the first and the third factors. Then

$$Tr(M) = Tr(N) \in \mathbb{R}.$$

Now, we extend this result to the case when A, B, C, D all are positive semidefinite matrices. Our proof is completely different from the proof for the positive definite case.

Theorem 11. *Let A, B, C and D be positive semidefinite with $R(A+C) = R(B+D)$. Set*

$$K = (A - B)(B + D)^{\#}(C - D)(A + C)^{\#}$$

and

$$L = (C - D)(B + D)^{\#}(A - B)(A + C)^{\#}.$$

Then

$$Tr(K) = Tr(L) \in \mathbb{R}.$$

Proof. Note that, since $A+C$ and $B+D$ are positive semidefinite, the matrices $(A + C)^{\#}$ and $(B + D)^{\#}$ are positive semidefinite. Since

$$R((A + C)^{\#}) = R(A + C) = R(B + D) = R((B + D)^{\#}),$$

we have

$$(B + D)^{\#}(A + C)(A + C)^{\#} = (B + D)^{\#}$$

and

$$(B + D)^{\#}(B + D)(A + C)^{\#} = (A + C)^{\#}.$$

Now

$$\begin{aligned}
K &= (A - B)(B + D)^{\#}(C - D)(A + C)^{\#} \\
&= (A - B)(B + D)^{\#}\{(A + C) - (B + D) - (A - B)\}(A + C)^{\#} \\
&= (A - B)(B + D)^{\#}\{(A + C) - (B + D)\}(A + C)^{\#} \\
&\quad - (A - B)(B + D)^{\#}(A - B)(A + C)^{\#} \\
&= (A - B)(B + D)^{\#} - (A - B)(A + C)^{\#} - (A - B)(B + D)^{\#}(A - B)(A + C)^{\#} \\
&= A(B + D)^{\#} - B(B + D)^{\#} - A(A + C)^{\#} + B(A + C)^{\#} \\
&\quad - (A - B)(B + D)^{\#}(A - B)(A + C)^{\#}.
\end{aligned}$$

By Theorem 2, the trace of each of the matrices above is real and nonnegative. Thus $Tr(K) \in \mathbb{R}$. Also,

$$\begin{aligned} Tr(K) &= \overline{Tr(K)} \\ &= Tr(K^*) \\ &= Tr\{(A+C)^{\#}(C-D)(B+D)^{\#}(A-B)\} \\ &= Tr(L). \end{aligned}$$

∎

Let us observe that if the second and the fourth factors in the expression for K as above are interchanged, then the resulting matrix has the same trace as the matrix L. In the result above, the condition that $R(A+C) = R(B+D)$ is indispensable. This is illustrated next.

Example 12. Let $A = C = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then A, B, C and D are positive semidefinite and $B + D = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$. Thus $R(A+C) = span\{(1,i)^*\} \neq R(B+D) = \mathbb{C}^2$. Also $(B+D)^{\#} = (B+D)^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1-i \\ -1+i & 2 \end{pmatrix}$ and $(A+C)^{\#} = (2A)^{\#} = \frac{1}{8}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Now $K = (A-B)(B+D)^{\#}(C-D)(A+C)^{\#} = \frac{1}{16}\begin{pmatrix} 0 & 0 \\ 2+i & -1+2i \end{pmatrix}$ and $L = (C-D)(B+D)^{\#}(A-B)(A+C)^{\#} = \frac{1}{16}\begin{pmatrix} -1-i & 1-i \\ -1 & -i \end{pmatrix}$. Thus $Tr(K) = \frac{-1+2i}{16}$ and $Tr(L) = \frac{-1-2i}{16}$.

The next result generalizes Theorem 2.1 and Theorem 3.1 of [5]. Let A, B be positive definite and C, D be positive semidefinite matrices of the same size. Set $F = (A-B)(B^{-1} - A^{-1})$, $S = (C-D)\{(B+D)^{-1} - (A+C)^{-1}\}$ and $T = (C-D)(B+D)^{-1}(A-B)(A+C)^{-1}$. Then $Tr(F+S) \geq |Tr(T)|$ and $Tr(F+4S) \geq 0$.

Theorem 13. *Let A, B, C and D be positive semidefinite with $R(C) \subseteq R(A) = R(B) \supseteq R(D)$. Set*

$$U = (A-B)(B^{\#} - A^{\#}),$$

$$V = (C-D)\{(B+D)^{\#} - (A+C)^{\#}\}$$

and

$$W = (C-D)(B+D)^{\#}(A-B)(A+C)^{\#}.$$

Then

(i) $Tr(U+V) \geq |Tr(W)|$

and

(ii) $Tr(U+4V) \geq 0.$

Proof. First, we observe that $R(A+C) = R(B+D)$. Then by Theorem 11, one has
$$Tr[(A-B)(B+D)^{\#}(C-D)(A+C)^{\#}] = Tr(W) \in \mathbb{R}.$$
Note that $U = (A-B)B^{\#}(A-B)A^{\#}$. Also,
$$\begin{aligned} Tr(U) &= Tr[(A-B)B^{\#}(A-B)A^{\#})] \\ &= Tr[(A-B)A^{\#}(A-B)B^{\#})]. \end{aligned}$$
By Theorem 9 (by taking $X = A-B$), one has
$$\begin{aligned} Tr(U) &\geq Tr[(A-B)(A+C)^{\#}(A-B)(B+D)^{\#}] \\ &= Tr[(A-B)(B+D)^{\#}(A-B)(A+C)^{\#}]. \end{aligned}$$
As before, since $R((A+C)^{\#}) = R(A+C) = R(B+D) = R((B+D)^{\#})$, we have
$$(B+D)^{\#}(A+C)(A+C)^{\#} = (B+D)^{\#}$$
and
$$(B+D)^{\#}(B+D)(A+C)^{\#} = (A+C)^{\#}.$$
Thus
$$\begin{aligned} V &= (C-D)(B+D)^{\#}\{(A+C)-(B+D)\}(A+C)^{\#} \\ &= (C-D)(B+D)^{\#}(C-D)(A+C)^{\#} \\ &\quad + (C-D)(B+D)^{\#}(A-B)(A+C)^{\#}. \end{aligned}$$
(i) We have,
$$\begin{aligned} Tr(U+V) \geq\ & Tr[(A-B)(B+D)^{\#}(A-B)(A+C)^{\#}] \\ & + Tr[(C-D)(B+D)^{\#}(C-D)(A+C)^{\#}] \\ & + Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}]. \end{aligned}$$
With $X_1 = A-B$, $X_2 = C-D$, $S_1 = (B+D)^{\#}$ and $S_2 = (A+C)^{\#}$ in Theorem 3 together with $a = b = 1$, one infers that the trace of the first two terms is greater than or equal to $2|Tr\{(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}\}|$. Thus

$$\begin{aligned}Tr(U+V) &\geq 2|Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}]| \\ &\quad + Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}] \\ &\geq |Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}]|,\end{aligned}$$

proving (i).

(ii) We have,

$$\begin{aligned}Tr(U+4V) &\geq Tr\{(A-B)(B+D)^{\#}(A-B)(A+C)^{\#}\} \\ &\quad + 4Tr[(C-D)(B+D)^{\#}(C-D)(A+C)^{\#}] \\ &\quad + 4Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}] \\ &\geq 4|Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}]| \\ &\quad + 4Tr[(C-D)(B+D)^{\#}(A-B)(A+C)^{\#}] \\ &\geq 0,\end{aligned}$$

where the second inequality was obtained by applying Theorem 3 with $a = 1$ and $b = 4$ and making suitable choices for X_1, X_2, S_1 and S_2. ■

To motivate our next result, let us recall that in the proof of Theorem 2 in [7], it was shown that if $U = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix}$ is positive definite, then $U = M^*M = KK^*$, where

$$M = \begin{pmatrix} (A_U - B_U C_U^{-1} B_U^*)^{1/2} & 0 \\ C_U^{-1/2} B_U^* & C_U^{1/2} \end{pmatrix}$$

and

$$K = \begin{pmatrix} A_U^{1/2} & 0 \\ B_U^* A_U^{-1/2} & (C_U - B_U^* A_U^{-1} B_U)^{1/2} \end{pmatrix}.$$

In the next result, we obtain an analogue for positive semidefinite matrices.

Theorem 14. *Let* $U = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix}$ *be positive semidefinite. Set* $P = \begin{pmatrix} G_U^{1/2} & 0 \\ (C_U^{\#})^{1/2} B_U^* & C_U^{1/2} \end{pmatrix}$ *and* $Q = \begin{pmatrix} A_U^{1/2} & 0 \\ B_U^* (A_U^{\#})^{1/2} & F_U^{1/2} \end{pmatrix}$, *where* $F_U = C_U - B_U^* A_U^{\#} B_U$ *and* $G_U = A_U - B_U C_U^{\#} B_U^*$. *Then*

$$U = P^*P = QQ^*.$$

Proof. Since U is positive semidefinite, it follows that A_U, C_U, F_U and G_U are positive semidefinite. Hence their square roots exist. By Theorem 5, one has

$$R(B_U^*) \subseteq R(C_U) = R\left(C_U^{1/2}\right)$$

and

$$R(B_U) \subseteq R(A_U) = R\left(A_U^{1/2}\right).$$

So,

$$C_U^{1/2}\left(C_U^{\#}\right)^{1/2} B_U^* = B_U^*,$$

$$B_U C_U^{1/2}\left(C_U^{\#}\right)^{1/2} = B_U,$$

$$A_U^{1/2}\left(A_U^{\#}\right)^{1/2} B_U = B_U$$

and

$$B_U^* A_U^{1/2}\left(A_U^{\#}\right)^{1/2} = B_U^*.$$

Now

$$P^*P = \begin{pmatrix} G_U^{1/2} & B_U\left(C_U^{\#}\right)^{1/2} \\ 0 & C_U^{1/2} \end{pmatrix}\begin{pmatrix} G_U^{1/2} & 0 \\ \left(C_U^{\#}\right)^{1/2} B_U^* & C_U^{1/2} \end{pmatrix} = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix} = U$$

and

$$QQ^* = \begin{pmatrix} A_U^{1/2} & 0 \\ B_U^*\left(A_U^{\#}\right)^{1/2} & F_U^{1/2} \end{pmatrix}\begin{pmatrix} A_U^{1/2} & \left(A_U^{\#}\right)^{1/2} B_U \\ 0 & F_U^{1/2} \end{pmatrix} = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix} = U.$$

■

As an application of Theorem 14, we obtain a generalization of the following result [9, Theorem 3.3]: Let $U = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix}$, $V = \begin{pmatrix} A_V & B_V \\ B_V^* & C_V \end{pmatrix}$ be positive semidefinite with A_U, C_U, A_V and C_V being invertible. For any positive integer m, set

$$K = \left[\left(A_U - B_U C_U^{-1} B_U^*\right)^{1/2} A_V^{1/2}\right]^{2m}$$

and

$$L = \left[C_U^{1/2}\left(C_V - B_V^* A_V^{-1} B_V\right)^{1/2}\right]^{2m}.$$

Then

$$Tr(K) + Tr(L) \leq Tr(UV)^{2m} \leq Tr(U^m V^m).$$

While this result was proved using singular values, the following proof of an extension does not make use of singular values.

Theorem 15. *Let*

$$U = \begin{pmatrix} A_U & B_U \\ B_U^* & C_U \end{pmatrix} \text{ and } V = \begin{pmatrix} A_V & B_V \\ B_V^* & C_V \end{pmatrix}$$

be positive semidefinite. Let F_U, G_U and F_V, G_V denote the pseudo Schur complements and the complementary pseudo Schur complements of U and V, respectively. Then for any positive integer m,

(i) $Tr\left[G_U^{1/2} A_V^{1/2}\right]^{2m} + Tr\left[C_U^{1/2} F_V^{1/2}\right]^{2m} \leq Tr(UV)^{2m} \leq Tr(U^m V^m)$.

(ii) $Tr\left[C_V^{1/2} F_U^{1/2}\right]^{2m} + Tr\left[G_V^{1/2} A_U^{1/2}\right]^{2m} \leq Tr(UV)^{2m} \leq Tr(U^m V^m)$.

Proof. (i) By Theorem 14, $U = P^*P$ and $V = QQ^*$, where

$$P = \begin{pmatrix} G_U^{1/2} & 0 \\ \left(C_U^{\#}\right)^{1/2} B_U^* & C_U^{1/2} \end{pmatrix} \text{ and } Q = \begin{pmatrix} A_V^{1/2} & 0 \\ B_V^* \left(A_V^{\#}\right)^{1/2} & F_V^{1/2} \end{pmatrix}.$$

Now,

$$PQ = \begin{pmatrix} G_U^{1/2} A_V^{1/2} & 0 \\ \left(C_U^{\#}\right)^{1/2} B_U^* A_V^{1/2} + C_U^{1/2} B_V^* \left(A_V^{\#}\right)^{1/2} & C_U^{1/2} F_U^{1/2} \end{pmatrix}$$

so that,

$$(PQ)^{2m} = \begin{pmatrix} \left(G_U^{1/2} A_V^{1/2}\right)^{2m} & 0 \\ * & \left(C_U^{1/2} F_V^{1/2}\right)^{2m} \end{pmatrix},$$

where the matrix at the bottom left corner is not relevant to our discussion. Now, by Theorem 1, for any positive integer m,

$$|Tr(PQ)^{2m}| \leq Tr(P^*PQQ^*)^m \leq Tr\{(P^*P)^m (QQ^*)^m\}.$$

Thus

$$Tr\left[G_U^{1/2} A_V^{1/2}\right]^{2m} + Tr\left[C_U^{1/2} F_V^{1/2}\right]^{2m} \leq Tr(UV)^{2m} \leq Tr(U^m V^m).$$

Similarly, (ii) may be shown. ∎

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