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ALL REGULAR-SOLID VARIETIES OF IDEMPOTENT SEMIRINGS

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Abstract

The lattice of all regular-solid varieties of semirings splits in two complete sublattices: the sublattice of all idempotent regular-solid varieties of semirings and the sublattice of all normal regular-solid varieties of semirings. In this paper, we discuss the idempotent part.

Keywords: semiring, hypersubstitution, regular hypersubstitution, regular hyperidentity, solid variety, regular-solid variety.

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1. INTRODUCTION

Varieties of semirings are varieties of algebras of type (2, 2), where both binary operations are associative and satisfy the two usual distributive laws. Single semirings as well as classes of semirings form important structures in Automata and Formal Languages Theories [5]. To get more insight into the complete lattice of all varieties of semirings, all solid and all pre-solid varieties of semirings were determined [1, 2]. Now, we are interested in the complete lattice of all regular-solid varieties of semirings by characterizing all regular-solid varieties of idempotent semirings. To achieve our aim, we recall some basic concepts.

Let F and G be the both binary operation symbols and let $W_{(2,2)}(X_2)$ be the set of all binary terms of type (2,2) built up by variables from the alphabet $X_2 = \{x, y\}$. Hypersubstitutions of type (2,2) are mappings

$$\sigma: \{F, G\} \to W_{(2,2)}(X_2).$$

The set of all hypersubstitutions of type (2, 2) will be denoted by Hyp. A hypersubstitution $\sigma \in Hyp$ can be extended on the set $W_{(2,2)}(X)$ of all terms of type (2, 2), where X is an arbitrary countably infinite alphabet of variables, by the following steps:

- (i) $\hat{\sigma}[t] := t$, if $t \in X$,
- (ii) $\hat{\sigma}[t] := \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$, if $t = f(t_1, t_2) \in W_{(2,2)}(X)$ with $f \in \{F, G\}$, where $\sigma(f)$ can be interpreted as the term operation $\sigma(f)^{\mathcal{F}_{(2,2)}(X)}$ induced by the term $\sigma(f)$ on the free algebra $\mathcal{F}_{(2,2)}(X) := (W_{(2,2)}(X); (\overline{F}, \overline{G}))$ with $\overline{f} : (W_{(2,2)}(X))^2 \to W_{(2,2)}(X), (t_1, t_2) \mapsto f(t_1, t_2).$

It is easy to prove that the algebra $(Hyp; \circ_h, \sigma_{id})$, is a monoid with \circ_h (where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and \circ is the usual mapping composition) as binary operation and σ_{id} , defined by $\sigma_{id}(f) := f(x, y)$ for all $f \in \{F, G\}$, as an identity element. Hypersubstitutions can be applied to algebras as follows: given an algebra $\mathcal{A} = (\mathcal{A}; (\mathcal{F}^{\mathcal{A}}, \mathcal{G}^{\mathcal{A}}))$ of type (2, 2) and a hypersubstitution $\sigma \in Hyp$, one defines the algebra $\sigma(\mathcal{A}) := (\mathcal{A}; (\sigma(\mathcal{F})^{\mathcal{A}}, \sigma(\mathcal{G})^{\mathcal{A}}))$. This algebra of type (2, 2) is called the derived algebra by \mathcal{A} and σ .

The hypersubstitution $\sigma \in Hyp$ such that $\sigma(F) = t$ and $\sigma(G) = s$ will be denoted by $\sigma_{t,s}$. For all variables u and v, the term F(u,v) and G(u,v) will be denoted by u + v and uv, respectively.

A hypersubstitution $\sigma \in Hyp$ is called a regular hypersubstitution if σ maps both F and G to binary terms containing both variables x and y. It is easy to verify that the set Reg of all regular hypersubstitutions of type (2,2) forms a submonoid of the monoid Hyp. An identity $s \approx t$ in a variety V of semirings is called a regular hyperidentity if for every $\sigma \in Reg$, the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ belongs to the set IdV of all identities satisfied in V. A variety V of semirings is called regular-solid if all identities in V are satisfied as regular hyperidentities. For more information about hypersubstitutions and varieties of algebras see in [3, 7].

In the next section, we will provide some necessary conditions for a variety of semirings to be a regular-solid one. This leads to a description of the lattice of all regular-solid varieties of semirings. The last section will be devoted to the determination of the lattice of all regular-solid varieties of idempotent semirings.

2. Some Properties

A variety V of semirings is medial if $x + y + z + u \approx x + z + y + u \in IdV$ and $xyzu \approx xzyu \in IdV$, idempotent if $x + x \approx x \approx x^2 \in IdV$, distributive if $xy + z \approx (x + z)(y + z) \in IdV$ and $x + yz \approx (x + y)(x + z) \in IdV$. An equation $s \approx t$ is called normal if either both terms s and t are equal to the same variable or none of them is a variable, that is, if s = t or the complexity (number of occurrences of operation symbols) of both terms s and t is greater or equal to 1. A variety in which all identities are normal is called a normal variety.

Now, we can derive some necessary conditions for varieties of semirings to be regular-solid.

Proposition 1. Let V be a regular-solid variety of semirings. The following properties are:

(1) V is medial, distributive and satisfies the identities:

i)
$$x^2yz \approx xy^2z \approx xyz^2 \approx xyz$$
,

(ii) $2x + y + z \approx x + 2y + z \approx x + y + 2z \approx x + y + z$.

(2) V is either idempotent or normal.

Proof. (1) It is clear that the usual distributive laws are satisfied in V. The application of the regular hypersubstitutions $\sigma_{xy,x+y}$ to them gives the other distributive laws since V is a regular-solid variety of semirings. Moreover, applying the regular hypersubstitutions $\sigma_{xy,xy}$ and $\sigma_{yx,yx}$ to the distributive law

$$x(y+z) \approx xy + xz,$$

of V, we get the identities

 $xyz \approx xyxz$ and $zyx \approx zxyx$, respectively, in V.

It is folklore that the identities $xyz \approx xyxz \approx xzyz$ imply the medial law $xyzu \approx xzyu$ and the identities $xyz \approx x^2yz \approx xy^2z \approx xyz^2$. The application of the regular hypersubstitution $\sigma_{xy,x+y}$ to these identities gives the remaining identities.

(2) Suppose that $t \approx x$ is an identity in V which is not normal. This provides $x^k \approx x \in IdV$ for some $k \geq 2$ (by using the regular hypersubstitution $\sigma_{xy,xy}$ and identifying all variables with x). From the identity $x^2yz \approx xyz \in IdV$, we get $x^4 \approx x^3 \in IdV$ and together with $x^k \approx x \in IdV$, we obtain the idenpotent law $x^2 \approx x \in IdV$. Therefore, V is idempotent by using the regular hypersubstitution $\sigma_{xy,x+y}$.

Proposition 1 (2), leads to a description of the complete lattice Reg(Sr) of all regular-solid varieties of semirings. Denoting by $\mathcal{L}(2,2)$ the lattice of all varieties of type (2,2), we have:

Corollary 2. The lattice Reg(Sr) splits into two complete sublattices of $\mathcal{L}(2,2)$, the sublattice $Reg_{Idem}(Sr)$ of all idempotent regular-solid varieties of semirings and the sublattice $Reg_N(Sr)$ of all normal regular-solid varieties of semirings.

Proof. The lattice $\mathcal{L}_N(2,2)$ of all normal varieties of type (2,2) and the lattice $\mathcal{L}_{Idem}(2,2)$ of all idempotent varieties of type (2,2) are complete sublattices of $\mathcal{L}(2,2)$ (see [4, 7]). Therefore, since $Reg_N(Sr) = Reg(Sr) \cap \mathcal{L}_N(2,2)$ (the intersection of two complete sublattices) and since $Reg_{Idem}(Sr) = Reg(Sr) \cap \mathcal{L}_{Idem}(2,2)$ (the intersection of two complete sublattices), it arises that both lattices $Reg_{Idem}(Sr)$ and $Reg_N(Sr)$ are complete sublattices. By Proposition 1 (2) the lattices $Reg_{Idem}(Sr)$ and $Reg_N(Sr)$ are disjoint and their union is Reg(Sr).

3. All Regular-Solid Varieties of Idempotent Semirings

In this section, the lattice of all regular-solid varieties of idempotent semirings will be determined. An equation $s \approx t$ is outermost if the terms s and t start with the same variable (we write leftmost(s) = leftmost(t)) and end also with the same variable (we write rightmost(s) = rightmost(t)). A variety V is called outermost if all equations in IdV are outermost. A variety V of semirings is commutative if $x + y \approx y + x \in IdV$ and $xy \approx yx \in IdV$. The following result gives a description of idempotent regular-solid varieties of semirings.

Proposition 3. Each idempotent regular-solid variety of semirings is either outermost or commutative.

Proof. Let V be an idempotent regular-solid variety of semirings. Assume that V is not outermost. We will show that V is commutative. Since V is not outermost, without loss of generality, we can assume that there exists an equation $s \approx t$ in IdV such that $leftmost(s) = x \neq y = leftmost(t)$. Applying the regular hypersubstitution $\sigma_{xy,xy}$ to the identity $s \approx t \in IdV$, we get the following identity $s_1 \approx t_1$ in V (where $leftmost(s_1) = x \neq y = leftmost(t_1)$). Let us consider the function $h: X \to W_{(2,2)}(X), w \mapsto \begin{cases} x & \text{if } w = x \\ y & \text{otherwise.} \end{cases}$

It is well known that this function can be uniquely extended to an endomorphism \overline{h} on $\mathcal{F}_{(\in,\in)}(\mathcal{X})$. Then, $\overline{h}(s_1) \approx \overline{h}(t_1) \in IdV$ and $\overline{h}(s_1)yx \approx \overline{h}(t_1)yx \in IdV$, so $xyx \approx yx \in IdV$ because of the idempotent law. Applying the regular hypersubstitution $\sigma_{yx,yx}$ to the latter identity, the following equations $xy \approx xyx \approx yx$ hold in V as identities. The application of $\sigma_{xy,x+y}$ to $xy \approx yx$ shows that V is commutative.

Now, we determine the commutative part of $Reg_{Idem}(Sr)$. Proposition 1 (1) shows that every regular-solid variety of idempotent semirings is a subvariety of the variety V_{MID} of all medial idempotent and distributive semirings. But the subvariety lattice of V_{MID} is fully described by Pastjin in [6] as follows:

Let us consider the two-element algebras (using the same notations as in [6]):

- $\mathcal{A} = (\{0,1\}; e_1^2, e_1^2), e_1^2$ is the binary projection $\{0,1\}^2 \rightarrow \{0,1\}$ on the first input;
- $\mathcal{A}^{\circ} = (\{0,1\}; e_2^2, e_2^2), e_2^2$ is the binary projection $\{0,1\}^2 \rightarrow \{0,1\}$ on the second input;
- $\mathcal{B} = (\{0,1\}; e_1^2, \wedge), \text{ where } \wedge \text{ denotes the conjunction};$

 $\begin{aligned} \mathcal{B}^{\circ} &= (\{0,1\}; e_2^2, \wedge); \\ \mathcal{B}^{\bullet} &= (\{0,1\}; \wedge, e_1^2); \\ \mathcal{B}^{\bullet\circ} &= (\{0,1\}; \wedge, e_2^2); \\ \mathcal{F} &= (\{0,1\}; e_1^2, \frac{2}{2}); \\ \mathcal{F}^{\circ} &= (\{0,1\}; e_2^2, e_1^2); \\ \mathcal{J} &= (\{0,1\}; \wedge, \vee), \text{ where } \vee \text{ denotes the disjunction; } \\ \mathcal{L} &= (\{0,1\}; \wedge, \wedge). \end{aligned}$

The algebra \mathcal{J} generates the variety DL of all distributive lattices and \mathcal{L} generates the variety SL of bi-semilattices. Then we have

Lemma 4 [6]. The subvariety lattice of the variety V_{MID} of all medial idempotent and distributive semirings is a Boolean lattice with 10 atoms and 10 dual atoms, i.e., with 2^{10} elements. The atoms are exactly the varieties $V(\mathcal{A}), V(\mathcal{A}^{\circ}), V(\mathcal{B}),$ $V(\mathcal{B}^{\circ}), V(\mathcal{B}^{\bullet}), V(\mathcal{B}^{\bullet}), V(\mathcal{F}), V(\mathcal{F}^{\circ}), DL$ and SL, where V(K) is the variety generated by a given algebra K of type (2,2).

Therefore, each subvariety of V_{MID} is a join of some of these 10 atoms. An equation $s \approx t$ is said to be regular if both terms s and t use the same variables and a variety of semirings is regular if all identities in that variety are regular. The lattice of all regular-solid varieties of commutative and idempotent semirings is determined as follows:

Theorem 5. The two-element lattice

 $\int_{-\tau}^{SL}$

is the lattice of all regular-solid varieties of commutative and idempotent semirings, where $\mathcal{T} = Mod\{x \approx y\}$ is the trivial variety of type (2,2).

Proof. Let V be a regular-solid variety of commutative and idempotent semirings. By Proposition 1 (1), the variety V is a commutative subvariety of V_{MID} . So V is either trivial or a join of some commutative atoms listed in Lemma 4. This means that either V is trivial or $V \in \{SL, DL, SL \lor DL\}$. But the varieties DL and $SL \lor DL$ are not regular-solid. Indeed, the application of $\sigma_{x+xy,x+xy}$ to the commutative identity $xy \approx yx$ gives the identity $x+xy \approx y+yx$ which cannot be satisfied in DL because of the absorption laws. IdSL is the set of all regular identities of type (2, 2). It is clear that applying regular hypersubstitution to any regular identity, one gets a regular identity. So SL is regular-solid.

We are now interested in the outermost part of $Reg_{Idem}(Sr)$. Some definitions and facts will be referred.

Definition. A variety V of semirings is s-outermost if for any identity $s \approx t \in IdV$, the equations $s \approx t$ as well as $\hat{\sigma}_{x+y,yx}[s] \approx \hat{\sigma}_{x+y,yx}[t]$ are outermost.

This definition coincides with that one given in [1] and it is clear that every outermost regular-solid variety of semirings is *s*-outermost since the hypersubstitution $\sigma_{x+y,yx}$ is regular.

A variety V of semirings is said to be a solid variety if for all $s \approx t \in IdV$ and for all $\sigma \in Hyp$, we get $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$. It is well known that the variety $RA_{(2,2)}$ generated by all projection algebras of type (2, 2) is a variety of semirings and it is defined by $RA_{(2,2)} = Mod\{(xy)z \approx x(yz) \approx xz, (x+y)+z \approx x+(y+z) \approx$ $x + z, (x + y)(z + u) \approx xz + yu, x^2 \approx x \approx x + x\}$ [1]. It is already proved:

Lemma 6 [1]. The lattice of all solid varieties of semirings is the four-element chain represented by $\mathcal{T} \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}$, where

$$\begin{aligned} RA_{(2,2)} &= V(\mathcal{A}) \lor V(\mathcal{A}^{\circ}) \lor V(\mathcal{F}) \lor V(\mathcal{F}^{\circ}) \quad and \\ V_{BE} &= RA_{(2,2)} \lor SL \lor V(\mathcal{B}) \lor V(\mathcal{B}^{\circ}) \lor V(\mathcal{B}^{\bullet}) \lor V(\mathcal{B}^{\bullet\circ}). \end{aligned}$$

Moreover, it holds

Lemma 7 [1]. The variety $RA_{(2,2)}$ is the least s-outermost variety of semirings.

Now, we can prove:

Lemma 8. Let V be an outermost regular-solid variety of idempotent semirings. If V is different from $RA_{(2,2)}$ then V is regular i.e all equations in IdV are regular.

Proof. We will prove that if V is not regular then $V = RA_{(2,2)}$. Since V is outermost regular-solid variety of semirings, V is s-outermost and we have $RA_{(2,2)} \subseteq V$ (Lemma 7). It left to prove that $V \subseteq RA_{(2,2)}$ i.e $Id(RA_{(2,2)}) \subseteq IdV$. Since V is not regular, there exists an identity $s \approx t$ in IdV such that, without loss of generality, a variable x_i occurs in s but not in t. Applying $\sigma_{xy,xy}$ to $s \approx t$ and identifying all variables different from x_i with x, we get $xx_ix \approx x \in IdV$ because V is outermost and idempotent. Therefore, $xyz \approx xz \in IdV$. The application of $\sigma_{xy,x+y}$ to this identity gives $x + y + z \approx x + z \in IdV$. Moreover, using the previous identity, the distributivity and the idempotency, the basis identities of $RA_{(2,2)}$ are also identities in V. This finishes the proof of $Id(RA_{(2,2)}) \subseteq IdV$.

Now, we have all tools to prove our main result:

Theorem 9. The lattice of all regular-solid varieties of idempotent semirings is the lattice



Proof. Let V be a regular-solid variety of idempotent semirings. Then V is either commutative or outermost (Proposition 3).

If V is commutative then $V \in \{\mathcal{T}, SL\}$ (Theorem 5). Otherwise, V is outermost. Then $V = RA_{(2,2)}$ or V is regular (Lemma 8). Therefore, $V = RA_{(2,2)}$ or $SL \subseteq V$ since Id(SL) is the set of all regular identities of type (2, 2). Moreover, V is s-outermost and thus $RA_{(2,2)} \subseteq V$ (Lemma 7). Altogether, we have $V = RA_{(2,2)}$ or $RA_{(2,2)} \lor SL \subseteq V$.

Let σ_i , i = 1, 2, 3, 4, be hypersubstitutions defined by

$$\sigma_1: F \mapsto F(y,x) \quad \sigma_2: F \mapsto G(x,y) \quad \sigma_3: F \mapsto G(x,y) \quad \sigma_4: F \mapsto G(y,x) \\ G \mapsto G(x,y) \quad G \mapsto F(x,y) \quad G \mapsto F(y,x) \quad G \mapsto F(x,y).$$

Then $\mathcal{B}^{\circ} = \sigma_1(\mathcal{B})$, $\mathcal{B} = \sigma_1(\mathcal{B}^{\circ})$, $\mathcal{B}^{\bullet} = \sigma_2(\mathcal{B})$, $\mathcal{B} = \sigma_2(\mathcal{B}^{\bullet})$, $\mathcal{B}^{\bullet \circ} = \sigma_3(\mathcal{B})$ and $\mathcal{B} = \sigma_4(\mathcal{B}^{\bullet \circ})$. Since a regular-solid variety has to contain all its derived algebras by using regular hypersubstitutions, all of the varieties $V(\mathcal{B})$, $V(\mathcal{B}^{\circ})$, $V(\mathcal{B}^{\bullet})$ and $V(\mathcal{B}^{\bullet \circ})$ are contained in the variety V if it contains one of them. It follows that V_{BE} is the only one dual atom of V_{MID} which is a regular-solid variety of semirings, since V_{BE} is solid and $DL \not\subset V_{BE}$ (Lemma 4 and Lemma 6).

Therefore, $V \in \{RA_{(2,2)}, RA_{(2,2)} \lor SL, V_{BE}, V_{MID}\}$. Each element of the previous set is a regular-solid variety of semirings (by using Theorem 5, Lemma 6 and the fact that $RA_{(2,2)} \lor SL$ is a join of two regular-solid varieties).

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