Discussiones Mathematicae
General Algebra and Applications 37 (2017) 59–74
doi:10.7151/dmgaa.1266

ON THE SECOND SPECTRUM OF LATTICE MODULES

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Abstract

The second spectrum $Spec^s(M)$ is the collection of all second elements of M. In this paper, we study the topology on $Spec^s(M)$, which is a generalization of the Zariski topology on the prime spectrum of lattice modules. Besides some properties, $Spec^s(M)$ is characterized and the interrelations between the topological properties of $Spec^s(M)$ and the algebraic properties of M, are studied.

Keywords: second element, prime element, maximal element, minimal element, spectral space.

2010 Mathematics Subject Classification: 06D10, 06E10, 06F10.

1. Introduction

The Zariski topology for second spectrum of a module over a commutative ring is being introduce and studied by Ansari-Toroghy, Farshadifar in [1]. As a generalization of most of the results in [1], we introduce the concept of second elements of a lattice module M over a C-lattice L and also study the Zariski topology on $Spec^s(M)$, the collection of all second elements of a lattice module M.

The concept of second element of a comultiplication lattice module was introduced in [10]. A lattice module M is said to be *comultiplication* if for every element N of M, there exists an element $a \in L$ such that $N = (0_M : a)$ and an element $0_M \neq N \in M$ is said to be *second*, if for each $a \in L$, either $aN = N$ or $aN = 0_M$.

There are many generalizations of the Zariski topology over the set of all prime submodules of a R-module M (see [1, 5, 8, 9, 15, 17]). In [5], the Zariski topology over the prime spectrum $Spec(M)$ of a lattice module M over a C-lattice L has been studied by Sachin Ballal and Villas Kharat. In [20], authors introduced and studied the concept of quasi-prime elements as a generalization of prime elements and also the Zariski topology on the quasi-prime spectrum of a lattice module M over a C-lattice L.

The Zariski topology on the set $Spec(L)$ of all prime elements in multiplicative lattices is being studied in [18] by Thakare, Manjarekar and Maeda, and in [19] by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.

A lattice L is said to be *complete*, if for any subset S of L, we have $\vee S, \wedge S \in L$. A complete lattice L is said to be a *multiplicative lattice*, if there is defined a binary operation "." called multiplication on L satisfying the following conditions:

(1) $a.b = b.a$, for all $a, b, c \in L$;

(2) $a.(b.c) = (a.b).c$, for all $a, b, c \in L$;

(3) $a.(\vee_\alpha b_\alpha) = \vee_\alpha (a.b_\alpha)$, for all $a, b_\alpha \in L$;

(4) $a.1 = a$, for all $a \in L$.

Henceforth, $a.b$ will be simply denoted by ab. An element $e \in L$ is said to be *meet principal* (respectively, *join principal*) if it satisfies the identity $a \wedge be = ((a : e) \wedge b)e$ (respectively, $((ae \vee b) : e) = a \vee (b : e)$), for all $a, b \in L$. An element $e \in L$ is said to be *principal* if it is both meet as well as join principal. If each element of L is the join of principal elements of L, then L is called *principally generated*.

An element a in L is called *compact* if $a \leq \bigvee_{\alpha \in I} b_\alpha (I$ is an indexed set) implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \cdots \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I. By a *C-lattice*, we mean a multiplicative lattice L, with least element 0_L and greatest element 1_L which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset C of compact elements of L. Throughout this paper, L will be a C-lattice.

An element $p \in L$ is said to be *proper* if $p < 1$. A proper element m of a multiplicative lattice L is said to be *maximal* if $m < x \leq 1$ implies $x = 1$, $x \in L$. A proper element m of a multiplicative lattice L is said to be *minimal* if $0 \leq x < m$ implies $x = 0$, $x \in L$. A proper element p of a multiplicative lattice L is said to be *prime* if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. A proper element p of a multiplicative lattice L is said to be *quasi-prime* if $a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$. For any $a \in L$, its radical is denoted by $\sqrt{a}$ and defined as $\sqrt{a} = \vee\{x \in L | x^n \leq a$, for some $n \in Z^+\} = \wedge\{p \in L | a \leq p$ and p is a prime$\}$.

An element $a \in L$ with $\sqrt{a} = a$ is called *semiprime* or *radical*.

A complete lattice M is said to be *lattice module* over a multiplicative lattice L, or L-module, if there is a multiplication between elements of M and L, denoted by $aN \in M$, for $a \in L$ and $N \in M$, which satisfies the following properties:

1. $(ab)N = a(bN)$;
2. $(\bigvee_\alpha a_\alpha)(\bigvee_\beta N_\beta) = (\bigvee_{\alpha\beta} a_\alpha N_\beta)$;
3. $1_L N = N$;
4. $0_L N = 0_M$; for all $a, b, a_\alpha \in L$, and for all $N, N_\beta \in M$.

The greatest element of M will be denoted by 1_M and the smallest element will be denoted by 0_M. For $N \in M$, $b \in L$, denote $(N : b) = \vee\{K \in M | bK \leq N\}$. For $a, b \in L$, we write $(a : b) = \vee\{x \in L | bx \leq a\}$ and for $A, B \in M$, $(A : B) = \vee\{x \in L | Bx \leq A\}$. An element $A \in M$ is said to be *weak meet principal* if $(B : A)A = B \wedge A$ for all $B \in M$; *weak join principal* if $(bA : A) = b \vee (0_M : A)$ for all $b \in L$; and *weak principal* if A is both weak meet principal and weak join principal. An element $N \in M$ is said to be *compact* if $N \leq \bigvee_{\alpha \in I} A_\alpha (I$ is an indexed set) implies $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I. If each element of M is the join of principal (compact) elements of M, then M is called *principally generated* (*compactly generated*).

An element $N < 1_M$ in M is said to be *prime* if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e., $a \leq (N : 1_M)$ for $a \in L$ and $X \in M$. An element $N < 1_M$ in M is said to be *quasi-prime* if $(N : 1_M)$ is a quasi-prime element of L. Note that, every prime element in M is quasi-prime. An element $N < 1_M$ of M is said to be *maximal* if $N \leq B$ implies either $N = B$ or $B = 1_M$, $B \in M$. A non-zero element $K \neq 1_M$ of M is said to be *minimal* if $0_M \leq N < K$ implies $N = 0_M$, $N \in M$.

Further, all these concepts and for more information on multiplicative lattices and lattice modules, the reader may refer ([3–7, 10–13, 18, 19]).

2. Topology on $Spec^s(M)$

Here, we define the second element for a lattice module M over a C-lattice L.

Definition 2.1. Let M be a lattice module over a C-lattice L. A non-zero element $N \in M$ is said to be *second*, if for $a \in L$, either $aN = N$ or $aN = 0_M$.

Note that, every minimal element of M is second.

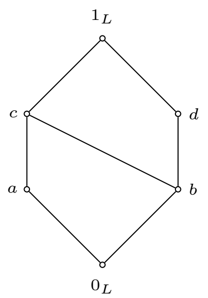
Lemma 2.2. *Let M be a lattice module over a C-lattice L and $N \in M$. If N is second then $(0_M : N)$ is a prime element of L.*

Proof. Suppose that N is a second element of M and $abN = 0_M$ for $a, b \in L$ with $bN \neq 0_M$. Since N is second, $bN = N$ and so $aN = 0_M$, i.e., $a \leq (0_M : N)$. Consequently, $(0_M : N)$ is a prime element of L. ■

Converse of Lemma 2.2 is true for comultiplication lattice module (see [10]).

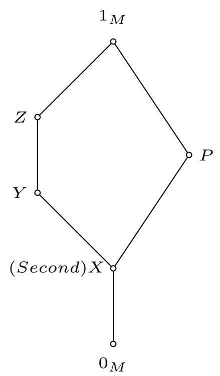
Lemma 2.3 [10]**.** *Let M be a comultiplication lattice module over a multiplicative lattice L and $N \in M$. Then N is second if and only if $(0_M : N)$ is a prime element in L.*

Example 2.4. The lattice depicted in Figure (a) is a multiplicative lattice L and the lattice depicted in Figure (b) is a lattice module M over a multiplicative lattice L. Note that, X is a second element of M but Y, Z, P and 1_M are not second elements of M.



.	0_L	a	b	c	d	1_L
0_L	0_L	0_L	0_L	0_L	0_L	0_L
a	0_L	a	0_L	a	0_L	a
b	0_L	0_L	0_L	0_L	b	b
c	0_L	a	0_L	a	b	c
d	0_L	0_L	b	b	d	d
1_L	0_L	a	b	c	d	1_L

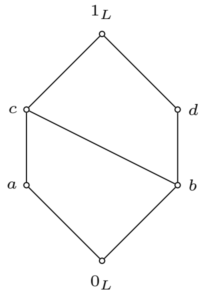
Figure (a). Multiplicative Lattice L.



.	0_M	X	Y	Z	P	1_M
0_L	0_M	0_M	0_M	0_M	0_M	0_M
a	0_M	0_M	0_M	0_M	0_M	0_M
b	0_M	0_M	X	X	X	X
c	0_M	0_M	X	X	X	X
d	0_M	X	Y	Z	P	1_M
1_L	0_M	X	Y	Z	P	1_M

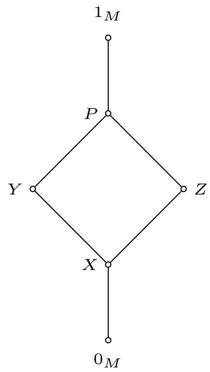
Figure (b). Lattice Module M over L.

Example 2.5. The lattice depicted in Figure (a) is a multiplicative lattice L and the lattice depicted in Figure (b) is a Lattice module M over a multiplicative lattice L. Note that, all non-zero elements of M are second elements of M.



.	0_L	a	b	c	d	1_L
0_L	0_L	0_L	0_L	0_L	0_L	0_L
a	0_L	a	0_L	a	0_L	a
b	0_L	0_L	0_L	0_L	b	b
c	0_L	a	0_L	a	b	c
d	0_L	0_L	b	b	d	d
1_L	0_L	a	b	c	d	1_L

Figure (a). Multiplicative Lattice L.



.	0_M	X	Y	Z	P	1_M
0_L	0_M	0_M	0_M	0_M	0_M	0_M
a	0_M	X	Y	Z	P	1_M
b	0_M	0_M	0_M	0_M	0_M	0_M
c	0_M	X	Y	Z	P	1_M
d	0_M	0_M	0_M	0_M	0_M	0_M
1_L	0_M	X	Y	Z	P	1_M

Figure (b). Lattice Module M over L.

The following result is useful throughout the paper.

Lemma 2.6 [14]**.** *Let M be a lattice module over a multiplicative lattice L. Then for $x \in L$ and $A, B, C \in M$, following holds:*

1. $x \leq (0_M : (0_M : x))$.
2. $A \leq (0_M : (0_M : A))$.
3. *If $A \leq B$ then $(C : B) \leq (C : A)$.*
4. $(0_M : A) = (0_M : (0_M : (0_M : A)))$.
5. $(A : B \vee C) = (A : B) \wedge (A : C)$.

Let M be a lattice module over a C-lattice L. Consider the set $Spec^s(M)$ of second elements of a lattice module M. Since every minimal element of M is second, $Min(M) \subseteq Spec^s(M)$, where $Min(M)$ is the set of all minimal elements of M. Also, $D^{s*}(N) = \{S \in Spec^s(M) | S \leq N\}$, for $N \in M$. Note that $D^{s*}(1_M) = Spec^s(M)$, and $D^{s*}(0_M)$ is an empty set.

Proposition 2.7. *Let M be a lattice module over a C-lattice L and $N, N_i, K \in M$ $(i \in I)$. Then the following statements hold.*

1. $\cap_{i \in I} D^{s*}(N_i) = D^{s*}(\wedge_{i \in I} N_i)$.
2. $D^{s*}(N) \cup D^{s*}(K) \subseteq D^{s*}(N \vee K)$.

Proof. (1) Note that, $D^{s*}(\wedge_{i \in I} N_i) \subseteq D^{s*}(N_i)$ for each i, since $\wedge_{i \in I} N_i \leq N_i$. Hence $D^{s*}(\wedge_{i \in I} N_i) \subseteq \cap_{i \in I} D^{s*}(N_i)$.

Now, suppose that $K \in \cap_{i \in I} D^{s*}(N_i)$. Then for each i, $K \in D^{s*}(N_i)$ therefore $K \leq N_i$. This implies $K \leq \wedge_{i \in I} N_i$ and so $\cap_{i \in I} D^{s*}(N_i) \subseteq D^{s*}(\wedge_{i \in I} N_i)$. Consequently, $\cap_{i \in I} D^{s*}(N_i) = D^{s*}(\wedge_{i \in I} N_i)$.

(2) Since $N, K \leq N \vee K$, we have $D^{s*}(N), D^{s*}(K) \subseteq D^{s*}(N \vee K)$ and so $D^{s*}(N) \cup D^{s*}(K) \subseteq D^{s*}(N \vee K)$. ■

We note from Proposition 2.7 that, the set $\zeta^{s*}(M) = \{D^{s*}(N) | N \in M\}$ forms a topology if and only if it is closed under finite union. In this case, $\zeta^{s*}(M)$ induces a topology τ^{s*} on $Spec^s(M)$, and we call it the *Zariski* topology.

Proposition 2.8. *Let M be a lattice module over a C-lattice L and $a, b \in L$. Then $D^{s*}((0_M : a)) \cup D^{s*}((0_M : b)) = D^{s*}((0_M : ab))$.*

Proof. Note that $D^{s*}((0_M : a)) \cup D^{s*}((0_M : b)) \subseteq D^{s*}((0_M : ab)$ for $a, b \in L$.

Now, suppose that $S \in D^{s*}((0_M : ab))$ with $S \notin D^{s*}((0_M : b))$. We claim that $S \in D^{s*}((0_M : a))$. By assumption $S \leq (0_M : ab)$ and $S \nleq (0_M : b)$, therefore $abS = 0_M$ and $bS \neq 0_M$. Since S is a second element of M and $bS \neq 0_M$, we have $bS = S$. Therefore $abS = aS = 0_M$ and so $S \leq (0_M : a)$. Consequently, $S \in D^{s*}((0_M : a))$. ■

From Proposition 2.7 and Proposition 2.8, we observe that, the set $\{D^{s*}((0_M : a)) | a \in L\}$ forms a topology, say τ'^{s} on $Spec^s(M)$.

It clear from Proposition 2.7 that, the collection $\{D^{s*}(N) | N \in M\}$ need not be closed under finite union. So for $N \in M$, we define a new set $D^s(N) = \{S \in Spec^s(M) | (0_M : N) \leq (0_M : S)\}$ and we have the following Theorem.

Theorem 2.9. *Let M be a lattice module over a C-lattice L and $N, N_i, K \in M$ $(i \in I)$. Then the following statements hold.*

1. *$D^s(1_M) = Spec^s(M)$, and $D^s(0_M)$ is an empty set.*
2. $\cap_{i \in I} D^s(N_i) = D^s(\wedge_{i \in I}(0_M : (0_M : N_i)))$.
3. $D^s(N) \cup D^s(K) = D^s(N \vee K)$.

Proof. (1) By definition $D^s(1_M) = \{S \in Spec^s(M) | (0_M : 1_M) = 0_L \leq (0_M : S)\} = Spec^s(M)$ and $D^s(0_M) = \{S \in Spec^s(M) | (0_M : 0_M) = 1_L \leq (0_M : S)\}$ is empty.

(2) Suppose that $S \in \cap_{i\in I} D^s(N_i)$. Then $S \in D^s(N_i)$, for each $i \in I$ therefore $(0_M : N_i) \leq (0_M : S)$, for each $i \in I$ and so $\vee_{i\in I}(0_M : N_i) \leq (0_M : S)$. Therefore $(0_M : (0_M : \vee_{i\in I}(0_M : N_i))) \leq (0_M : (0_M : (0_M : S))) = (0_M : S)$ by Lemma 2.31(3) and Lemma 2.31(4) and hence $S \in D^s(\wedge_{i\in I}(0_M : (0_M : N_i)$ by Lemma 2.31(5).

Now, suppose that $K \in D^s(\wedge_{i\in I}(0_M : (0_M : N_i))$. Then $(0_M : \wedge_{i\in I}(0_M : (0_M : N_i)) \leq (0_M : K)$, and hence $(0_M : (0_M : K)) \leq (0_M : (0_M : \wedge_{i\in I}(0_M : (0_M : N_i))) = \wedge_{i\in I}(0_M : (0_M : N_i))$ by Lemma 2.31 (3) and Lemma 2.31(4). Therefore $(0_M : (0_M : K)) \leq (0_M : (0_M : N_i))$ for each $i \in I$ and so $(0_M : N_i) \leq (0_M : K)$, for each $i \in I$ by Lemma 2.31(3) and Lemma 2.31(4). Thus $K \in D^s(N_i)$ for each $i \in I$ and consequently, $K \in \cap_{i\in I} D^s(N_i)$.

(3) Note that $D^s(N) \cup D^s(K) \subseteq D^s(N \vee K)$ for $N, K \in M$.

Now, suppose that $S \in D^s(N \vee K)$. Then $(0_M : N \vee K) \leq (0_M : S)$ and so $(0_M : N) \wedge (0_M : K) \leq (0_M : S)$ by Lemma 2.31(5). Since S is second, $(0_M : S)$ is a prime element of L by Lemma 2.2, and hence quasi-prime. Therefore $(0_M : N) \leq (0_M : S)$ or $(0_M : N) \leq (0_M : S)$ by definition of quasi-prime element and so $S \in D^s(N)$ or $S \in D^s(K)$. Consequently, $S \in D^s(N) \cup D^s(K)$. ■

Theorem 2.9 shows that, there exists a topology, say τ^s on $Spec^s(M)$ having $\{D^s(N) | N \in M\}$ as a family of closed sets.

We denote $Spec^s_p(M) = \{N \in M | N \text{ is second and } (0_M : N) = p\}$, where p is a prime element of L and for $a \in L$, $D^s((0_M : a)) = \{S \in Spec^s(M) | (0_M : (0_M : a)) \leq (0_M : S)\}$.

Lemma 2.10. *Let M be a lattice module over a C-lattice L and $N, K \in M$. Then the following statements hold.*

1. *If $(0_M : N) = (0_M : K)$, then $D^s(N) = D^s(K)$. Also, the converse is true if $N, K \in Spec^s(M)$.*
2. *$D^s(N) = \cup_{p\in D^s((0_M:N))} Spec^s_p(M)$.*
3. *$D^s(N) = D^s((0_M : (0_M : N))) = D^{s*}((0_M : (0_M : N)))$. In particular, we have $D^s((0_M : a)) = D^{s*}((0_M : a))$ for $a \in L$.*

Proof. (1) Suppose that $(0_M : N) = (0_M : K)$ and $S \in D^s(N)$. Then $(0_M : N) \leq (0_M : S)$ and so $(0_M : K) \leq (0_M : S)$. Therefore $S \in D^s(K)$ and so $D^s(N) \subseteq D^s(K)$. Similarly, $D^s(K) \subseteq D^s(N)$.

Conversely, suppose that $D^s(N) = D^s(K)$ and $N, K \in Spec^s(M)$. Given $N \in D^s(N)$ and $D^s(N) = D^s(K)$, therefore $(0_M : K) \leq (0_M : N)$ and $(0_M : N) \leq (0_M : K)$. Consequently, $(0_M : N) = (0_M : K)$.

(2) Suppose that $P \in D^s(N)$. Then $(0_M : N) \leq (0_M : P) = p$. Therefore $P \in \cup_{p\in D^s((0_M:N))} Spec^s_p(M)$. Consequently, $D^s(N) \subseteq \cup_{p\in D^s((0_M:N))} Spec^s_p(M)$.

Now, suppose that $K \in \cup_{p\in D^s((0_M:N))} Spec^s_p(M)$. Then there exists $a \in D^s((0_M : N))$ with $(0_M : N) \leq a = (0_M : K)$ and hence $K \in D^s(N)$, therefore $\cup_{p\in D^s((0_M:N))} \subseteq D^s(N)$. Consequently, $D^s(N) = \cup_{p\in D^s((0_M:N))} Spec^s_p(M)$.

(3) Suppose that $S \in D^s(N)$. Then $(0_M : N) \leq (0_M : S)$ and so $(0_M : (0_M : (0_M : N))) \leq (0_M : (0_M : (0_M : S))) = (0_M : S)$ by Lemma 2.31(3) and Lemma 2.31(4), therefore $S \in D^s((0_M : (0_M : N)))$. Thus $D^s(N) \subseteq D^s((0_M : (0_M : N)))$.

Now, suppose that $S \in D^s((0_M : (0_M : N)))$. Then $(0_M : (0_M : (0_M : N))) \leq (0_M : S)$, i.e., $(0_M : N) \leq (0_M : S)$ by Lemma 2.31(3) and hence $S \in D^s(N)$. Consequently, $D^s(N) = D^s((0_M : (0_M : N)))$.

Next, suppose that $K \in D^s(N)$. Then $(0_M : N) \leq (0_M : K)$ and so $K \leq (0_M : (0_M : K)) \leq (0_M : (0_M : N))$ by Lemma 2.31(2) and Lemma 2.31(3), therefore $K \in D^{s*}((0_M : (0_M : N)))$. Thus $D^s(N) \subseteq D^{s*}((0_M : (0_M : N)))$.

Now, $P \in D^{s*}((0_M : (0_M : N)))$ implies $P \leq (0_M : (0_M : N))$ and hence $(0_M : (0_M : (0_M : N))) \leq (0_M : P)$ by Lemma 2.31(3). Therefore $(0_M : N) \leq (0_M : P)$ by Lemma 2.31(4) and so $P \in D^s(N)$. Thus $D^{s*}((0_M : (0_M : N))) \subseteq D^s(N)$. Consequently, $D^s(N) = D^{s*}((0_M : (0_M : N)))$. ∎

In what follows and thereafter, the map $\psi^s : Spec^s(M) \to Spec(L/(0_M : 1_M))$ defined by $\psi^s(N) = \overline{(0_M : N)}$ is called the *natural map* of $Spec^s(M)$, where M is a lattice module over a C-lattice L.

Lemma 2.11. *Let M be a lattice module over a C-lattice L. Then the natural map ψ^s is continuous; more precisely, $(\psi^s)^{-1}D(\overline{a}) = D^s((0_M : a))$ for $a \in L$ with $(0_M : 1_M) \leq a$.*

Proof. Given $S \in (\psi^s)^{-1}(D(\overline{a}))$, there exists $\overline{b} \in D(\overline{a})$ with $S = (\psi^s)^{-1}(\overline{b})$. Therefore $\psi^s(S) = \overline{b}$ and so $\overline{(0_M : S)} = \overline{b}$. Thus $\overline{a} \leq \overline{(0_M : S)} = \overline{b}$ and hence $a \leq (0_M : S) = b$. We conclude that, $(0_M : (0_M : a)) \leq (0_M : S)$ by Lemma 2.31(3) and Lemma 2.31(4). Which implies that $S \in D^s((0_M : a))$. Thus $(\psi^s)^{-1}(D(\overline{a})) \subseteq D^s((0_M : a))$.

Now, suppose that $K \in D^s((0_M : a))$. Then $(0_M : (0_M : a)) \leq (0_M : K)$. But by Lemma 2.31(1), $a \leq (0_M : (0_M : a))$, therefore $\overline{a} \leq \overline{(0_M : (0_M : a))} \leq \overline{(0_M : K)}$. Hence $K \in (\psi^s)^{-1}D(\overline{a})$. Consequently, $(\psi^s)^{-1}D(\overline{a}) = D^s((0_M : a))$. ∎

Theorem 2.12. *Let M be a lattice module over a C-lattice L. Then the following statements are equivalent.*

1. *The natural map $\psi^s : Spec^s(M) \to Spec(L/(0_M : 1_M))$ is injective.*
2. *For $N, K \in Spec^s(M)$, if $D^s(N) = D^s(K)$ then $N = K$.*
3. *$|Spec^s_p(M)| \leq 1$ for $p \in Spec(L)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that the natural map ψ^s is injective and $D^s(N) = D^s(K)$ for $N, K \in Spec^s(M)$. Then $(0_M : N) = (0_M : K)$, by Lemma 2.10(1).

Therefore $\overline{(0_M : N)} = \overline{(0_M : K)}$, and hence $\psi^s(N) = \psi^s(K)$, consequently, $K = N$, since ψ^s is injective.

(2) $\Rightarrow$ (3) Suppose that $K, N \in Spec^s_p(M)$ for some $p \in Spec(L)$. Then $(0_M : N) = (0_M : K) = p$ and hence $D^s(N) = D^s(K)$ by Lemma 2.10 (1) and $N = K$ by (2).

(3) $\Rightarrow$ (1) Suppose that $\psi^s(K) = \psi^s(N) = \overline{a}$ for $K, N \in Spec^s(M)$. Then $\overline{(0_M : K)} = \overline{(0_M : N)} = \overline{a}$. Therefore $(0_M : K) = (0_M : N) = a$ and so $K = N$ by (3). Thus, ψ^s is injective. ■

Theorem 2.13. *Let M be a lattice module over a C-lattice L. If the natural map ψ^s is surjective, then it is both closed and open. More precisely, for every $N \in M$, $\psi^s(D^s(N)) = D(\overline{(0_M : N)})$ and $\psi^s(Spec^s(M) - D^s(N)) = Spec(L/(0_M : 1_M)) - D(\overline{(0_M : N)})$.*

Proof. Suppose that ψ^s is surjective. By Lemma 2.11, we have $(\psi^s)^{-1}(D(\overline{(0_M : N)})) = D^s((0_M : (0_M : N)))$ again by Lemma 2.10(3), we have $D^s((0_M : (0_M : N))) = D^s(N)$, therefore $(\psi^s)^{-1}(D(\overline{(0_M : N)})) = D^s(N)$. Since ψ^s is surjective, $\psi^s \circ (\psi^s)^{-1}(D(\overline{(0_M : N)})) = \psi^s(D^s(N))$, therefore $\psi^s(D^s(N)) = D(\overline{(0_M : N)})$ and hence ψ^s is closed. Similarly, $\psi^s(Spec^s(M) - D^s(N)) = Spec(L/(0_M : 1_M)) - D(\overline{(0_M : N)})$, i.e., ψ^s open. ■

Corollary 2.14. *Let M be a lattice module over a C-lattice L. If the natural map ψ^s is surjective, then it is bijective if and only if it is homeomorphism.*

Now, we introduce an open base for the Zariski topology on $Spec^s(M)$. For each $r \in L$, define $X^s(r) = Spec^s(M) - D^s((0_M : r))$. Then $X^s(r)$ is an open set of $Spec^s(M)$.

Lemma 2.15. *Let M be a lattice module over a C-lattice L. Then the set $B = \{X^s(a) | a \in L\}$ forms an open base for the Zariski topology on $Spec^s(M)$.*

Proof. Suppose that $Spec^s(M)$ is non-empty and U is an open subset of $Spec^s(M)$. Then for $N \in M$, $U = Spec^s(M) - D^s(N) = Spec^s(M) - D^s((0_M : (0_M : N)))$ by Lemma 2.10(3). Therefore $U = Spec^s(M) - D^s(N) = Spec^s(M) - D^s((0_M : (0_M : N))) = Spec^s(M) - D^s((0_M : \vee\{x \in L | xN = 0_M\})) = Spec^s(M) - D^s(\wedge_{\{x \in L | xN = 0_M\}}(0_M : x))$ by Lemma 2.6(5). By Theorem 2.9(2), we have $D^s(\wedge_{\{x \in L | xN = 0_M\}}(0_M : x)) = \cap_{\{x \in L | xN = 0_M\}} D^s((0_M : x))$, therefore $U = Spec^s(M) - D^s(\wedge_{\{x \in L | xN = 0_M\}}(0_M : x)) = Spec^s(M) - \cap_{\{x \in L | xN = 0_M\}} D^s((0_M : x)) = \cup_{\{x \in L | xN = 0_M\}}(Spec^s(M) - D^s((0_M : x))) = \cup_{\{x \in L | xN = 0_M\}} X^s(x)$. ■

Theorem 2.16. *Let M be a lattice module over a C-lattice L. Then for $a \in L$ and the natural map $\psi^s : Spec^s(M) \to Spec(L/(0_M : 1_M))$, the following statements hold.*

1. $(\psi^s)^{-1}(X(\overline{a})) = X^s(a)$, *where* $X(\overline{a}) = Spec(\overline{L}) - D(\overline{a})$.
2. $\psi^s(X^s(a)) \subseteq X(\overline{a})$ *and if* ψ^s *is surjective, then* $\psi^s(X^s(a)) = X(\overline{a})$.

Proof. (1) Consider $(\psi^s)^{-1}(X(\overline{a})) = (\psi^s)^{-1}(Spec(\overline{L}) - D(\overline{a})) = Spec^s(M) - (\psi^s)^{-1}(D(\overline{a})) = Spec^s(M) - D^s((0_M : a)) = X^s(a)$, where $(\psi^s)^{-1}D(\overline{a}) = D^s((0_M : a))$ for $a \in L$ with $(0_M : 1_M) \leq a$ by Lemma 2.11.

(2) Follows from (1). ∎

Theorem 2.17. *Let* M *be a lattice module over a* C*-lattice* L*. Then* $X^s(ab) = X^s(a) \cap X^s(b)$ *for* $a, b \in L$.

Proof. By Theorem 2.16(1), we have $X^s(ab) = (\psi^s)^{-1}(X(\overline{ab}))$. Therefore $X^s(ab) = (\psi^s)^{-1}(X(\overline{ab})) = (\psi^s)^{-1}(Spec^s(\overline{L}) - D(\overline{ab})) = Spec^s(M) - (\psi^s)^{-1}(D(\overline{ab})) = Spec^s(M) - D^s((0_M : ab)) = Spec^s(M) - D^{s*}((0_M : ab))$ by Lemma 2.11 and Lemma 2.10(3). But by Proposition 2.8, we have $D^{s*}((0_M : a)) \cup D^{s*}((0_M : b)) = D^{s*}((0_M : ab))$, therefore $X^s(ab) = Spec^s(M) - D^{s*}((0_M : ab)) = Spec^s(M) - (D^{s*}((0_M : a)) \cup D^{s*}((0_M : b))) = (Spec^s(M) - D^{s*}((0_M : a)) \cap (Spec^s(M) - D^{s*}((0_M : b))) = (Spec^s(M) - D^s((0_M : a)) \cap (Spec^s(M) - D^s((0_M : b))) = X^s(a) \cap X^s(b)$. ∎

A topological space Z is called *quasi-compact* if each of its open covers has a finite subcover (see [16]). We recall that $Spec(L)$ is quasi-compact if L is compactly generated multiplicative lattice with 1_L compact (see[18]).

Theorem 2.18. *Let* M *be a lattice module over a* C*-lattice* L *and the natural map* ψ^s *is surjective. Then for* $r \in L$*, the open set* $X^s(r)$ *is quasi-compact. In particular, the space* $Spec^s(M)$ *is quasi-compact.*

Proof. Suppose that the natural map ψ^s is surjective. By Lemma 2.15, the set $B = \{X^s(a) | a \in L\}$ is an open base for the Zariski topology on $Spec^s(M)$. Let $\{a_\lambda \in L | \lambda \in \Lambda\}$ be such that $Spec^s(M) = \cup_{\lambda\in\Lambda} X^s(a_\lambda)$. Then by Theorem 2.16(2), $Spec(\overline{L}) = X(\overline{1_L}) = \psi^s(X^s(1_L)) = \psi^s(Spec^s(M) - D^s((0_M : 1_L))) = \psi^s(Spec^s(M)) = \psi^s(\cup_{\lambda\in\Lambda} X^s(a_\lambda)) = \cup_{\lambda\in\Lambda}\psi^s(X^s(a_\lambda)) = \cup_{\lambda\in\Lambda} X(\overline{a_\lambda})$. Since $Spec(\overline{L})$ is quasi-compact, there exists a finite subset Λ' of Λ such that $Spec(\overline{L}) \subseteq \cup_{\lambda\in\Lambda'} X(\overline{a_\lambda})$ therefore by Theorem 2.16(1), $Spec^s(M) = X^s(1_L) = (\psi^s)^{-1}(X(\overline{1_L})) = (\psi^s)^{-1}(Spec(\overline{L})) \subseteq (\psi^s)^{-1}(\cup_{\lambda\in\Lambda'} X(\overline{a_\lambda})) \subseteq \cup_{\lambda\in\Lambda'} (\psi^s)^{-1}(X(\overline{a_\lambda})) = \cup_{\lambda\in\Lambda'} X^s(a_\lambda)$. Consequently, $Spec^s(M)$ is a quasi-compact space. ∎

The following Theorem follows from Lemma 2.15, Theorem 2.17 and Theorem 2.18.

Theorem 2.19. *Let* M *be a lattice module over a* C*-lattice* L *and the natural map* ψ^s *is surjective. Then the family of quasi-compact open sets of* $Spec^s(M)$ *is closed under finite intersection and forms an open base.*

Note that, by Theorem 2.9(3), the collection $\{D^s(N)|N \in M\}$ is closed under finite union. Therefore each closed set is of the form of $D^s(N)$ for $N \in M$.

A topological space Z is T_0 if and only if the closures of distinct points are distinct and a topological space Z is T_1 if and only if every singleton subset is closed (see [16]). Denote the closure of $Y \subseteq Spec^s(M)$ by $Cl(Y)$, and the join of all elements in Y by $T(Y)$.

Lemma 2.20. *Let M be a lattice module over a C-lattice L and $Y \subseteq Spec^s(M)$. Then $D^s(T(Y)) = Cl(Y)$. Hence, Y is closed if and only if $D^s(T(Y)) = Y$.*

Proof. Suppose that $Y \subseteq Spec^s(M)$ is closed. Clearly, $Y \subseteq D^s(T(Y))$. Now, suppose that $D^s(N)$ is a closed subset of $Spec^s(M)$ with $Y \subseteq D^s(N)$. Then $(0_M : N) \leq (0_M : K)$ for each $K \in Y$ and so $(0_M : N) \leq \wedge_{K \in Y}(0_M : K)$. But by Lemma 2.31(5), we have $\wedge_{K \in Y}(0_M : K) = (0_M : \vee_{K \in Y} K)$, therefore $(0_M : N) \leq (0_M : \vee_{K \in Y} K) = (0_M : T(Y))$. Thus $(0_M : N) \leq (0_M : T(Y)) \leq (0_M : Q)$ for $Q \in D^s(T(Y))$. This implies $D^s(T(Y)) \subseteq D^s(N)$ and hence $D^s(T(Y))$ is the smallest closed subset of $Spec^s(M)$ containing Y. Consequently, $D^s(T(Y)) = Cl(Y)$. ■

Lemma 2.21 [10]**.** *Let M be a lattice L-module. Then M is a comultiplication lattice L-module if and only if $N = (0_M : (0_M : N))$ for every $N \in M$.*

Theorem 2.22. *Let M be a comultiplication lattice module over a C-lattice L. Then $Spec^s(M)$ is a T_0-space.*

Proof. Suppose that $N, K \in Spec^s(M)$. Then by Lemma 2.20 and Lemma 2.10(1), we have $Cl(\{N\}) = Cl(\{K\})$ if and only if $D^s(N) = D^s(K)$ if and only if $(0_M : N) = (0_M : K)$ and by Lemma 2.6 and Lemma 2.21 we have $(0_M : N) = (0_M : K)$ if and only if $N = K$. This implies closures of distinct points are distinct, and so $Spec^s(M)$ is a T_0-space. ■

Lemma 2.23. *Let M be a lattice module over a C-lattice L and $S \in Spec^s(M)$. Then the following statements hold.*

1. *$Cl(\{S\}) = D^s(S)$.*
2. *$K \in Cl(\{S\})$ implies $D^s(K) \subseteq D^s(S)$. Also, the converse is true if $K \in Spec^s(M)$.*

Proof. (1) Suppose that $Y = \{S\}$. Then $T(Y) = \vee_{S \in Y} S = S$ and therefore by Lemma 2.20, $Cl(\{S\}) = D^s(T(Y)) = D^s(S)$.

(2) Suppose that $K \in Cl(\{S\})$. Then by (1), we have $K \in D^s(S)$ and so by definition $(0_M : S) \leq (0_M : K)$. If $P \in D^s(K)$, then $(0_M : K) \leq (0_M : P)$ and so, we have $(0_M : S) \leq (0_M : K) \leq (0_M : P)$ which implies $P \in D^s(S)$ and therefore $D^s(K) \subseteq D^s(S)$. Conversely, suppose that $K \in Spec^s(M)$ and $D^s(K) \subseteq D^s(S)$. Then $K \in D^s(K) \subseteq D^s(S)$. Therefore $(0_M : S) \leq (0_M : K)$ and hence $K \in Cl(\{S\})$ by (1). ■

Lemma 2.24. *Let M be a principally generated lattice module over a C-lattice L. If $K \in M$ is minimal then $(0_M : K)$ is a maximal element of L.*

Proof. Suppose that $K \in M$ is minimal and $c \in L$ with $(0_M : K) \leq c$. Since K is minimal and $cK \leq K$, we have either $cK = K$ or $cK = 0_M$. If $cK = K$, then $1_L = (cK : K)$. Since M is principally generated, we have $(cK : K) = c \vee (0_M : K)$, therefore $1_L = (cK : K) = c \vee (0_M : K) = c$. Now, if $cK = 0_M$, then $c \leq (0_M : K)$ and hence $c = (0_M : K)$. This implies, for $c \in L$ with $(0_M : K) \leq c$, either $1_L = c$ or $c = (0_M : K)$. Consequently, $(0_M : K)$ is a maximal element of L. ■

Lemma 2.25 [10]**.** *Let M be a principally generated comultiplication lattice module over a multiplicative lattice L. Then M has a minimal element. In particular, every nonzero element of M has a minimal element.*

Lemma 2.26 [10]**.** *Let M be a principally generated comultiplication lattice module over a multiplicative lattice L. Then $K \in M$ is minimal if and only if $K = (0_M : p) \neq 0_M$ for some maximal element $p \in L$.*

Theorem 2.27. *Let M be a principally generated comultiplication lattice module over a C-lattice L and $S \in Spec^s(M)$. Then $\{S\}$ is closed in $Spec^s(M)$ if and only if S is minimal element of M and $Spec^s_p(M) = \{S\}$.*

Proof. Suppose that S is a minimal element of M and $Spec^s_p(M) = \{S\}$. Then by Lemma 2.24, $(0_M : S)$ is a maximal element of L. Now, suppose that $K \in Cl(\{S\})$. Then by Lemma 2.23 (1), $K \in D^s(S)$, and so $(0_M : S) \leq (0_M : K)$. Since $(0_M : S)$ is a maximal element of L, we have $p = (0_M : S) = (0_M : K)$. Therefore $S, K \in Spec^s_p(M) = \{S\}$. This implies $S = K$ and hence $Cl(\{S\}) = \{S\}$.

Conversely, suppose that $\{S\}$ is closed in $Spec^s(M)$ and S is not minimal. Then by Lemma 2.25, there exists a minimal element $N \leq S$ and so $(0_M : N)$ is a maximal element of L by Lemma 2.24. Since every maximal element is prime, we have $(0_M : N)$ is a prime element of L and therefore $N \in Spec^s(M)$ by Lemma 2.3. Now, we have $N, S \in Spec^s(M)$ with $N \leq S$, therefore $(0_M : S) \leq (0_M : N)$ by Lemma 2.6(3) and so $N \in D^s(S) = Cl(\{S\}) = \{S\}$ by Lemma 2.23. Hence $N = S$, and so by Lemma 2.6(3) $(0_M : N) = (0_M : S)$. Consequently, S is a minimal element of M and $Spec^s_p(M) = \{S\}$. ■

A topological space Z is irreducible if for any decomposition $Z \subseteq A_1 \cup A_2$ with closed subsets A_i of Z with $i = 1, 2$, we have $A_1 = Z$ or $A_2 = Z$. A subset Y of Z is irreducible if it is irreducible as a subspace of Z. An irreducible component of a topological space Z is a maximal irreducible subset of Z. A singleton subset and its closure in Z are irreducible (see [2]).

Lemma 2.28. *Let M be a lattice module over a C-lattice L and $S \in Spec^s(M)$. Then $D^s(S)$ is an irreducible closed subset of $Spec^s(M)$.*

Proof. Note that, for $S \in Spec^s(M)$, the set $\{S\}$ is irreducible and also that $Cl(\{S\})$ irreducible. But by Lemma 2.23(1), we have $Cl(\{S\}) = D^s(S)$. Therefore $D^s(S)$ is an irreducible closed subset of $Spec^s(M)$. ∎

Lemma 2.29 [11]**.** *Let L be a multiplicative lattice and $S \subseteq Spec(L)$. Then S is irreducible if and only if the meet of all elements of S is prime.*

Theorem 2.30. *Let M be a lattice module over a C-lattice L and $Y \subseteq Spec^s(M)$. If $T(Y)$ is a second element of M, then Y is irreducible. Conversely, if Y is irreducible, then $K = \{(0_M : S) | S \in Y\}$ is an irreducible subset of $Spec(L)$ such that $T'(K) = (0_M : T(Y))$ is a prime element of L, where $T'(K)$ is the meet of all elements of K.*

Proof. Suppose that $Y \subseteq Y_1 \cup Y_2$, where Y_1 and Y_2 are two closed subsets of $Spec^s(M)$. Then by Lemma 2.23(1), and Lemma 2.28, there exist $N, K \in Spec^s(M)$ such that $Y_1 = D^s(N)$ and $Y_2 = D^s(K)$. Therefore $Y \subseteq D^s(N) \cup D^s(K)$. By Theorem 2.9(3), we have $D^s(N) \cup D^s(K) = D^s(N \vee K)$, so $Y \subseteq D^s(N \vee K)$. This implies $(0_M : (N \vee K)) \leq (0_M : P)$ for $P \in Y$ and hence $(0_M : (N \vee K)) \leq \wedge_{P \in Y}(0_M : P)$. But by Lemma 2.2(3), we have $\wedge_{P \in Y}(0_M : P) = (0_M : \vee_{P \in Y} P) = (0_M : T(Y))$, therefore $(0_M : (N \vee K)) = (0_M : N) \wedge (0_M : K) \leq (0_M : T(Y))$. Since $T(Y)$ is second, $(0_M : T(Y))$ is prime by Lemma 2.2 and hence quasi-prime, therefore $(0_M : N) \wedge (0_M : K) \leq (0_M : T(Y))$ implies either $(0_M : N) \leq (0_M : T(Y))$ or $(0_M : K) \leq (0_M : T(Y))$. Hence for $P \in Y$, $(0_M : N) \leq (0_M : T(Y)) \leq (0_M : P)$ or $(0_M : K) \leq (0_M : T(Y)) \leq (0_M : P)$. This implies $P \in D^s(N)$ or $P \in D^s(K)$ and hence $Y \subseteq D^s(N) = Y_1$ or $Y \subseteq D^s(K) = Y_2$. Consequently, Y is irreducible.

Conversely, suppose that Y is irreducible. Then $\psi^s(Y) = K' = \{\overline{(0_M : S)} | S \in Y\}$ is an irreducible subset of $Spec(L/(0_M : 1_M))$, since ψ^s is continuous. Therefore $K = \{(0_M : S) | S \in Y\}$ is an irreducible subset of $Spec(L)$ and so $T'(K) = \wedge_{S \in Y}(0_M : S)$ is a prime element of L by Lemma 2.29. But by Lemma 2.31(5), $\wedge_{S \in Y}(0_M : S) = (0_M : \vee_{S \in Y} S) = (0_M : T(Y))$, therefore $T'(K) = (0_M : T(Y))$ is a prime element of L and so $K = \{(0_M : S) | S \in Y\}$ is an irreducible subset of $Spec(L)$ by Lemma 2.29. ∎

Corollary 2.31. *Let M be a comultiplication lattice module over a C-lattice L and $Spec^s_p(M)$ is non-empty, for $p \in Spec(L)$. Then the following statements hold.*

1. *$Spec^s_p(M)$ is irreducible.*
2. *$Spec^s_p(M)$ is an irreducible closed subset of $Spec^s(M)$, if p is a maximal element of L.*

Proof. (1) Suppose that $Spec^s_p(M)$ is non-empty. Then $(0_M : T(Spec^s_p(M)) = (0_M : \vee_{S\in Spec^s_p(M)} S) = \wedge_{S\in Spec^s_p(M)}(0_M : S)$ by Lemma 2.31(5). But $(0_M : S) = p$ for $S \in Spec^s_p(M)$, therefore $(0_M : T(Spec^s_p(M)) = \wedge_{S\in Spec^s_p(M)}(0_M : S) = \wedge_{S\in Spec^s_p(M)} p = p$ and hence $(0_M : T(Spec^s_p(M))$ is a prime element of L. Therefore $T(Spec^s_p(M)$ is a second element of M by Lemma 2.3. Consequently, $Spec^s_p(M)$ is irreducible by Theorem 2.30.

(2) Note that, $Spec^s_p(M)$ is irreducible by (1).
Now, suppose that $Spec^s_p(M)$ is non-empty with maximal element $p \in L$. Then $Spec^s_p(M) = \{S \in Spec^s(M) | (0_M : S) = p\}$. By Lemma 2.6(1), we have $p \leq (0_M : (0_M : p))$, therefore $Spec^s_p(M) = \{S \in Spec^s(M) | p = (0_M : (0_M : p)) = (0_M : S)\}$ by maximality of p and so $Spec^s_p(M) = D^s((0_M : p))$ is closed by Theorem 2.9. Consequently, $Spec^s_p(M)$ is an irreducible closed subset of $Spec^s(M)$. ■

Theorem 2.32. *Let M be a lattice module over a C-lattice L and $Y \subseteq Spec^s(M)$ with $(0_M : T(Y)) = p$ is a prime element of L. Then Y is irreducible if $Spec^s_p(M)$ is non-empty.*

Proof. Suppose that $Spec^s_p(M)$ is non-empty and $Y \subseteq Spec^s(M)$ with $(0_M : T(Y)) = p$ is a prime element of L. Then $(0_M : T(Y)) = p = (0_M : S)$ for each $S \in Spec^s_p(M)$. Therefore $D^s(S) = D^s(T(Y))$ by Lemma 2.10(1) and so $D^s(S) = D^s(T(Y)) = Cl(\{Y\})$ by Lemma 2.20. Hence $Cl(\{Y\})$ is irreducible by Lemma 2.28. Consequently, Y is irreducible. ■

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y, if $Y = Cl(\{y\})$ (see [2]). By Proposition 2.23(1), we observe that, $S \in Spec^s(M)$ is a generic point of the irreducible closed subset $D^s(S)$.

Theorem 2.33. *Let M be a lattice module over a C-lattice L with the surjective natural map ψ^s and $Y \subseteq Spec^s(M)$. Then Y is an irreducible closed subset of $Spec^s(M)$ if and only if $Y = D^s(S)$ for some $S \in Spec^s(M)$. Hence, every irreducible closed subset of $Spec^s(M)$ has a generic point.*

Proof. By Lemma 2.28, $Y = D^s(S)$ is an irreducible closed subset of $Spec^s(M)$.

Conversely, suppose that Y is an irreducible closed subset of $Spec^s(M)$. Then by Theorem 2.30, $(0_M : T(Y)) = p$ is a prime element of L. Since ψ^s is surjective, there exists $S \in Spec^s(M)$ with $(0_M : S) = (0_M : T(Y)) = p$, therefore $D^s(S) = D^s(T(Y))$ by Lemma 2.10(1) and hence $D^s(T(Y)) = Cl(Y)$ by Lemma 2.20. Thus $D^s(S) = Cl(Y)$. Since Y is closed, $Cl(Y) = Y$. Consequently, $D^s(S) = Y$ for some $S \in Spec^s(M)$. ■

Theorem 2.34. *Let M be a principally generated comultiplication lattice module over a C-lattice L. Then $Spec^s(M)$ is a T_1-space if and only if $Spec^s(M) = Min(M)$.*

Proof. Note that, $Min(M) \subseteq Spec^s(M)$. Suppose that $Spec^s(M)$ is a T_1-space. Then for $S \in Spec^s(M)$, $\{S\}$ is closed in $Spec^s(M)$. Therefore $S \in Min(M)$ by Theorem 2.27 and so $Spec^s(M) \subseteq Min(M)$. Consequently, $Spec^s(M) = Min(M)$.

Conversely, suppose that $Spec^s(M) = Min(M)$ and $S \in Spec^s(M)$. Then $(0_M : S) = p$ is a prime element of L by Lemma 2.2, therefore $S \in Spec^s_p(M)$. Now, suppose that $N \in Spec^s_p(M)$. Then N is second element with $(0_M : N) = p$. Since $Spec^s(M) = Min(M)$, N is a minimal element of M. By Lemma 2.6(5), $(0_M : S \vee N) = (0_M : S) \wedge (0_M : N) = p \wedge p = p$, therefore $S \vee N \in Spec^s(M)$ by Lemma 2.3 and hence $S \vee N \in Min(M)$ since $Spec^s(M) = Min(M)$. Thus $N = S \vee N$ and so $S \leq N$. Since N is a minimal element of M, $N = S$ and so $\{S\}$ is closed in $Spec^s(M)$ by Theorem 2.27. Thus every singleton subset is closed and consequently, $Spec^s(M)$ is T_1-space. ∎

Definition 2.35 [12]**.** Topological space Z is spectral space if Z satisfy the conditions: (1) Z is a T_0-space, (2) Z is quasi-compact, (3) The quasi-compact open subsets of Z are closed under finite intersection and form an open base and (4) Each irreducible closed subset of Z has a generic point.

The following Theorem follows immediately from Theorem 2.17, Theorem 2.18 and Theorem 2.33.

Theorem 2.36. *Let M be a lattice module over a C-lattice L and ψ^s be the surjective natural map. Then $Spec^s(M)$ is spectral if and only if it is T_0-space.*

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Received 31 August 2016
Revised 8 January 2017