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ZERO-DIVISOR GRAPHS OF REDUCED RICKART $*$-RINGS

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Abstract

For a ring A with an involution $*$, the *zero-divisor graph* of A, $\Gamma^*(A)$, is the graph whose vertices are the nonzero left zero-divisors in A such that distinct vertices x and y are adjacent if and only if $xy^* = 0$. In this paper, we study the zero-divisor graph of a Rickart $*$-ring having no nonzero nilpotent element. The distance, diameter, and cycles of $\Gamma^*(A)$ are characterized in terms of the collection of prime strict ideals of A. In fact, we prove that the clique number of $\Gamma^*(A)$ coincides with the cellularity of the hull-kernel topological space $\Sigma(A)$ of the set of prime strict ideals of A, where cellularity of the topological space is the smallest cardinal number m such that every family of pairwise disjoint non-empty open subsets of the space have cardinality at most m.

Keywords: reduced ring, Rickart $*$-ring, zero-divisor graph, prime strict ideals.

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1. Introduction

An *involution* '$*$', on an associative ring A is a mapping $*: A \to A$ such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$, for all $a, b \in A$. A ring with an involution $*$ is called a $*$-*ring*. Clearly, identity mapping is an involution if and only if the ring is commutative. An element e in a $*$-ring A is a *projection* if it is self-adjoint (i.e., $e = e^*$) and idempotent (i.e., $e^2 = e$). By $\tilde{A}$, we denote the set of all projections in A. Let S be a nonempty subset of A. We write $r(S) = \{a \in A \mid sa = 0, \ \forall\ s \in S\}$, the *right annihilator* of S in A. A *Rickart* $*$-*ring* is a $*$-ring in which right annihilator of every element is generated, as a right ideal, by a projection in A. Every Rickart $*$-ring contains a unity. For each element a in a Rickart $*$-ring, there is a unique projection e such that $ae = a$ and $ax = 0$ if and only if $ex = 0$; called the *right projection* of a, $RP(a)$. In fact, $r(\{a\}) = (1 - RP(a))A$. Similarly, the left projection, $LP(a)$, is defined for each element a in a Rickart $*$-ring. The set of projections in a Rickart $*$-ring A forms a lattice, denoted by $L(\tilde{A})$, under the partial order '$e \leq f$ if and only if $e = fe$'. In that case, $e \vee f = f + RP(e(1-f))$ and $e \wedge f = e - LP(e(1-f))$ (see Berberian [5]). An ideal(subring) I is a $*$-*ideal*($*$-*subring*) if $x \in I$ implies $x^* \in I$. An ideal I of a Rickart $*$-ring A is called a *strict ideal* if $x \in I$ implies $RP(x) \in I$. A proper strict ideal P of a Rickart $*$-ring A is called *prime strict*, if for strict ideals I, J of A, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. Let $\Sigma(A)$ denote the set of all prime strict ideals of a Rickart $*$-ring A. Thakare *et al.* [19] studied the hull-kernel topology on $\Sigma(A)$, where A is a reduced Rickart $*$-ring (i.e., a Rickart $*$-ring having no nonzero nilpotent element).

Being motivated from Beck [4], Anderson *et al.* [3] defined the zero-divisor graph for a commutative ring. Further study of this graph, such as connectedness, diameter, girth, etc. is found in [1, 2, 11]. Later on, Redmond [18] generalized the concept of the zero-divisor graph to a non-commutative ring in the following way: Let R be a ring. Then the undirected zero-divisor graph of R, denoted by $\overline{\Gamma(R)}$, is the graph whose vertices are the non-zero zero-divisors of R, and there is an edge between two distinct vertices a and b if and only if either $ab = 0$ or $ba = 0$. This concept is then generalized to semigroups by DeMeyer *et al.* in [7, 8], to semirings by Dolžan in [6]. Further, this concept is well studied in ordered structures such as lattices, meet-semilattices, posets (see [9, 13, 15, 16]). Recently, Sen Gupta *et al.* [10] defined a new graph for a ring with unity by extending the definition of the usual zero-divisor graph. In [17], the authors extended the concept of a zero-divisor graph to $*$-rings as follows: Let A be a $*$-ring. We associate a simple undirected graph $\Gamma^*(A)$ to A whose vertex set is $V(\Gamma^*(A)) = \{a(\neq 0) \in A \mid ab = 0$, for some nonzero $b \in A\}$ (i.e., nonzero left zero-divisors) and two distinct vertices x and y are adjacent if and only if $xy^* = 0$. The zero-divisor graph of a $*$-ring A is denoted by $\Gamma^*(A)$. In the case of reduced Rickart $*$-rings, this graph is the same as the graph introduced by Nimbhorkar in [12].

In this paper, we continue our study of the zero-divisor graphs of Rickart $*$-rings that was started in [14]. Here particularly we deal with the reduced Rickart $*$-rings. The open subsets of the hull-kernel topological space $\Sigma(A)$ of the set of prime strict ideals of A are used to give algebraic and topological characterizations of distances, diameters, and cycles. We show that the clique number of $\Gamma^*(A)$ and the cellularity of $\Sigma(A)$ coincide. In fact, we have obtained the main results of Samei [11] to reduced Rickart $*$-rings.

For undefined concepts in Rickart $*$-rings and graphs, see [5, 20] respectively.

2. Main results

We recall some definitions that are used in the sequel.

Let A be a reduced Rickart $*$-ring and $\Sigma(A)$ be the set of all prime strict ideals in A. Then for $x \in A$ and any subset S of A, we define $H(S) = \{P \in \Sigma(A) \mid S \subseteq P\}$, $H(x) = H(\{x\})$, $B(S) = \{P \in \Sigma(A) \mid S \nsubseteq P\}$, $B(x) = B(\{x\})$ and $< x >=$ the ideal generated by x. For any subset T of $\Sigma(A)$, the kernel of T is the set $K(T) = \bigcap\{P \mid P \in T\}$, and the hull of $C \subseteq A$ is $H(C)$.

At the outset, we list the following observations made in [19, page 67].

Lemma 2.1. *Let A be a reduced Rickart $*$-ring and $\Sigma(A)$ is the set of all prime strict ideals in A. Then*

(1) $B(x) = B(RP(x)) = B(< RP(x) >)$.

(2) $\bigcup_{i\in\Lambda} B(x_i) = B(\{x_i | i \in \Lambda\})$, $B(I \cap J) = B(I) \cap B(J)$, $B(0) = \emptyset$ *and* $B(A) = B(1) = \Sigma(A)$.

(3) $H(x) = H(< x >) = H(< RP(x) >)$, $H(0) = \Sigma(A), H(A) = H(1) = \emptyset$, $H(\cup E_i) = \cap H(E_i)$ *and* $H(I \cap J) = H(I) \cup H(J)$, *where each* E_i, I *and* J *are ideals of* A.

(4) *Every prime strict ideal contains* e *or* $1-e$ *but not both, for any projection* e.

(5) $\bigcap\{P \mid P \in \Sigma(A)\} = \{0\}$.

We note from the above Lemma 2.1 that the sets $\{B(x) \mid x \in A\}$ form a basis for open sets and define a topology on $\Sigma(A)$ called the *hull-kernel topology.* The complement of $B(x)$ is $H(x)$ for any $x \in A$.

Remark 2.2. Following are the simple properties of a reduced Rickart $*$-ring A.

(1) For $a, b \in A$, $ab = 0$ if and only if $ba = 0$. For, let $ab = 0$. Then $(ba)^2 = baba = 0$. Since A is reduced, we get $ba = 0$.

(2) Every projection in A is central. For, if e is a projection, then $e(1-e) = (1-e)e = 0$ and $(ex(1-e))^2 = ex(1-e)ex(1-e) = 0$ for any $x \in A$.

As is reduced, $ex(1-e) = 0$. Therefore $ex = exe$ and $ex^* = ex^*e$, i.e., $xe = exe = ex$, for all $x \in A$. Thus e is a central projection.

(3) For any $a \in A$, $RP(a) = RP(a^*)$, since every projection is central, we have $RP(a) = LP(a)$. This together with $RP(a^*) = LP(a)$ gives $RP(a) = RP(a^*)$.

(4) $RP(xy) = RP(x)RP(y)$, for any $x, y \in A$. For, since $RP(x)$ and $RP(y)$ both are central, we get $xyRP(x)RP(y) = xRP(x)yRP(y) = xy$. Let $xyz = 0$. By definition of $RP(x)$, we get $RP(x)yz = 0$. This together with centrality of $RP(x)$ imply that $yRP(x)z = 0$. Again by definition of right projection, we get $RP(y)RP(x)z = 0$, i.e., $RP(x)RP(y)z = 0$. Note that product of two commuting projections is a projection. Therefore by Remark 2.2(2), the projection $RP(x)RP(y)$ satisfies the conditions of $RP(xy)$, hence $RP(xy) = RP(x)RP(y)$.

(5) For any projection $e \in A$, the ideal $< e >$ is a strict ideal. For, if $x \in < e >$, then $xe = x$. Hence $RP(x) = RP(xe) = RP(x)e \in < e >$.

Lemma 2.3. *Let A be a reduced Rickart $*$-ring. Then the following statements hold.*

(1) *Every strict ideal is a $*$-ideal.*

(2) *Let P be a strict ideal. Then P is a prime strict ideal if and only if $ab \in P$ implies $a \in P$ or $b \in P$.*

Proof. (1) Let I be a strict ideal of A and $a \in I$. Hence $RP(a) \in I$. Then $a^* = a^*RP(a^*) = a^*RP(a) \in I$, by Remark 2.2(3). Thus I is a $*$-ideal.

(2) Let P be a strict ideal. Suppose that P is a prime strict ideal and $ab \in P$ with $a \notin P$. We claim that $b \in P$. Since $ab \in P$ and P is a strict ideal, we get $RP(ab) \in P$. By Remark 2.2(4), $RP(ab) = RP(a)RP(b)$. Observe that $< RP(a) > \cdot < RP(b) >=< RP(a)RP(b) >$. Thus $I =< RP(a) >$ and $J =< RP(b) >$ are two strict ideals (by Remark 2.2(5)) such that $IJ \subseteq P$ with $I \nsubseteq P$. Since P is a prime strict ideal, we get $J \subseteq P$. Hence $RP(b) \in P$. Thus $b = bRP(b) \in P$. Conversely, suppose that $ab \in P$ implies $a \in P$ or $b \in P$. Let I and J be strict ideals of A such that $IJ \subseteq P$ with $I \nsubseteq P$. We claim that $J \subseteq P$. On the contrary assume that $J \nsubseteq P$. Let $a, b \in A$ be such that $a \in I \setminus P$ and $b \in J \setminus P$. Then $ab \in IJ \subseteq P$ with $a, b \notin P$, a contradiction. Hence $J \subseteq P$. Thus P is a prime strict ideal. ■

For a $*$-ring A, two graphs $\Gamma^*(A)$ and $\overline{\Gamma(A)}$ need not be isomorphic, see Example 2.6.

Theorem 2.4. *For a reduced Rickart $*$-ring A, $\Gamma^*(A) \cong \overline{\Gamma(A)}$.*

Recall that, a graph G is said to be *connected* if there is a path between any two distinct vertices of G. For two vertices x and y of G, we define $d(x, y)$ to be the length of the shortest path from x to y (($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is defined as $diam(G) = sup\{d(x, y) \mid x$ and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of the shortest cycle in G ($gr(G) = \infty$ if G contains no cycle). A subset S of $V(G)$ is called a *clique* if any two distinct vertices of S are adjacent; the *clique number*, $\omega(G)$, is the least upper bound of the size of the cliques in G.

Theorem 2.5. *Let A be a reduced Rickart $*$-ring such that $V(\Gamma^*(A)) \neq \emptyset$. Then $\Gamma^*(A)$ is connected with $diam(\Gamma^*(A)) \leq 3$.*

Proof. Since A is a reduced ring, by Remark 2.2(1), $ab = 0$ if and only if $ba = 0$ if and only if $ab^* = 0$. Hence the proof follows by Redmond [18, Theorem 3.2]. ∎

Following example shows that Rickartness is necessary in Theorem 2.4 and Theorem 2.5.

Example 2.6. Let $A = \mathbb{Z}_3 \times \mathbb{Z}_3$ with $(a, b)^* = (b, a)$ as an involution. Then A is a reduced but not a Rickart $*$-ring (since $(0, 1)^*(0, 1) = (0, 0)$ with $(0, 1) \neq (0, 0)$). Here $V(\Gamma^*(A)) = V(\overline{\Gamma(A)}) = \{a = (0, 1), b = (1, 0), c = (0, 2), d = (2, 0)\}$. Figure 1 shows that $\Gamma^*(A)$ and $\overline{\Gamma(A)}$ are non-isomorphic.

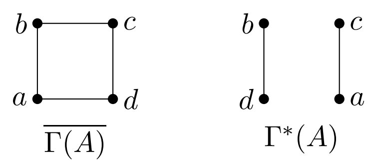

Figure 1.

Next result gives an equivalent condition for adjacency in $\Gamma^*(A)$.

Lemma 2.7. *Let A be a reduced Rickart $*$-ring such that $V(\Gamma^*(A)) \neq \emptyset$. Then the vertices a and b are adjacent in $\Gamma^*(A)$ if and only if $B(a) \cap B(b) = \emptyset$.*

Proof. Suppose a and b are adjacent in $\Gamma^*(A)$. Let P be any prime strict ideal of A. Then $ab^* = 0 \in P$. Since P is a prime strict ideal, we get either $a \in P$ or $b^* \in P$. Since P is a strict ideal, $b^* \in P$ whenever $b \in P$. Hence every prime strict ideal either contains a or b. Therefore $B(a) \cap B(b) = \emptyset$. Conversely, suppose that $B(a) \cap B(b) = \emptyset$. Hence for any prime strict ideal P, we have either $a \in P$ or $b \in P$, i.e., $a \in P$ or $b^* \in P$. Consequently $ab^* \in P$, for all $P \in \Sigma(A)$, hence $ab^* \in \bigcap\{P \mid P \in \Sigma(A)\} = \{0\}$. Thus $ab^* = 0$. ∎

Observe that, for each projection e in a reduced Rickart $*$-ring, by Lemma 2.1(4), we have $B(e) = H(1-e)$. This leads to the following:

Corollary 2.8. *Let A be a reduced Rickart $*$-ring such that $V(\Gamma^*(A)) \neq \emptyset$. Then the vertices a and b are adjacent in $\Gamma^*(A)$ if and only if $B(a) \subseteq H(b)$.*

Remark 2.9. In a reduced Rickart $*$-ring, $H(1 - RP(a))$ = set of prime strict ideals containing $1 - RP(a) = B(RP(a)) = B(a)$.

The following result characterizes the distance between two vertices. We use the notation $a \leftrightarrow b$ to indicate that a and b are adjacent.

Proposition 2.10. *Let A be a reduced Rickart $*$-ring and $a, b, c \in V(\Gamma^*(A))$ be distinct vertices. Then*

(1) *c is adjacent to both a and b if and only if $B(a) \cup B(b) \subseteq H(c)$.*

(2) *$d(a, b) = 1$ if and only if $B(a) \cap B(b) = \emptyset$.*

(3) *$d(a, b) = 2$ if and only if $B(a) \cap B(b) \neq \emptyset$ and $B(a) \cup B(b) \neq \Sigma(A)$.*

(4) *$d(a, b) = 3$ if and only if $B(a) \cap B(b) \neq \emptyset$ and $B(a) \cup B(b) = \Sigma(A)$.*

Proof. (1) Suppose that c is adjacent to both a and b. By Corollary 2.8, $B(a) \subseteq H(c)$ and $B(b) \subseteq H(c)$. Hence $B(a) \cup B(b) \subseteq H(c)$. Conversely, suppose that $B(a) \cup B(b) \subseteq H(c)$. Hence $B(a) \subseteq H(c)$ and $B(b) \subseteq H(c)$. Consequently, $B(a) \cap B(c) = \emptyset$ and $B(b) \cap B(c) = \emptyset$. By Lemma 2.7, c is adjacent to a and c is adjacent to b.

(2) Follows from Lemma 2.7.

(3) Suppose that $d(a, b) = 2$. Let $a \leftrightarrow d \leftrightarrow b$ be a path of length 2. Since a and b are non-adjacent, $B(a) \cap B(b) \neq \emptyset$. Here d is adjacent to both a and b, hence by (1) above, $B(a) \cup B(b) \subseteq H(d)$. Therefore $B(a) \cup B(b) \neq \Sigma(A)$, otherwise $H(d) = \Sigma(A)$ which yields $d \in \bigcap_{P \in \Sigma(A)} P = \{0\}$ giving $d = 0$, a contradiction. Thus $B(a) \cap B(b) \neq \emptyset$ and $B(a) \cup B(b) \neq \Sigma(A)$. Conversely, suppose that $B(a) \cap B(b) \neq \emptyset$ and $B(a) \cup B(b) \neq \Sigma(A)$. By (2) above, a and b are non-adjacent. Let $P \in \Sigma(A)$ be such that $P \notin B(a) \cup B(b)$. Since $a \in P$, we get $RP(a) \in P$. By Lemma 2.1(4), $1 - RP(a) \notin P$. Similarly, $RP(b) \in P$ and $1 - RP(b) \notin P$. Therefore $(1 - RP(a))(1 - RP(b)) \notin P$. Consequently, $c = (1 - RP(a))(1 - RP(b)) \neq 0$. Then $a \leftrightarrow c \leftrightarrow b$ is a path. Therefore $d(a, b) = 2$.

(4) Follows from (2), (3) above and Theorem 2.5. ∎

Now, in the following result, we characterize the diameter of $\Gamma^*(A)$.

Theorem 2.11. *For a finite reduced Rickart $*$-ring A following statements hold:*

(1) *$diam(\Gamma^*(A)) = 1$ if and only if $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with identity mapping as an involution.*

(2) *$diam(\Gamma^*(A)) = 2$ if and only if $|V(\Gamma^*(A))| \geq 3$ and $L(\tilde{A})$ contains exactly two atoms.*

(3) *$diam(\Gamma^*(A)) = 3$ if and only if $L(\tilde{A})$ contains at least three atoms.*

Proof. (1) One way is clear. Conversely, suppose that $diam(\Gamma^*(A)) = 1$. Therefore $\Gamma^*(A)$ is a complete graph. Let e be a non-trivial projection in A (since $V(\Gamma^*(A)) \neq \emptyset$, such e exists). Then $A = eA \oplus (1-e)A$. By completeness of $\Gamma^*(A)$, we get $eA = \mathbb{Z}_2$ and $(1-e)A = \mathbb{Z}_2$. Thus $A = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Further, by Remark 2.2(5), eA is strict ideal. Hence for $x \in eA \setminus \{e\}$, we have $RP(x) = RP(x^*) = e$. This gives $x^* = x^*e \in eA$, which gives $x^* = x$. Therefore the involution on eA and $(1-e)A$, and hence on A is identity.

(2) Let $diam(\Gamma^*(A)) = 2$. Clearly $|V(\Gamma^*(A))| \geq 3$. Also, for a finite Rickart $*$-ring A with $V(\Gamma^*(A)) \neq \emptyset$, the lattice $L(\tilde{A})$ contains at least two atoms. Suppose e, f and g are three atoms in $L(\tilde{A})$, hence $ef = fg = eg = 0$. If $e + f = 1$, then $g = eg + fg = 0$, a contradiction. Hence $e + f \neq 1$. Similarly, $f + g \neq 1$ and $e + g \neq 1$. This gives $(1-e)(1-f) \neq 0$, otherwise $e + f = 1$. If there is a projection h such that $(1-e) \leftrightarrow h \leftrightarrow (1-f)$ is a path, then $h = he = hf$ implies $h = hf = hef = 0$. Thus $1-e$ and $1-f$ are nonadjacent and there is no path of length 2 joining them, hence $d(1-e, 1-f) \geq 3$, a contradiction to the fact that $diam(\Gamma^*(A)) = 2$. Therefore $L(\tilde{A})$ contains exactly two atoms. Conversely, suppose that $|V(\Gamma^*(A))| \geq 3$ and $L(\tilde{A})$ contains exactly two atoms, say e_1 and e_2. Let a and b be any two vertices of $\Gamma^*(A)$. We consider the following two cases.

Case (i) Suppose $RP(a)$ and $RP(b)$ have common atom, say e_1, i.e., $e_1 \leq RP(a)$ and $e_1 \leq RP(b)$. Then $e_1 \leq RP(a)RP(b)$, hence $RP(a)RP(b) \neq 0$. On the other hand, $e_2 \leq 1 - RP(a)$ and $e_2 \leq 1 - RP(b)$ which gives $(1-RP(a))(1-RP(b)) \neq 0$. Hence $RP(a) \leftrightarrow (1-RP(a))(1-RP(b)) \leftrightarrow RP(b)$ is a path joining $RP(a)$ and $RP(b)$. Thus $d(a,b) = 2$.

Case (ii) Suppose $RP(a)$ and $RP(b)$ don't have common atom. Without loss of generality, assume that $e_1 \leq RP(a)$ and $e_2 \leq RP(b)$. Then $e_2RP(a) = e_1RP(b) = 0$. This gives $e_1RP(a)RP(b) = e_2RP(a)RP(b) = 0$, hence $e_1 \nleq RP(a)RP(b)$ and $e_2 \nleq RP(a)RP(b)$. Since $L(\tilde{A})$ contains exactly two atoms, we must have $RP(a)RP(b) = 0$. Hence $d(a,b) = 1$.

Thus $diam(\Gamma^*(A)) \leq 2$. Since $|V(\Gamma^*(A))| \geq 3$ and $L(\tilde{A})$ contains exactly two atoms, there exists two vertices x and y such that $RP(x)$ and $RP(y)$ contains a common atom in $L(\tilde{A})$. Hence by Case(i) above, we get $d(x,y) = 2$. Therefore $diam(\Gamma^*(A)) = 2$.

(3) If $diam(\Gamma^*(A)) = 3$, then by (2), $L(\tilde{A})$ contains at least three atoms. Conversely, if $L(\tilde{A})$ contains three atoms, then as in the proof of (2), $d(1-e, 1-f) = 3$. By Theorem 2.5, $diam(\Gamma^*(A)) = 3$. ∎

It is known that the girth of the zero-divisor graph is either 3 or 4 (if it contains a cycle); see [17]. The following theorem characterizes the diameter and the girth of $\Gamma^*(A)$ in terms of the number of prime strict ideals of A.

Theorem 2.12. *Let A be a reduced Rickart $*$-ring such that $V(\Gamma^*(A)) \neq \emptyset$.*

(1) *If $A \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with identity involution, then $diam(\Gamma^*(A)) = \min\{|\Sigma(A)|, 3\}$.*

(2) *If $|\Sigma(A)| = 2$, then $gr(\Gamma^*(A)) = 4$ or ∞; otherwise $gr(\Gamma^*(A)) = 3$.*

Proof. (1) Since $V(\Gamma^*(A)) \neq \emptyset$, we get $|\Sigma(A)| \geq 2$. Also, $A \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ which gives $diam(\Gamma^*(A)) \neq 1$. Further, by Theorem 2.5, $diam(\Gamma^*(A)) \leq 3$. It is enough to show that $diam(\Gamma^*(A)) = 3$ if and only if $|\Sigma(A)| \geq 3$. Suppose that $|\Sigma(A)| \geq 3$ and P_1, P_2, P_3 be three distinct prime strict ideals of A. Since $\Sigma(A)$ is Hausdorff [19, Lemma 3.4], there are $a_i \in A$ such that $P_i \in B(a_i)$ and $B(a_i) \cap B(a_j) = \emptyset$, for $i \neq j$ and $i, j \in \{1, 2, 3\}$. By Lemma 2.7, we have a_i and a_j are adjacent in $\Gamma^*(A)$ for $i \neq j$. This implies that $gr(\Gamma^*(A)) = 3$. By Remark 2.9, $P_1 \in H(1 - RP(a_1))$ and $P_2 \in H(1 - RP(a_2))$. Since $a_1 a_3 = a_2 a_3 = 0 \in P_3$ and $a_3 \notin P_3$, we have $a_1, a_2 \in P_3$. Hence $1 - RP(a_1), 1 - RP(a_2) \notin P_3$. Therefore $P_3 \in B(1 - RP(a_1)) \cap B(1 - RP(a_2))$. Since $B(a_1) \cap B(a_2) = \emptyset$, we get $H(RP(a_1)) \cup H(RP(a_2)) = \Sigma(A)$. This together with $H(RP(a_i)) = B(1 - RP(a_i))$ for $i = 1, 2$, gives $B(1 - RP(a_1)) \cup B(1 - RP(a_2)) = \Sigma(A)$. By Proposition 2.10, $d(1 - RP(a_1), 1 - RP(a_2)) = 3$. Therefore $diam(\Gamma^*(A)) = 3$. Conversely, suppose that $diam(\Gamma^*(A)) = 3$. Let $a \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow b$ is a path of length 3 in $\Gamma^*(A)$. Since $V(\Gamma^*(A)) \neq \emptyset$ and $A \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, A has at least two prime strict ideals. On the contrary assume that P_1 and P_2 are the only prime strict ideals of A. Observe that, for no vertex x we have $B(x) = \{P_1, P_2\}$. Without loss of generality, assume that $B(a) = \{P_1\}$. Since a is adjacent to a_1 and nonadjacent to b, we must have $B(a_1) = \{P_2\}$ and $B(b) = \{P_1\}$. Which gives $B(a_1) \cap B(b) = \emptyset$, i.e., a_1 is adjacent to b, a contradiction to the fact that $d(a, b) = 3$. Therefore $|\Sigma(A)| \geq 3$.

(2) By the proof of part (1), $|\Sigma(A)| \geq 3$ implies that $gr(\Gamma^*(A)) = 3$. Now suppose $|\Sigma(A)| = 2$, say $\Sigma(A) = \{P_1, P_2\}$. Since $V(\Gamma^*(A)) \neq \emptyset$, A contains at least one non-trivial projection, say, e. Further, A is a reduced ring, hence e is central. Therefore $A = eA \oplus (1 - e)A$. Next, we claim that eA does not contain nonzero zero-divisor. On the contrary assume that, $xy = 0$ for some nonzero $x, y \in eA$. Then x and y are adjacent in $\Gamma^*(A)$, giving $B(x) \cap B(y) = \emptyset$. Since $\Sigma(A) = \{P_1, P_2\}$ and $H(e) \notin \{\emptyset, \Sigma(A)\}$, without loss of generality assume that, $H(e) = \{P_1\}$. Then $x, y \in eA \subseteq P_1$. Since $H(x), H(y) \notin \{\emptyset, \Sigma(A)\}$, we get $H(x) = H(y) = \{P_1\}$, hence $B(x) = B(y) = \{P_2\}$, a contradiction to the fact that $B(x) \cap B(y) = \emptyset$. Therefore eA does not contain nonzero zero-divisor. Similarly, $(1 - e)A$ does not contain nonzero zero-divisor. If either $|eA| = 2$ or $|(1 - e)A| = 2$, then $gr(\Gamma^*(A)) = \infty$; else it is easy to see that $gr(\Gamma^*(A)) = 4$. ■

The associated number $e(a)$- the *eccentricity of* a denoted by $e(a)$, of a vertex of a graph G is defined to be $e(a) = max\{d(a,b) \mid a \neq b\}$.

Remark 2.13. Note that, for any vertex a of $\Gamma^*(A)$, $d(a,b) \leq 3$ for any $b \in V(\Gamma^*(A)) \setminus \{a\}$. Therefore $e(a) \leq 3$.

Theorem 2.14. *In a reduced Rickart $*$-ring A,*

(1) $e(a) = 1$ *if and only if a is a projection and* $|\Sigma(A)| = |<a>| = 2$.
If $e(a) \neq 1$. *Then*

(2) $e(a) = 2$ *if and only if* $|B(a)| = 1$.

(3) $e(a) = 3$ *if and only if* $|B(a)| > 1$.

Proof. (1) Let $e(a) = 1$, i.e., a is adjacent to all the other vertices of $\Gamma^*(A)$. If a is not a projection, then we get a contradiction since a and $RP(a)$ are always nonadjacent in $\Gamma^*(A)$. Therefore a is a projection. Let $b \in <a>$ with $b \neq a$, hence $b = ab$. Since a and b are adjacent, we get $ab = 0$. Consequently, $b = ab = 0$. Therefore $|<a>| = 2$. Also, by Remark 2.2(5), $<a>$ is a strict ideal of A. Next, let P be a prime strict ideal containing a and $y \in P \setminus \{a\}$. Since P is prime strict ideal, we have $RP(y) \neq 1$. If $R(y) \neq 0$, then a is adjacent to both $RP(y)$ and $1 - RP(y)$, which gives $a = 0$, a contradiction. Hence $RP(y) = 0$, consequently $y = 0$. Therefore $P = \{0, a\}$. Thus $\{0, a\}$ is a prime strict ideal. Now, let $P_1 \in B(a)$. Then for any $x \in V(\Gamma^*(A)) \setminus \{a\}$, $aRP(x) = 0$. Hence $aRP(x) \in P_1$ with $a \notin P_1$ and since P_1 is a prime strict ideal, we have $RP(x) \in P_1$ which yields $x \in P_1$, for all $x \in V(\Gamma^*(A)) \setminus \{a\}$. Therefore $V(\Gamma^*(A)) \setminus \{a\} \subseteq P_1$. On the other hand, for any nonzero $y \in P_1$, $RP(y) \neq 1$ and $y(1 - RP(y)) = 0$. Hence $y \in V(\Gamma^*(A))$. Therefore $P_1 = \{0\} \cup [V(\Gamma^*(A)) \setminus \{a\}]$. Thus $|\Sigma(A)| = 2$. Conversely, suppose that a is a projection and $|\Sigma(A)| = |<a>| = 2$. Let P_1 and P_2 be two prime strict ideals of A with $a \in P_1 \setminus P_2$. Hence $1 - a \in P_2$. Then for any $x \in P_1, RP(x)(1-a) \in P_1 \cap P_2 = \{0\}$, which gives $RP(x) = RP(x)a$. This yields $x = xa \in <a>$. Therefore $P_1 = \{0, a\}$. Since $|<a>| = 2$, there is no b different from a such that $RP(b) = a$. Hence for any $b \in V(\Gamma^*(A)) \setminus \{a\}$, we get $RP(b) \in P_2$. This gives $RP(b)a \in P_1 \cap P_2 = \{0\}$, giving $RP(b)a = 0$, i.e., b and a are adjacent. Thus $e(a) = 1$.

For (2) and (3), suppose that $e(a) \neq 1$. Then by Theorem 2.11, $e(a) = 2$ or 3.

Let $|B(a)| = 1$ and $P \in \Sigma(A)$ be such that $a \notin P$. As $e(a) \neq 1$ there exists $b \in V(\Gamma^*(A)) \setminus \{a\}$ such that a and b are non-adjacent. Hence $B(a) \cap B(b) \neq \emptyset$. Now $B(a) \cap B(b) \subseteq B(a)$ which is a singleton set, hence we get $B(a) \subseteq B(b)$. Therefore $B(a) \cup B(b) = B(b) \neq \Sigma(A)$. By Theorem 2.11, $e(a) = 2$.

Now suppose $|B(a)| > 1$ and $P_1, P_2 \in \Sigma(A)$ such that $a \notin P_1 \cup P_2$. Let $b \in P_1 \setminus P_2$ then $b \in P_1$ and $b \notin P_2$. Therefore $P_2 \in B(a) \cap B(b)$, which gives

$d(a,b) \neq 1$. Let $P \in B(ab)$, i.e., $ab \notin P$. Since P is a prime strict ideal, we have $a \notin P$ and $b \notin P$. Hence $P \in B(a)$. Thus $B(ab) \subseteq B(a)$ which gives $B(a) \cup B(1 - RP(a)RP(b)) = \Sigma(A)$. If $RP(a)(1 - RP(a)RP(b)) = 0$, then $RP(a) = RP(a)RP(b) \in P_1$ giving $a \in P_1$, a contradiction. Hence a and $1 - RP(a)RP(b)$ are non-adjacent in $\Gamma^*(A)$. By Theorem 2.11, $d(a,(1 - aRP(b))) = 3$. Thus $e(a) = 3$. ■

A graph G is *triangulated* if each vertex of G is a vertex of a triangle. Let $Z(A)$ be the set of zero-divisors in A. Now, we characterize the triangulated zero-divisor graphs.

Theorem 2.15. *For a reduced Rickart $*$-ring A,*

(1) *$\Gamma^*(A)$ is triangulated if and only if $|H(a)| > 1$, for all $a \in V(\Gamma^*(A))$.*

(2) *If $2 \notin Z(A)$, then every vertex of $\Gamma^*(A)$ is a 4-cycle-vertex.*

Proof. (1) Suppose that $\Gamma^*(A)$ is triangulated. On the contrary, assume that there is a vertex a such that a belongs to unique prime strict ideal, say, Q of A, i.e., $|H(a)| = 1$. Since $\Gamma^*(A)$ is triangulated, there exists $b, c \in V(\Gamma^*(A))$ such that $ab^* = bc^* = ca^* = 0$. Therefore $B(a)\cap B(b) = B(a)\cap B(c) = B(b)\cap B(c) = \emptyset$. This gives $B(a) \subseteq H(b)$ and $B(c) \subseteq H(a) \cap H(b)$, hence $H(a) \cap H(b) \neq \emptyset$. Also, $H(a) \cap H(b) \subseteq H(a)$ and $H(a)$ is a singleton set. Therefore $H(a) \subseteq H(b)$. Now $\Sigma(A) = B(a) \cup H(a) \subseteq H(b)$ gives $H(b) = \Sigma(A)$. Hence $b \in \bigcap_{P\in\Sigma(A)} P = \{0\}$ giving $b = 0$, a contradiction. Therefore $|H(a)| > 1$, for all $a \in V(\Gamma^*(A))$.

Conversely, suppose that $|H(a)| > 1$, for all $a \in V(\Gamma^*(A))$. Let $a \in V(\Gamma^*(A))$ and P_1, P_2 be two distinct prime strict ideals containing a. Let $e = RP(a)$ and $I = (1 - e)A$, i.e., $I = r(\{a\})$. Then $e \in P_1, P_2$. Since $A = eA \oplus (1 - e)A$, if $P_1 \cap I = \{0\}$ and $P_2 \cap I = \{0\}$, we get $P_1 \subseteq eA$ and $P_2 \subseteq eA$. Then for any $x \in P_1$, we get $x = xe \in P_2$, i.e; $P_1 \subseteq P_2$. Similarly $P_2 \subseteq P_1$. This gives $P_1 = P_2$, a contradiction. Hence $P_1 \cap I \neq \{0\}$ or $P_2 \cap I \neq \{0\}$. Without loss of generality, suppose $P_1 \cap I \neq \{0\}$. Let $b \in P_1 \cap I$ with $b \neq 0$ and $f = RP(b)$. Then $(1 - e)f = f$ giving $ef = 0$, hence a and b are adjacent. If $(1 - e)(1 - f) = 0$, then $1 - e = (1 - e)f \in P_1$. Which gives $1 = 1 - e + e \in P_1$, a contradiction. Hence $(1 - e)(1 - f) \neq 0$ and $a \leftrightarrow b \leftrightarrow (1 - e)(1 - f) \leftrightarrow a$ is a triangle. Thus every vertex is a vertex of a triangle.

(2) Suppose that $2 \notin Z(A)$. Let $a \in V(\Gamma^*(A))$. Then there exists nonzero $b \in A$ such that $ab^* = 0$. Since A is reduced and $2 \notin Z(A)$, we get $2a \neq b$, $a \neq 2b$. Therefore $a, b, 2a, 2b$ all are distinct with $ab^* = 0 = (2a)b^* = (2a)(2b)^* = 2a(2b)^*$. Thus $a \leftrightarrow b \leftrightarrow 2a \leftrightarrow 2b \leftrightarrow a$ is a 4-cycle containing a. ■

If a and b are two vertices in $\Gamma^*(A)$, by $c(a,b)$ we mean the length of the smallest cycle containing a and b. For every two vertices a and b, all possible cases for $c(a,b)$ are given in the following theorem.

Theorem 2.16. *Let A be a reduced Rickart $*$-ring, $a, b \in V(\Gamma^*(A))$ and $2 \notin Z(A)$.*

(1) $c(a,b) = 3$ *if and only if* $B(a) \cap B(b) = \emptyset$ *and* $RP(a) + RP(b) \neq 1$.

(2) $c(a,b) = 4$ *if and only if either* $B(a) \cap B(b) = \emptyset$ *and* $RP(a) + RP(b) = 1$ *or* $B(a) \cap B(b) \neq \emptyset$ *and* $d(a,b) = 2$.

(3) $c(a,b) = 6$ *if and only if* $d(a,b) = 3$.

Proof. (1) Suppose that $B(a) \cap B(b) = \emptyset$ and $RP(a) + RP(b) \neq 1$. Then a and b are adjacent, hence $RP(a)RP(b) = 0$. If $(1 - RP(a))(1 - RP(b)) = 0$, then $RP(a) + RP(b) = 1$, a contradiction. Therefore $(1 - RP(a))(1 - RP(b)) \neq 0$ which yields $a \leftrightarrow (1 - RP(a))(1 - RP(b)) \leftrightarrow b \leftrightarrow a$ a $3-$cycle containing a and b. Thus $c(a,b) = 3$. Conversely, suppose that $c(a,b) = 3$. Then a and b are adjacent, hence $B(a) \cap B(b) = \emptyset$. Let $a \leftrightarrow b \leftrightarrow c$ be a 3-cycle. Then $RP(a)RP(b) = RP(a)RP(c) = RP(b)RP(c) = 0$. If $RP(a) + RP(b) = 1$, then $RP(c) = RP(a)RP(c) + RP(b)RP(c) = 0$, giving $c = 0$, a contradiction. Therefore $RP(a) + RP(b) \neq 1$.

(2) Let $c(a,b) = 4$. Suppose $B(a) \cap B(b) = \emptyset$. If $RP(a) + RP(b) \neq 1$, then by (1) above, $c(a,b) = 3$, a contradiction. Hence $RP(a) + RP(b) = 1$. Now suppose that $B(a) \cap B(b) \neq \emptyset$. Then a and b are non-adjacent. This together with $c(a,b) = 4$ gives $d(a,b) = 2$. Conversely, suppose that $B(a) \cap B(b) = \emptyset$ and $RP(a) + RP(b) = 1$. Since $2 \notin Z(A)$, we get $a \leftrightarrow b \leftrightarrow 2a \leftrightarrow 2b \leftrightarrow a$ a 4-cycle containing a and b. Next, we will show that there is no cycle of length 3 which contains a and b. If $c(a,b) = 3$, then by (1) above, we have $RP(a) + RP(b) \neq 1$, a contradiction. Therefore $c(a,b) = 4$. Suppose $B(a) \cap B(b) \neq \emptyset$ and $d(a,b) = 2$. Let $a \leftrightarrow c \leftrightarrow b$ be a path. Then $a \leftrightarrow c \leftrightarrow b \leftrightarrow 2c \leftrightarrow a$ is a 4-cycle containing a and b. Hence $c(a,b) = 4$.

(3) Suppose $d(a,b) = 3$. Let $a \leftrightarrow c \leftrightarrow d \leftrightarrow b$ be a path. Then $RP(c)RP(b) \neq 0$ (as $d(a,b) = 3$) which gives $a \leftrightarrow c \leftrightarrow d \leftrightarrow b \leftrightarrow (1 - RP(b)) \leftrightarrow RP(c)RP(b) \leftrightarrow a$ a 6-cycle containing a and b. Since $d(a,b) = 3$, we get $c(a,b) = 6$. Conversely, suppose that $c(a,b) = 6$. Then by (1) and (2) above, we get $d(a,b) \neq 1, 2$. Therefore $d(a,b) = 3$. ■

The *cellularity* of a topological space X is denoted by $c(X)$ and it is the smallest cardinal number m such that every family of pairwise disjoint non-empty open subsets of X has cardinality at most m. Next we show that the clique number of $\Gamma^*(A)$ and the cellularity of $\Sigma(A)$ coincide.

Theorem 2.17. *Let A be a reduced Rickart $*$-ring. Then* $\omega(\Gamma^*(A)) = c(\Sigma(A))$.

Proof. Let C be a clique in $\Gamma^*(A)$. Then for every $a, b \in C, ab^* = 0$, i.e., $B(a) \cap B(b) = \emptyset$. Then the collection $\mathcal{C} = \{B(a) \mid a \in C\}$ is a family of pairwise disjoint non-empty open subsets of $\Sigma(A)$. Hence $\omega(\Gamma^*(A)) \leq c(\Sigma(A))$. Next, suppose that $\mathcal{C} = \{A_\lambda \mid \lambda \in \Lambda\}$ is a collection of pairwise disjoint non-empty open subsets of $\Sigma(A)$. For every $A_\lambda \in \mathcal{C}$ there exists $a_\lambda \in A$ such that $\emptyset \neq B(RP(a_\lambda)) \subseteq A_\lambda$. Clearly for every $A_\lambda, A'_\lambda \in \mathcal{C}$, we have $a_\lambda {a'_\lambda}^* = 0$. Hence $B = \{a_\lambda \mid \lambda \in \Lambda\}$ is a clique in $\Gamma^*(A)$. This implies that $c(\Sigma(A)) \leq \omega(\Gamma^*(A))$. Therefore $\omega(\Gamma^*(A)) = c(\Sigma(A))$. ■

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