Discussiones Mathematicae
General Algebra and Applications 37 (2017) 45–57
doi:10.7151/dmgaa.1264

SOME HOMOLOGICAL PROPERTIES OF SKEW PBW EXTENSIONS ARISING IN NON-COMMUTATIVE ALGEBRAIC GEOMETRY

Oswaldo Lezama

Helbert Venegas
Seminario de Álgebra Constructiva – SAC^2
Departamento de Matemáticas
Universidad Nacional de Colombia, Sede Bogotá

e-mail: jolezamas@unal.edu.co

Abstract

In this short paper we study for the skew PBW (Poincar-Birkhoff-Witt) extensions some homological properties arising in non-commutative algebraic geometry, namely, Auslander-Gorenstein regularity, Cohen-Macaulayness and strongly noetherianity. Skew PBW extensions include a considerable number of non-commutative rings of polynomial type such that classical PBW extensions, quantum polynomial rings, multiplicative analogue of the Weyl algebra, some Sklyanin algebras, operator algebras, diffusion algebras, quadratic algebras in 3 variables, among many others. Parametrization of the point modules of some examples is also presented.

Keywords: Auslander regularity condition, Cohen-Macaulay rings, strongly noetherian algebras, skew PBW extensions, filtered-graded rings, point modules.

2010 Mathematics Subject Classification: Primary: 16S38; Secondary: 16W50, 16S80, 16S36.

1. Introduction

In the study of non-commutative algebraic geometry, many important classes of non-commutative rings and algebras have been investigated intensively in the last years. Actually, the non-commutative algebraic geometry consists in generalizing some classical results of commutative algebraic geometry to some special non-commutative rings and algebras. Probably, the most important types of such rings are the Artin-Schelter regular algebras, Auslander regular algebras,

the Auslander Gorenstein rings, the Cohen-Macaulay rings, and the strongly Noetherian algebras. In non-commutative algebraic geometry, Artin-Schelter regular algebras are the analog of the commutative polynomials in commutative algebraic geometry, and in addition, all studied examples of Artin-Schelter regular algebras are Auslander regular, Auslander-Gorenstein, Cohen-Macaulay, and strongly Noetherian. In the present paper, we are interested in the Auslander-Gorenstein, Cohen-Macaulay, and strongly Noetherian conditions for skew PBW extensions. Skew PBW extensions represent a way of describing many important non-commutative algebras such as classical PBW extensions, quantum polynomial rings, multiplicative analogue of the Weyl algebra, some Sklyanin algebras, operator algebras, diffusion algebras, quadratic algebras in 3 variables, among many others. It is important to remark that other authors have studied these types of algebras using alternative descriptions, for example, interpreting them as solvable algebras [18, 25], PBW rings [12, 13, 14, 15], G-algebas [24], etc; moreover, the properties studied in the present paper have been considered in some of the cited previous works for many of examples of algebras covered by the skew PBW extensions, but the novelty of the paper consists in investigating the Auslander-Gorenstein, Cohen-Macaulay, and strongly Noetherian conditions for skew PBW extensions.

The paper is organized as follows: In the first section we recall the definition of skew PBW extensions and some basic properties of this type of non-commutative rings. In the second section we discuss the Auslander-Gorenstein condition. The next two sections are dedicated to investigate the Cohen-Macaulay and the strongly Noetherian properties. In the last section we present an application of the strongly noetherianity to parametrize the point modules of some remarkable skew PBW extensions.

Definition 1.1 [21]**.** Let R and A be rings. We say that A is a *skew PBW extension of* R (also called a $\sigma-PBW$ extension of R), if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist finitely many elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis

$$\text{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}, \text{ with}$$
$$\mathbb{N} := \{0, 1, 2, \dots\}.$$

The set $\text{Mon}(A)$ is called the set of standard monomials of A.

(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$, there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1.1}$$

(iv) For every $1 \leq i, j \leq n$, there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n. \tag{1.2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

Associated to a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$, there are n injective endomorphisms $\sigma_1, \ldots, \sigma_n$ of R and σ_i-derivations δ_i, as the following proposition shows.

Proposition 1.2. *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [21], Proposition 3. ∎

Next, we present particular cases of skew PBW extensions which are very important in Theorem 1.5. We recall the following definition (see [21]).

Definition 1.3. Let A be a skew PBW extension.

(a) A is quasi-commutative if the conditions (iii) and (iv) in Definition 1.1 are replaced by

(iii') For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r = c_{i,r} x_i. \tag{1.3}$$

(iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{1.4}$$

(b) A is bijective if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Definition 1.4. Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 1.2.

(i) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in Mon(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) Let $0 \neq f \in A$, $t(f)$ is the finite set of terms that conform f, i.e., if $f = c_1X_1 + \cdots + c_tX_t$, with $X_i \in Mon(A)$ and $c_i \in R - \{0\}$, then $t(f) := \{c_1X_1, \ldots, c_tX_t\}$.

(iv) Let f be as in (iii), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

The next theorems establish some results for skew PBW extensions that we will use later, for their proofs see [22].

Theorem 1.5. *Let A be an arbitrary skew PBW extension of the ring R. Then, A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A | \deg(f) \leq m\}, & \text{if } m \geq 1 \end{cases}$$

and the corresponding graded ring $Gr(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $Gr(A)$ is quasi-commutative bijective skew PBW extension of R.

Theorem 1.6. *Let A be a quasi-commutative skew PBW extension of a ring R. Then,*

1. *A is isomorphic to an iterated skew polynomial ring of endomorphism type, i.e.,*
$$A \cong R[z_1; \theta_1] \cdots [z_n; \theta_n].$$
2. *If A is bijective, then each endomorphism θ_i is bijective, $1 \leq i \leq n$.*

Theorem 1.7 (Hilbert Basis Theorem)**.** *Let A be a bijective skew PBW extension of R. If R is a left (right) Noetherian ring then A is also a left (right) Noetherian ring.*

A long list of examples of skew PBW extensions can be found in [22] and [29].

2. Auslander regularity conditions

In this section we will study the Auslander regularity and the Auslander-Gorenstein conditions for skew PBW extensions. The first condition was studied by Björk in [10] and [11], and by Ekström in [16] for filtered Zariski rings and Ore extensions (see also [15] and [29]). We will consider the second condition, but for completeness, we will integrate the first one in the statements of the results below.

Definition 2.1. Let A be a ring.

(i) The grade $j(M)$ of a left (or right) A-module M is defined by

$$j(M) := \min\left\{i \geq 0 \,|\, Ext^i_A(M, A) \neq 0\right\}$$

or ∞ if no such i exists.

(ii) A satisfies the *Auslander* condition if for every Noetherian left (or right) A-module M and for all $i \geq 0$

$$j(N) \geq i \text{ for all submodules } N \subseteq Ext^i_A(M, A).$$

(iii) A is *Auslander-Gorenstein* (AG) if A is Noetherian (i.e., two-sided Noetherian), which satisfies the Auslander condition, $id({}_AA) < \infty$, and $id(A_A) < \infty$ (injective dimension).

(iv) A is *Auslander regular* (AR) if it is AG and $gld(A) < \infty$ (global dimension).

Remark 2.2. (i) We recall that if a ring A is Noetherian, then $gld(A) := rgld(A) = lgld(A)$.

(ii) If A is Noetherian and if $id({}_AA) < \infty$ and $id(A_A) < \infty$, then $id({}_AA) = id(A_A)$ (see [31]).

Definition 2.3. Let A be a filtered ring with filtration $\{F_n(A)\}_{n\in\mathbb{Z}}$.

1. The Rees ring associated to A is a graded ring defined by

$$\widetilde{A} := \bigoplus_{n\in\mathbb{Z}} F_n(A).$$

2. The filtration $\{F_n(A)\}_{n\in\mathbb{Z}}$ is left (right) Zariskian, and A is called a left (right) Zariski ring, if $F_{-1}(A) \subseteq Rad(F_0(A))$ and the associated Rees ring $\widetilde{A}$ is left (right) Noetherian.

The following result is a characterizations of the Zariski property (see [26]).

Proposition 2.4. *Let A be a $\mathbb{N}$-filtered ring such that $Gr(A)$ is left (right) Noetherian. Then, A is left (right) Zariskian.*

Proposition 2.5. *Let A be a left and right Zariski ring. If its associated graded ring $Gr(A)$ is AG, respectively AR, then so too is A.*

Proof. See [11], Theorem 3.9. ∎

Proposition 2.6. *If A is AG, respectively AR, then the skew polynomial ring $A[x;\sigma,\delta]$ with σ bijective is also AG, respectively AR.*

Proof. [16], Theorem 4.2. ∎

The following proposition states that the AG (AR) conditions are preserved under arbitrary localizations.

Proposition 2.7. *Let A be an AG ring, respectively AR, and S a multiplicative Ore set of regular elements of A. Then so too is $S^{-1}A$ (and also AS^{-1}).*

Proof. See [3], Proposition 2.1. ∎

Lemma 2.8. *If A is a bijective skew PBW extension of a Noetherian ring R, then A is a left and right Zariski ring.*

Proof. Since A is $\mathbb{N}$-filtered, $0 = F_{-1}(A) \subseteq Rad(F_0(A)) = Rad(R)$. By Theorem 1.6, $Gr(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$, with θ_i is bijective, $1 \leq i \leq n$. Whence $Gr(A)$ is Noetherian. Proposition 2.4 says that A is a left and right Zariski ring. ∎

Theorem 2.9. *Let A be a bijective skew PBW extension of a ring R such that R is AG, respectively AR, then so too is A.*

Proof. Acoording to Theorem 1.5, $Gr(A)$ is a quasi-commutative skew PBW extension, and by the hypothesis, $Gr(A)$ is also bijective. By Theorem 1.6, $Gr(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$ such that each θ_i is bijective, $1 \leqslant i \leqslant n$. Proposition 2.6 says that $Gr(A)$ is AG, respectively AR. From Lemma 2.8, A is a left and right Zariski ring, so by Proposition 2.5, A is AG, respectively AR. ∎

Corollary 2.10. *If R is AG, respectively AR, then the ring of skew quantum polynomials*

$$Q_{q,\sigma}^{r,n} := R_{q,\sigma}[x_1^{\pm 1},\ldots,x_r^{\pm 1},x_{r+1},\ldots,x_n]$$

is AG, respectively AR.

Proof. Recall that the ring of skew quantum polynomials $Q_{q,\sigma}^{r,n}$ is a generalization of the algebra of classical quantum polynomials over fields (compare with [4] and [5]); in addition, $Q_{q,\sigma}^{r,n}(R)$ is a localization of a bijective skew PBW extension A of the ring R by a multiplicative Ore set of regular elements of A (see [22]). Let R be AG, from Theorem 2.9 and Proposition 2.7 we get that $Q_{q,\sigma}^{r,n}(R)$ is AG. If R is AR, then R is AG and $gld(R) < \infty$, then $gld(Q_{q,\sigma}^{r,n}(R)) < \infty$ and $Q_{q,\sigma}^{r,n}(R)$ is AG, so $Q_{q,\sigma}^{r,n}(R)$ is AR. ∎

Remark 2.11. All examples of skew PBW extensions listed in Section 3 of [22] are AG rings (respectively AR) assuming that the ring of coefficients (R or K) is AG (AR). If the ring the coefficients is a field, then of course, it is an AG ring (respectively AR). For many of these examples the AR and the AG conditions have been investigated before for other authors (see [15]), however, the novelty here is to prove these properties for skew PBW extensions which include some examples probably not considered before.

3. Cohen-Macaulayness

In this section we study the Cohen- Macaulay property for skew PBW extensions.

Definition 3.1. Let A be an algebra over a field K. We say that A is *Cohen-Macaulay* (CM) with respect to the classical Gelfand-Kirillov dimension if

$$GK\dim(A) = j_A(M) + GK\dim(M)$$

for every non-zero Noetherian A-module M.

Recall that if A is a K-algebra, then the *classical Gelfand-Kirillov dimension* of A is defined by

$$\mathrm{GKdim}(A) := \sup_V \overline{\lim_{n\to\infty}} \log_n \dim_K V^n, \tag{3.1}$$

where V vanishes over all frames of A and $V^n := {}_K\langle v_1 \cdots v_n | v_i \in V\rangle$ (a *frame* of A is a finite dimensional K-subspace of A such that $1 \in V$; since A is a K-algebra, then $K \hookrightarrow A$, and hence, K is a frame of A of dimension 1). For a K-algebra A, an automorphism σ of A is said to be *locally algebraic* if for any $a \in A$ the set $\{\sigma^m(a) | m \in \mathbb{N}\}$ is cointained in a finite dimensional subspace of A.

The classical Gelfand-Kirillov dimension for skew PBW extensions was studied in [28] (in [23], it has been studied the Gelfand-Kirillov dimension of skew PBW extensions over a K-algebra R).

Proposition 3.2. *Let R be a K-algebra with a finite dimensional generating subspace V and let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension of R. If $\sigma_n(V) \subseteq V$ or σ_n is locally algebraic, then*

$$GK\dim(A) = GK\dim(R) + n.$$

The following proposition in the classical case is also known.

Proposition 3.3. *Let A be a K-algebra with a finite filtration $\{A_i\}_{i\in\mathbb{Z}}$ such that $Gr(A)$ is finitely generated. Then $GK\dim({}_{Gr(A)}Gr(A)) = GK\dim({}_AA)$.*

Proof. See [17], Proposition 6.6. ∎

Proposition 3.4. *Suppose that A is a left and right Zariskian ring, and $Gr(A)$ is AG. Then $j_A(M) = j_{Gr(A)}(Gr(M))$ for every non-zero Noetherian A-module M.*

Proof. If M is a finitely generated A-module and if $\{F_n(M)\}$ is a good filtration on M then in general $j_A(M) \leq j_{Gr(A)}(Gr(M))$, but if A is left and right Zariski ring and $Gr(A)$ is AG then $j_A(M) \geq j_{Gr(A)}(Gr(M))$, so $j_A(M) = j_{Gr(A)}(Gr(M))$ (see [11], Proof of Theorem 3.9). ∎

Now, it is possible to prove the following proposition.

Proposition 3.5. Let A be a left and right Zariski ring with finite filtration and such that $Gr(A)$ is AG. If $Gr(A)$ is CM, then so too is A.

Proof. Let M be a noetherian A-module, then

$$\begin{aligned} GKdim(A) &= GKdim(Gr(A)) \\ &= GKdim({}_{Gr(A)}Gr(M)) + j_{Gr(A)}(M) \\ &= GKdim({}_AM) + j_A(M). \end{aligned}$$

Therefore A is CM. ∎

Proposition 3.6. *Suppose that R is AR (AG) and CM ring. Let $R[x;\sigma,\delta]$ be an Ore extension with σ bijective. If $R = \oplus_{i\geq 0} R_i$ is a connected graded K-algebra (i.e., $R_0 = K$) such that $\sigma(R_i) \subseteq R_i$ for each $i \geq 0$, then $R[x;\sigma,\delta]$ is CM.*

Proof. See [20], Lemma, Part (ii). ∎

Definition 3.7. Let A be a K-algebra, it is said that $x \in A$ is a local normal element if for every frame $V \subset A$, there is a frame $V' \supset V$ such that $xV' = V'x$.

It is clear that every central element is local. The next proposition says that CM property is preserved under certain localizations.

Proposition 3.8. *Let A be an AG ring, and S a multiplicatively closed set of local normal elements in A. If A is CM, so is $S^{-1}A$.*

Proof. See [3], Theorem 2.4. ∎

Theorem 3.9. *Let A be a bijective skew PBW extension of a ring R such that R is AG, CM, and $R = \sum_{i\geq 0} \oplus R_i$ is a connected graded K-algebra such that $\sigma_j(R_i) \subseteq R_i$ for each $i \geq 0$ and $1 \leq j \leq n$, then A is CM.*

Proof. From Theorem 1.5 it is clear that A is a K-algebra with a finite filtration and $Gr(A)$ is a quasi-commutative skew PBW extensions, and by the hypothesis, $Gr(A)$ is also bijective. By Theorem 1.6, $Gr(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1]\cdots[z_n;\theta_n]$ such that each θ_i is bijective, $1 \leqslant i \leqslant n$ (according to the proof of Theorem 1.6 in [22], $\theta_j(r) = \sigma_j(r)$ for every $r \in R$). Proposition 2.6 says that $Gr(A)$ is AG. From Lemma 2.8, A is a left and right Zariski ring, and by Proposition 3.6 $Gr(A)$ is CM, so by Proposition 3.5 A is CM. ∎

Corollary 3.10. *Let $R = \sum_{i\geq 0} \oplus R_i$ be a connected graded K-algebra such that $\sigma_j(R_i) \subseteq R_i$ for every $i \geq 0$ and all σ_j in definition of skew quantum polynomials $Q^{r,n}_{q,\sigma}$. If R is AG and CM, then $Q^{r,n}_{q,\sigma}$ is CM.*

Proof. As was observed above, $Q^{r,n}_{q,\sigma}$ is a localization of a bijective skew PBW extension A of R by a multiplicative Ore set of regular elements of A, and the multiplicative set generated by $x_1, \ldots, x_r$ consists of monomials, which are local normal elements. From Theorem 3.9 and Proposition 3.8 we get that $Q^{r,n}_{q,\sigma}$ is CM. ∎

4. Strongly noetherian algebras

Now we will consider the strongly noetherian property for skew PBW extensions. This condition was studied by Artin, Small and Zhang in [6], and appears naturally in the study of point modules in non-commutative algebraic geometry (see [7, 8] and [30]).

Definition 4.1. Let K be a commutative ring and let A be a left Noetherian K-algebra. We say that A is left strongly Noetherian if for any commutative Noetherian K-algebra C, $C \otimes_K A$ is left Noetherian.

Some examples of strongly Noetherian algebras include Weyl algebras, Sklyanin algebras over a field K and universal enveloping algebras of finite dimensional Lie algebras (see [6], Corollaries 4.11 and 4.12). Moreover, all known examples of Artin-Shelter regular algebras are strongly Noetherian. It is an open question if every Artin-Shelter regular algebra is strongly Noetherian.

Proposition 4.2. *Let K be a commutative ring and let A be a K-algebra.*

(i) *If A is left strongly Noetherian, then $A[x;\sigma,\delta]$ is left strongly Noetherian when σ is bijective.*

(ii) *If A is $\mathbb{N}$-filtered and $Gr(A)$ is left strongly Noetherian, then A is left strongly Noetherian.*

(iii) *If S is a multiplicative Ore set of regular elements of A, then $S^{-1}A$ is left strongly Noetherian.*

Proof. See [6], Proposition 4.1. and Proposition 4.10. ∎

With the previous result we get the main result of the present section.

Theorem 4.3. *Let K be a commutative ring and let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective skew PBW extension of a left strongly Noetherian K-algebra R. Then A is left strongly Noetherian.*

Proof. According to Theorem 1.5, $Gr(A)$ is a quasi-commutative skew PBW extension, and by the hypothesis, $Gr(A)$ is also bijective. By Theorem 1.6, $Gr(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1, \theta_1] \cdots [z_n, \theta_n]$ such that each θ_i is bijective, $1 \leq i \leq n$. Proposition 4.2 part (i) says that $Gr(A)$ is left strongly Noetherian, and by part (ii) A is left strongly Noetherian. ∎

Corollary 4.4. *Let R be a left strongly Noetherian K-algebra, then the ring of skew quantum polynomials*

$$Q_{q,\sigma}^{r,n} := R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$$

is left strongly Noetherian.

Proof. From Theorem 4.3 and Proposition 4.2 part (iii) we get that $Q_{q,\sigma}^{r,n}(R)$ is left strongly Noetherian. ∎

Remark 4.5. All examples of skew PBW extensions listed in Section 3 of [22] are left strongly Noetherian assuming that the ring of coefficients is left strongly Noetherian K-algebra.

5. Point modules of some skew PBW extensions

As application of strongly Noetherian algebras, next we present some examples of finitely graded algebras which are bijective skew PBW extensions and such that they have nice spaces parametrizing its point modules. It is important to remark that Artin, Tate and Van den Bergh studied point modules in order to complete the classification of Artin-Shelter regular algebras of dimension 3 (see [7]).

Recall that a K-algebra A is *finitely graded* if it is connected $\mathbb{N}$-graded and finitely generated as a K-algebra.

Definition 5.1. Let A be a finitely graded K-algebra that is generated in degree 1. A point module for A is a graded left module M such that M is cyclic, generated in degree 0, and $\dim_K(M_n) = 1$ for all $n \geq 0$.

If A is commutative, then its point modules naturally correspond to the (closed) points of the scheme $proj(A)$. Similarly, many non-commutative graded rings also have nice parameter spaces of point modules. For example, the point modules of the quantum plane or Jordan plane are parametrized by $\mathbb{P}^1$, i.e., there exists a bijective correspondence between the projective space $\mathbb{P}^1$ and the collection of isomorphism classes of point modules of the quantum plane or Jordan plane (see [30], Example 3.2). The following proposition states that for finitely graded strongly Noetherian algebras is possible to find a projective scheme that parametrizes the set of its point modules.

Proposition 5.2 ([8], Corollary 4.12)**.** *Let A be a finitely graded K-algebra which is generated in degree* 1*. If A is strongly Noetherian, then the point modules of A are naturally parametrized by a commutative projective scheme over K.*

The following bijective skew PBW extensions are finitely graded K-algebras generated in degree 1:

(i) Polynomial ring, $K[x_1, \ldots, x_n]$.

(ii) Skew polynomial ring $K[x;\sigma]$ with σ bijective. In general, $K[x_1;\sigma_1]\cdots[x_n;\sigma_n]$ with σ_i bijective for $1 \leq i \leq n$.

(iii) Quantum polynomial ring.

(iv) Multiplicative analogue of the Weyl algebra.

(v) The Sklyanin algebra S is the K-algebra

$$S = K\langle x, y, z\rangle/\langle ayx + bxy + cz^2, axz + bzx + cy^2, azy + byz + cx^2\rangle,$$

where $a, b, c \in K$. Consider the case when $c = 0$ and $a, b \neq 0$, then in S we have $yx = -\frac{b}{a}xy$; $zx = -\frac{a}{b}xz$ and $zy = -\frac{b}{a}yz$, therefore $S \cong \sigma(K)\langle x, y, z\rangle$ is a skew PBW extension of K

According to Theorem 4.3 they are left strongly noetherian and by Proposition 5.2 there exists a commutative projective scheme that parametrizes the set of point modules of A.

Corollary 5.3. *Let A be a bijective skew PBW extension as above, then there exists a commutative projective scheme over K that parametrizes the set of point modules of A.*

References

[1] J.P. Acosta, C. Chaparro, O. Lezama, I. Ojeda and C. Venegas, *Ore and Goldie theorems for skew PBW extensions*, Asian-European J. Math. **6** (2013). doi:10.1142/S1793557113500617

[2] J.P. Acosta, O. Lezama and M.A. Reyes, *Prime ideals of skew PBW extensions*, Revista de la Unión Matemtica Argentina **56** (2015) 39–55.

[3] K. Ajitabh, S.P. Smith and J.J. Zhang, *Injective resolutions of some regular rings*, J. Pure Appl. Algebra **140** (1999) 1–21.
doi:10.1016/S0022-4049(99)00049-3

[4] V. Artamonov, Quantum Polynomials (WSPC Proceedings, 2008).

[5] V. Artamonov, *Serre's quantum problem*, Russian Math. Surveys **53** (1998) 657–730.
doi:10.1070/RM1998v053n04ABEH000056

[6] M. Artin, L.W. Small and J.J. Zhang, *Generic flatness for strongly Noetherian algebras*, J. Algebra **221** (1999) 579–610.
doi:10.1006/jabr.1999.7997

[7] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festchrift, Vol. I, Birkhuser Boston (1990) 33–85.
doi:10.1007/978-0-8176-4574-8-3

[8] M. Artin and J.J. Zhang, *Noncommutative projective schemes*, Adv. Mah. **109** (1994) 228–287.
doi:10.1006/aima.1994.1087

[9] A. Bell and K. Goodearl, *Uniform rank over differential operator rings and Poincar-Birkhoff-Witt extensions*, Pacific J. Math. **131** (1988) 13–37.
doi:10.2140/pjm.1988.131.13

[10] J.E. Björk, Rings of Differential Operators, North-Holland Mathematical Library, vol. 21 (North-Holland Publishing Co., Amsterdam-New York, 1979).

[11] J.E. Björk, *The Auslander Condition on Noetherian Rings*, Sm. d'algbre P. Dubreil et M.-P. Malliavin, Lect. Notes Math. **1404** (1989) 137–173.

[12] J. Bueso, J. Gomez-Torrecillas and A. Verschoren, Algorithmic Methods in Noncommutative Algebra: Applications to Quantum Groups (Kluwer, 2003).

[13] J. Gomez-Torrecillas, *Basic Module Theory over Non-Commutative Rings with Computational Aspects of Operator Algebras*, in: M. Barkatou *et al.* (Eds.), AADIOS 2012, Lecture Notes in Computer Science **8372** (2014) 23–82.

[14] J. Gomez-Torrecillas and F.J. Lobillo, *Global homological dimension of Multifiltered rings and quantized enveloping algebras*, J. Algebra **225** (2000) 522–533.
doi:10.1006/jabr.1999.8101

[15] J. Gomez-Torrecillas and F.J. Lobillo, *Auslander-regular and Cohen-Macaulay Quantum groups*, Algebras and Representation Theory **7** (2004) 35–42.
doi:10.1023/B:ALGE.0000019384.36800.fa

[16] E.K. Ekström, *The Auslander condition on graded and filltered noetherian rings*, Sminaire Dubreil-Malliavin 1987–88, Lect. Notes Math. 1404 (Springer Verlag, 1989) 220–245.

[17] G. Krause and T.H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, 2nd ed., Graduate Studies in Mathematics, 22 (AMS, Providence, USA, 2000).

[18] H. Kredel, Solvable Polynomial Rings (Shaker, 1993).

[19] T. Levasseur, *Some properties of non-commutative regular rings*, Glasglow J. Math. **34** (1992) 277–300.
doi:10.1017/S0017089500008843

[20] T. Levasseur and J.T. Stafford, *The quantum coordinate ring of the special linear group*, J. Pure Appl. Algebra **86** (1993) 181–186.
doi:10.1016/0022-4049(93)90102-Y

[21] O. Lezama and C. Gallego, *Gröbner bases for ideals of sigma-PBW extensions*, Communications in Algebra **39** (2011) 50–75.
doi:10.1080/00927870903431209

[22] O. Lezama and M. Reyes, *Some homological properties of skew PBW extensions*, Communications in Algebra **42** (2014) 1200–1230.
doi:10.1080/00927872.2012.735304

[23] O. Lezama and E. Latorre, *Non-commutative algebraic geometry of semi-graded rings*, arXiv:1605.09057 [math.RA].

[24] V. Levandovskyy, Non-Commutative Computer Algebra for Polynomial Algebras: Gröbner Bases, Applications and Implementation (Dissertation, University of Kaiserslautern, 2005).

[25] H. Li, Noncommutative Gröbner Bases and Filtered-Graded Transfer (Springer, 2002).

[26] H. Li, Non-Commutative Zariskian Rings, Ph.D Thesis (Universiteit Antwerpen, Antwerp, Belgium, 1990).

[27] H. Li and F.V. Oystaeyen, *Zariskian filtrations*, Comm. Algebra **17** (1989) 2945–2970.
doi:10.1080/00927878908823888

[28] A. Reyes, *Gelfand-Kirillov dimension of skew PBW extensions*, Revista Colombiana de Matemáticas **47** (2013) 95–111.

[29] A. Reyes, Ring and Module Theoretic Properties of $\sigma - PBW$ Extensions, Tesis de Doctorado (Universidad Nacional de Colombia, Bogotá, 2013).

[30] D. Rogalski, *An introduction to non-commutative projective algebraic geometry*, arXiv:1403.3065 [math.RA].

[31] A. Zaks, *Injective dimension of semiprimary rings*, J. Algebra **13** (1969) 73–89.
doi:10.1016/0021-8693(69)90007-6

Received 19 July 2016
Revised 3 February 2017