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WEAK-HYPERLATTICES DERIVED FROM FUZZY CONGRUENCES

Blériot Blaise Njionou Koguep

Department of Mathematics and Computer science
University of Dschang, BP 67

e-mail: koguep@yahoo.com

and

Celestin Lele

Department of Mathematics and Computer science
University of Dschang, BP 67

e-mail: celestinlele@yahoo.com

Abstract

In this paper we explore the connections between fuzzy congruence relations, fuzzy ideals and homomorphisms of hyperlattices. Indeed, we introduce the concept of fuzzy quotient set of hyperlattices as it was done in the case of rings [19]. We prove that a fuzzy congruence induces a fuzzy ideal of the fuzzy quotient hyperlattice. In particular, we establish necessary and sufficient conditions for a zero-fuzzy congruence class to be a fuzzy ideal of a hyperlattice.

Keywords: hyperlattice, ideal, prime ideal, fuzzy ideal, fuzzy prime ideal, fuzzy congruence relation.

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1. Introduction

Hyperstructures theory was firstly introduced by F. Marty in the eighth congress of Scandinavian Mathematicians in 1934 [13]. Nowadays, a number of different types of hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics [7]. Particularly, the theory of hyperlattices was introduced by Konstantinidou in

1977 [12]. Barghi considered the prime ideal theorem for distributive hyperlattices [15, 16]. Koguep, Nkuimi, and Lele studied ideals and filters in hyperlattices [11]. Furthermore, they introduce normal hyperlattices and pure ideals of hyperlattices and studied their connections [10]. Rasouli and Davvaz defined fundamental relations on a hyperlattice and obtained a lattice from a hyperlattice. Moreover, they defined a topology on the set of prime ideals of a distributive hyperlattice [17, 18] and later Ameri *et al.* [1] consider the relationship between prime ideals and prime filters in hyperlattices.

The study of congruences is important both, from a theoretical standpoint and for its applications in the field of logic-based approaches to uncertainty. Regarding applications, the notion of congruence is intimately related to the foundations of fuzzy reasoning and its relationships with other logics of uncertainty. In the recent literature, several authors have presented different approaches to fuzzy congruence relations on some algebraic structures [2, 14, 20]. In [3, 5], the notion of fuzzy congruence relation on hypergroupoid is introduced.

This paper follows the current trend of providing suitable fuzzifications of crisp concepts, as a theoretical tool to the development of new method of reasoning under uncertainty, imprecision and lack of information. We follow one of our research lines which is aimed at investigating fuzzy subsets of hyperlattices [10]. In this paper, we focus not only on the notions of fuzzy congruence, fuzzy ideal and homomorphism on the generalized structure of hyperlattice, but also on their traditional connections. We provide suitable definitions of these notions in order to guarantee the classical relationship between these concepts.

The structure of the paper is as follows: in Section 2, the definition and preliminary theoretical results about ideals, congruences and homomorphism of hyperlattices are recalled, including the selection of their desirable computational properties. Then, in Section 3 the main contribution of the paper is presented: the fuzzy equivalence class and the fuzzy quotient set are given. We prove that in a bounded distributive hyperlattice, the fuzzy equivalence class of the least element is always a fuzzy ideal. We establish that any fuzzy congruence relation on a bounded distributive hyperlattice can induce a fuzzy ideal of the fuzzy quotient set. The section is ended by the isomorphism theorem for fuzzy quotient set of hyperlattices. Finally, concluding remarks and future work are presented.

2. Backgrounds on hyperlattices

For the basic notion of lattice one can see [8], and a complete overview on hyperlattices may be found in [12].

In this section, we gather some definitions and basic properties of hyperlattices that are needed.

Definition [15]**.** Let L be a non empty set and $\vee : L \times L \to \mathcal{P}(L)^*$ be a hyperoperation, where $\mathcal{P}(L)$ is a power set of L and $\mathcal{P}(L)^* = \mathcal{P}(L)\backslash\{\emptyset\}$ and $\wedge : L\times L \to L$ be an operation. Then $\mathcal{L} = (L, \vee, \wedge)$ is a **hyperlattice** if for all a, b, $c \in L$:

(i) $a \in a \vee a$, $a \wedge a = a$,

(ii) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

(iii) $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,

(iv) $a \in [a \wedge (a \vee b)] \cap [a \vee (a \wedge b)]$,

(v) $a \in a \vee b \Rightarrow a \wedge b = b$,

where for all non empty subsets A and B of L and $a \in L$, $A \wedge B = \{a \wedge b;\ a \in A, b \in B\}$, $A \vee B = \bigcup\{a \vee b;\ a \in A,\ b \in B\}$, $A \vee a = A \vee \{a\}$ and $a \vee B = \{a\} \vee B$. If $\mathcal{L}$ satisfies conditions (i) to (iv), then $\mathcal{L}$ is called **weak-hyperlattice**. In a weak-hyperlattice, we always have $a \wedge b = b \Rightarrow a \in a \vee b$, for all a, $b \in L$.

We define on L the relation $\leq$ by, $x \leq y$ if and only if $x = x \wedge y$. Then, an element $0 \in L$ such that $0 \leq x$, for all $x \in L$ is a zero of L. A unit of a hyperlattice L is an element 1 of L such that $x \leq 1$, for all $x \in L$. A hyperlattice with 0 and 1 is a bounded hyperlattice [15].

Example 1 [11]**.** Let $\mathcal{L} = (L, \vee, \wedge)$ be a lattice. If we define on $\mathcal{L}$ the hyperoperation $\sqcup$ by : $a \sqcup b = \{x \in L; a \wedge x = a \text{ and } b \wedge x = b\}$, for all $a, b \in L$, then $(L; \sqcup, \wedge)$ is a hyperlattice.

Definition [15]**.** A hyperlattice $\mathcal{L} = (L, \vee, \wedge)$ is said to be **distributive** if for all a, b, $c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Proposition 2 [11]**.** *Let $\mathcal{L} = (L, \vee, \wedge)$ be a distributive hyperlattice with a least element* 0*. Then,* $0 \vee 0 = \{0\}$.

Remark 3. The converse of the above proposition is not true, i.e. we could have $0 \vee 0 = \{0\}$ in a non-distributive hyperlattice.

Let us recall the notions of ideal and filter in hyperlattices.

Definition [15]**.** Let $\mathcal{L} = (L, \vee, \wedge)$ be a hyperlattice. A nonempty subset J of L is called an **ideal** of $\mathcal{L}$ if for all $x, y \in L$,

(i) $x, y \in J$ implies $x \vee y \subseteq J$;

(ii) if $x \in J$ and $y \leq x$, then $y \in J$.

A nonempty subset F of L is called a *filter* of $\mathcal{L}$ if for all $x, y \in L$,

(i) $x, y \in F$ implies $x \wedge y \in F$;

(ii) if $x \in F$ and $x \leq y$, then $y \in F$.

We shall denote by $I(\mathcal{L})$ and $F(\mathcal{L})$ the set of all ideals and all filters of the hyperlattice $\mathcal{L}$ respectively.

Proposition 4 [11]**.** *Let $\mathcal{L} = (L, \vee, \wedge)$ be a hyperlattice. For all $x, y \in L$, if $x \vee y$ is an ideal of $\mathcal{L}$, then $x = y$.*

Proposition 5 [15]**.** *Let $\mathcal{L} = (L, \vee, \wedge)$ be a distributive hyperlattice. If $a \in L$ then $I = (a] = \{x \in L \mid x \leq a\}$ is an ideal.*

Corollary 6 [15]**.** *Let $\mathcal{L} = (L, \vee, \wedge)$ be a distributive hyperlattice and a, $b \in L$, then we have $a \vee b \subseteq \bigcap\limits_{\substack{c \geq a \\ c \geq b}} (c]$*

As usual, a non-empty intersection of ideals of a hyperlattice $\mathcal{L}$ is an ideal of $\mathcal{L}$. If $\mathcal{L} = (L, \vee, \wedge)$ is a bounded distributive hyperlattice and A a non empty subset of $\mathcal{L}$. The ideal of $\mathcal{L}$ induced by A is the intersection of all ideals of $\mathcal{L}$ containing A and is denoted by $\langle A \rangle$. For each $a \in L$, $\langle \{a\} \rangle$ is denoted by $\langle a \rangle$. One should observe that if $\mathcal{L}$ is a bounded hyperlattice, I and J ideals of $\mathcal{L}$ and $a \in L$, then $\langle a \rangle = (a]$ and $\langle I \cap J \rangle = I \cap J$.

Proposition 7 [15]**.** *Let $\mathcal{L} = (L, \vee, \wedge)$ be a distributive hyperlattice. If I is an ideal of $\mathcal{L}$, and $a \in L$ then $I \vee (a]$ is an ideal of $\mathcal{L}$ and $\langle I \cup \{a\} \rangle = I \vee (a]$.*

We recall a characterization of the ideal generated by a subset in hyperlattices.

Proposition 8 [10]**.** *Let X be a nonempty subset of a distributive hyperlattice $\mathcal{L}$, we have $\langle X \rangle = \{x : x \in (a_1] \vee (a_2] \vee \cdots \vee (a_n]$, for some $a_1, \ldots, a_n \in X$ and $n \geq 1\}$.*

Corollary 9 [10]**.** *Let I be an ideal of a distributive hyperlattice $\mathcal{L}$ and $a \in L$, we have $\langle I \cup \{a\} \rangle = \{x \in L : x \in \alpha \vee \beta$ for some $\alpha \in I; \beta \leq a\}$.*

A number of papers have been published about congruence relations on different algebraic hyperstructures (see for example [6]). Since our aim is to generalize the results about congruences on hyperlattices, let us recall some notions about the concepts that we will use.

Through this section, $\mathcal{L}$ stands for the hyperlattice $(L, \wedge, \vee)$.

We recall the following notation (from [16]) which will be very useful. Let $\mathcal{R}$ be a binary relation on L and A, $B \subseteq L$, then $A\mathcal{R}B$ means that, for all $a \in A$, there exists $b \in B$ such that $a\mathcal{R}b$ and for all $b \in B$, there exists $a \in A$ such that $a\mathcal{R}b$. For simplicity, the notations $A\mathcal{R}B$ and $(A, B) \in \mathcal{R}$ will be used interchangeably.

Definition [16]**.** A ***congruence relation*** on the hyperlattice $\mathcal{L}$ is any equivalence relation θ which satisfies, for all a, b, c, $d \in \mathcal{L}$, if $a\theta b$, $c\theta d$, then $(a \wedge c)\theta(b \wedge d)$ and $(a \vee c)\theta(b \vee d)$.

The following remark is useful when computing with congruences.

Remark 10. An equivalence relation on a lattice $\mathcal{L}$ is a congruence relation if and only if, for all a, b, $c \in L$, $a\theta b$ implies $((a \vee c)\theta(b \vee c)$ and $(a \wedge c)\theta(b \wedge c))$.

Let L be a non empty set and θ an equivalence relation on L. L/θ denote the set of blocks of the partition of L induced by θ, that is, $L/\theta = \{[x]_\theta, x \in L\}$, where $[x]_\theta$ is the θ-equivalence class of the element x in L.

Proposition 11 [16]**.** *Let $\mathcal{L} = (L; \vee, \wedge)$ be a hyperlattice and θ a congruence relation on $\mathcal{L}$. The hyperoperation $\vee$ and the operation $\wedge$ on L/θ by $[a] \vee [b] = \{[x]; x \in a \vee b\}$ and $[a] \wedge [b] = [a \wedge b]$, for all a, $b \in L$, are well defined.*

From the work done by Jakubik [9] and according to the definition of a congruence relation on hyperlattices, it is easy to see that the quotient of a hyperlattice by a congruence relation is not always a hyperlattice.

Proposition 12. *If $\mathcal{L} = (L; \vee, \wedge)$ is a hyperlattice and θ a congruence relation on $\mathcal{L}$, then the quotient hyperstructure $(L/\theta; \vee, \wedge)$ is a weak-hyperlattice.*

Proof. Let a, b, $c \in L$.

(i) $a \in a \vee a$, then $[a] \in [a] \vee [a]$. Also, $[a] = [a \wedge a] = [a] \wedge [a]$.

(ii) $[a] \vee [b] = [b] \vee [a]$ and $[a] \wedge [b] = [b] \wedge [a]$.

(iii) Recall that $([a] \vee [b]) \vee [c] = \bigcup\limits_{x \in a \vee b} \{[t];\ t \in x \vee c\}$ and $[a] \vee ([b] \vee [c]) = \bigcup\limits_{y \in b \vee c} \{[s]; s \in a \vee y\}$

$$\begin{aligned} ([a] \wedge [b]) \wedge [c] &= [a \wedge b] \wedge [c] \\ &= [(a \wedge b) \wedge c] \\ &= [a \wedge (b \wedge c)] \\ &= [a] \wedge [b \wedge c] \\ &= [a] \wedge ([b] \wedge [c]) \end{aligned}$$

$$\begin{aligned} [t] \in ([a] \vee [b]) \vee [c] &\Leftrightarrow \exists x \in a \vee b, t \in x \vee c \\ &\Leftrightarrow t \in (a \vee b) \vee c \\ &\Leftrightarrow t \in a \vee (b \vee c) \\ &\Leftrightarrow \exists y \in b \vee c, t \in a \vee y \\ &\Leftrightarrow t \in (a \vee b) \vee c \\ &\Leftrightarrow \exists y \in b \vee c, [t] \in [a] \vee [b] \\ &\Leftrightarrow [t] \in [a] \vee ([b] \vee [c]) \end{aligned}$$

Hence, $([a] \vee [b]) \vee [c] = [a] \vee ([b] \vee [c])$.

(iv) $[a]\wedge([a]\vee[b]) = \{[a]\wedge[x]; x \in a\vee b\} = \{[a\wedge x]; x \in a\vee b\}$. Since $a \in a\wedge(a\vee b)$, there exists $x \in a \vee b$ such that $a = a \wedge x$. Therefore, $[a] \in [a] \wedge ([a] \vee [b])$.

$[a] \vee ([a] \wedge [b]) = [a] \vee [a \wedge b] = \{[x]; x \in a \vee (a \wedge b)\}$. Since $a \in a \vee (a \wedge b)$, we have $[a] \in [a] \vee ([a] \wedge [b])$. Hence $([a] \in [a] \wedge ([a] \vee [b])) \cap ([a] \vee ([a] \wedge [b]))$. ■

Definition [16]**.** Let $f : \mathcal{L}_1 \to \mathcal{L}_2$ be a map between two (weak)hyperlattices. f is said to be a ***homomorphism*** if for all $a, b \in L_1$, we have $f(a\vee b) = f(a)\vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$.

Proposition 13 [16]**.** *Let θ be a congruence relation on $\mathcal{L}$. Then the map $\widetilde{\theta} : L \to L/\theta$ defined by $\widetilde{\theta}(x) = [x]$, for all $x \in L$, is a surjective homomorphism.*

Let us now move to the context of fuzzy congruences on hyperlattices.

3. Fuzzy congruence relations, fuzzy ideals and homomorphisms on hyperlattices

In the recent literature, several authors have presented different approaches to fuzzy congruence relations on some algebraic structures [3, 5, 14, 20]. In [3], the notion of fuzzy congruence relation on hypergroupoid is introduced.

Our aim here is to study the connections between fuzzy congruences, fuzzy ideals and homomorphisms of hyperlattices.

We will assume that $\mathcal{L} = (L, \vee, \wedge)$ is a hyperlattice (recall that $\vee$ is a binary hyperoperation and $\wedge$ is a binary operation).

We recall the definition of fuzzy ideal and fuzzy filter of hyperlattice.

Definition [11]**.** Let μ be a fuzzy subset of L. Then:

(i) μ is **a fuzzy ideal** of $\mathcal{L}$ if, for all $x, y \in L$,

- $\star$ $\inf\limits_{a\in x\vee y} \mu(a) \geq \mu(x) \wedge \mu(y)$
- $\star$ $x \leq y \Rightarrow \mu(x) \geq \mu(y)$

(ii) μ is a **fuzzy filter** of $\mathcal{L}$ if for all $x, y \in L$,

- $\star$ $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$
- $\star$ $x \leq y \Rightarrow \mu(x) \leq \mu(y)$

First let us analyse fuzzy congruences and fuzzy ideals connections.

3.1. Fuzzy congruences and fuzzy ideals of hyperlattices

In [19], the concepts of fuzzy quotient set modulo a fuzzy congruence relation of ring is studied, in this section we applied it to the case of hyperlattices.

We will use the following definition of fuzzy equivalence relation due to Zadeh [21].

Definition [21]**.** Let X, Y and Z be non-empty sets.

(i) A function $\rho : X \times Y \to [0,1]$ (i.e., a fuzzy subset of $X \times Y$) is called a **fuzzy relation** between a set X and a set Y.

(ii) Let $\rho : X \times Y \to [0,1]$ and $\sigma : Y \times Z \to [0,1]$ be two fuzzy relations, the product $\sigma \circ \rho$ of ρ and σ is defined by $(\sigma \circ \rho)(x,z) = \sup\limits_{y\in Y}\{\min\{\rho(x,y), \sigma(y,z)\}\}$.

(iii) Let ρ be a fuzzy relation on a nonempty set X. ρ is said to be:

- **reflexive** if, $\rho(x,x) = \sup\limits_{y,\, z\in X} \rho(y,z)$, for all $x \in L$;
- **symmetric** if, $\rho(x,y) = \rho(y,x)$, for all $x,\ y \in X$;
- **transitive** if, $\rho(x,y) \wedge \rho(y,z) \le \rho(x,z)$, for all $x,\ y,\ z \in X$.

(iv) A **fuzzy equivalence relation** is a reflexive, symmetric and transitive fuzzy relation.

(v) A fuzzy relation ρ on a nonempty set X satisfies the **sup property** if for every subset Y of X there exists $x,\ y \in Y$ such that $\rho(x,y) = \sup\limits_{a,\, b\in Y} \rho(a,b)$.

(vi) Let ρ be a fuzzy relation on a nonempty set X. For $t \in [0,1]$, a t-cut (level relation) of ρ is the corresponding level subset : $\rho_t = \{(x,y);\ \rho(x,y) \ge t\}$ (it is a crisp relation on X).

Since a fuzzy relation on a nonempty set X is a fuzzy subset of $X \times X$, we can define the inclusion, intersection and union of fuzzy relations as follows : $\rho \subseteq \sigma$ if $\rho(x,y) \le \sigma(x,y)$, for all $x,\ y \in X$. For all family $\{\rho_i, i \in \Lambda\}$ of fuzzy relations on X, $(\bigcap\limits_{i\in\Lambda} \rho_i)(x,y) = \inf\limits_{i\in\Lambda} \rho_i(x,y)$ and $(\bigcup\limits_{i\in\Lambda} \rho_i)(x,y) = \sup\limits_{i\in\Lambda} \rho_i(x,y)$, for all $x,\ y \in X$.

We denote by χ_R the characteristic function of a binary relation R on a nonempty set X.

The set of all fuzzy equivalence relations on a non empty set X will be denoted by $FEq(X)$. In [14] Murali proved that $(FEq(X); \subseteq)$ is a complete lattice where the meet is the intersection and the join is the transitive closure of the union.

For every fuzzy relation ρ on $\mathcal{L}$, the powerset extension of ρ is defined as: $\widehat{\rho} : \mathcal{P}(L) \times \mathcal{P}(L) \to [0;1]$, with $\widehat{\rho}(X,Y) = (\bigwedge\limits_{x\in X} \bigvee\limits_{y\in Y} \rho(x,y)) \wedge (\bigwedge\limits_{y\in Y} \bigvee\limits_{x\in X} \rho(x,y))$ ([5], for more details).

Definition [5]**.** A fuzzy equivalence relation ρ on a hyperlattice $\mathcal{L} = (L, \vee, \wedge)$ is said to be a fuzzy congruence relation if for all a, b, c, $d \in L$ we have the following:

(i) $\widehat{\rho}(a \vee c, b \vee d) \geq \rho(a,b) \wedge \rho(c,d)$,

(ii) $\rho(a \wedge c, b \wedge d) \geq \rho(a,b) \wedge \rho(c,d)$.

Proposition 14 [5]**.** *Let ρ be a fuzzy equivalence relation on a hyperlattice $\mathcal{L}$. Then ρ is a fuzzy congruence relation if only if for all a, b, $c \in L$ we have the following:*

(i) $\widehat{\rho}(a \vee c, b \vee c) \geq \rho(a,b)$,

(ii) $\rho(a \wedge c, b \wedge c) \geq \rho(a,b)$.

In [4], Borzooei et al. have established that the transfer principle is satisfied for the notion of fuzzy equivalence relation on hypergroupoids and hyper BCK-algebras. The following proposition shows that the result is still true in the case of hyperlattice.

Proposition 15 [4]**.** *Let ρ be a fuzzy relation on a hyperlattice $\mathcal{L}$ that satisfies the sup property. Then the following statements are equivalent:*

(i) *ρ is a fuzzy equivalence relation on $\mathcal{L}$,*

(ii) *$\rho_t \neq \emptyset$ is an equivalence relation on $\mathcal{L}$ for all $t \in [0;1]$.*

Theorem 16. *Let ρ be a fuzzy relation on $\mathcal{L}$ that satisfies the sup property. ρ is a fuzzy congruence relation if and only if every nonempty level subset ($\rho_t = \{(x,y) \in L \times L;\ \rho(x,y) \geq t\}$, $t \in [0;1]$) is a congruence relation.*

Proof. According to the above Proposition15, we need only to prove that ρ is compatible with $\vee$ and $\wedge$ if and only if ρ_t is compatible with $\vee$ and $\wedge$, for all $t \in [0;1]$ such that $\rho_t \neq \emptyset$.

Suppose that ρ is compatible with $\vee$ and $\wedge$. Let $t \in [0;1]$ such that $\rho_t \neq \emptyset$. For a, b, $c \in L$ such that $a\rho_t b$. We have $\widehat{\rho}(a \vee c, b \vee c) \geq \rho(a,b) \geq t$ and $\rho(a \wedge c, b \wedge c) \geq \rho(a,b) \geq t$. this implies that $a \wedge c \rho_t b \wedge c$.

Also $\bigwedge\limits_{x \in a \vee c} \bigvee\limits_{y \in b \vee c} \rho(x,y) \geq \rho(a,b) \geq t$ and $\bigwedge\limits_{y \in b \vee c} \bigvee\limits_{x \in a \vee c} \rho(x,y) \geq \rho(a,b) \geq t$.

So for all $x \in a \vee c$, $\bigvee\limits_{y \in b \vee c} \rho(x,y) \geq t$ and for all $y \in b \vee c$, $\bigvee\limits_{x \in a \vee c} \rho(x,y) \geq t$.

Because ρ satisfies the sup property, there exist $y_0 \in b \vee c$ and $x_0 \in a \vee c$ such that for all $x \in a \vee c$, $\rho(x, y_0) = \bigvee\limits_{y \in b \vee c} \rho(x,y) \geq t$ and for all $y \in b \vee c$, $\rho(x_0, y) = \bigvee\limits_{x \in a \vee c} \rho(x,y) \geq t$.

Hence for all $x \in a \vee c$ there exist $y_0 \in b \vee c$ such that $x\rho_t y_0$ and for all $y \in b \vee c$ there exist $x_0 \in a \vee c$ such that $x_0 \rho_t y$. This implies that $a \vee c\rho_t b \vee c$.

Conversely suppose that ρ_t is compatible with $\vee$ and $\wedge$, for all $t \in [0;1]$ such that $\rho_t \neq \emptyset$. Let $a,\ b,\ c \in L$, for $t = \rho(a,b)$, we have $(a,b) \in \rho_t$, i.e., $\rho_t \neq \emptyset$. Therefore, $a \wedge c\rho_t b \wedge c$ and $a \vee c\rho_t b \vee c$. This implies $\rho(a \wedge c, b \wedge c) \geq t = \rho(a,b)$.

And also, for all $x \in a \vee c$ there exist $y \in b \vee c$ such that $\rho(x,y) \geq t$. Then $\bigwedge\limits_{x \in a \vee c} \bigvee\limits_{y \in b \vee c} \rho(x,y) \geq t = \rho(a,b)$. And for all $y \in b \vee c$ there exist $x \in a \vee c$ such that $\rho(x,y) \geq t$. This shows that $\bigwedge\limits_{y \in b \vee c} \bigvee\limits_{x \in a \vee c} \rho(x,y) \geq t = \rho(a,b)$. Therefore, $\widehat{\rho}(a \vee c, b \vee c) = \big(\bigwedge\limits_{x \in a \vee c} \bigvee\limits_{y \in b \vee c} \rho(x,y)\big) \wedge \big(\bigwedge\limits_{y \in b \vee c} \bigvee\limits_{x \in a \vee c} \rho(x,y)\big) \geq \rho(a,b)$.

Thus ρ is compatible with $\vee$ and $\wedge$. ■

Corollary 17. *Let R be a crisp relation on $\mathcal{L}$. Then R is an equivalence relation (resp. congruence relation) on $\mathcal{L}$ iff, χ_R is a fuzzy equivalence relation (resp. fuzzy congruence relation) on $\mathcal{L}$.*

Proof. It is a direct application of Theorem 16, since $\chi_R : L \times L \to [0;1]$ is fuzzy relation on $\mathcal{L}$ that satisfy the sup property (for all $x \in L$, $\chi_R(x,x) = 1 = \bigvee\limits_{x,\, y \in L} \chi_R(x,y)$). ■

Let ρ be a fuzzy equivalence relation on a hyperlattice $\mathcal{L}$. For each $a \in L$, we define a fuzzy subset ρ^a of L as follow: $\rho^a(x) = \rho(a,x)$, for all $x \in L$. The fuzzy subset ρ^a of L is called the **fuzzy ρ-equivalence class** of a in L. The quotient set $L/\rho = \{\rho^x;\ x \in L\}$ is called the **ρ-fuzzy quotient set** of L.

Proposition 18. *Let ρ be a fuzzy equivalence relation on a hyperlattice $\mathcal{L}$. Then for all $a,\ b \in L$, $\rho^a = \rho^b$ if and only if $\rho(a,b) = \rho(a,a)$.*

Proof. Let $a,\ b \in L$. Suppose that $\rho^a = \rho^b$, then $\rho^a(a) = \rho^b(a)$, i.e., $\rho(a,a) = \rho(b,a) = \rho(a,b)$. Conversely, if $\rho(a,b) = \rho(a,a)$, then for all $x \in L$, $\rho^a(x) = \rho(a,x) \geq (\rho(a,b) \wedge \rho(b,x)) = \rho(b,x) = \rho^b(x)$ and $\rho^b(x) = \rho(b,x) \geq (\rho(b,a) \wedge \rho(a,x)) = \rho(a,x) = \rho^a(x)$, thus $\rho^a(x) = \rho^b(x)$.

Therefore, $\rho^a = \rho^b$ if and only if $\rho(a,b) = \rho(a,a)$. ■

Let us prove that we can defined suitable operation and hyperoperation on a fuzzy quotient set.

Proposition 19. *Let ρ be a fuzzy congruence relation on a hyperlattice $\mathcal{L}$. We define $\overline{\wedge}$ and $\veebar$ on L/ρ as follow:*

For all $a,\ b \in L$, $\rho^a \,\overline{\wedge}\, \rho^b = \rho^{a \wedge b}$ and $\rho^a \veebar \rho^b = \{\rho^x;\ x \in a \vee b\} = \rho^{a \vee b}$. Then, $\overline{\wedge}$ and $\veebar$ are well defined.

Proof. Let a, b, a', $b' \in L$ such that $\rho^a = \rho^b$ and $\rho^{a'} = \rho^{b'}$. By Proposition 18 we have $\rho(a,b) = \rho(a,a) = \rho(a',a') = \rho(a',b') = \sup\limits_{y,\, z \in L} \rho(y,z)$. Further, $\rho(a \wedge a', b \wedge b') \geq \rho(a,b) \wedge \rho(a',b') = \rho(a,a) = \rho(a \wedge a', a \wedge a')$. Hence, $\rho^{a \wedge a'} = \rho^{b \wedge b'}$. That is, $\rho^a \,\overline{\wedge}\, \rho^{a'} = \rho^b \,\overline{\wedge}\, \rho^{b'}$.

Let $t = \sup\limits_{y,\, z \in L} \rho(y,z)$. We have $a\rho_t a'$ and $b\rho_t b'$. Then, $(a \vee b)\rho_t(a' \vee b')$, since ρ_t is a congruence relation on $\mathcal{L}$. Now, let $\rho^x \in \rho^a \,\underline{\vee}\, \rho^b$, there exists $x' \in a \vee b$ such that $\rho^x = \rho^{x'}$. Since, $x' \in a \vee b$ and $(a \vee b)\rho_t(a' \vee b')$, there exists $y \in a' \vee b'$ such that $x'\rho_t y$. Therefore, $\rho(x',y) \geq t = \sup\limits_{y,\, z \in L} \rho(y,z) \geq \rho(x',y)$. Hence, $\rho(x',y) = t$. As ρ is a fuzzy equivalence relation, for all $s \in L$, we have

$$\begin{aligned} \rho^x(s) &= \rho^{x'}(s) \\ &= \rho(x',s) \\ &\geq \rho(x',y) \\ &\geq \rho(x',y) \wedge \rho(y,s) \\ &= t \wedge \rho(y,s) \\ &= \rho^y(s) \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \rho^y(s) &= \rho(y,s) \\ &\geq \rho(x',y) \\ &\geq \rho(y,x') \wedge \rho(x',s) \\ &= t \wedge \rho(x',s) \\ &= \rho^{x'}(s) \\ &= \rho^x(s). \end{aligned}$$

Hence, $\rho^x = \rho^y$ and $y \in a' \vee b'$, i.e., $\rho^x \in \rho^{a'} \,\underline{\vee}\, \rho^{b'}$. Therefore, $\rho^a \,\underline{\vee}\, \rho^b \subseteq \rho^{a'} \,\underline{\vee}\, \rho^{b'}$.

The converse, $\rho^{a'} \,\underline{\vee}\, \rho^{b'} \subseteq \rho^a \,\underline{\vee}\, \rho^b$ is establish similarly. Thus $\rho^a \,\underline{\vee}\, \rho^b = \rho^{a'} \,\underline{\vee}\, \rho^{b'}$ and $\overline{\wedge}$ and $\underline{\vee}$ are well defined. ■

The following examples show that the quotient hyperstructure $(L/\rho; \overline{\wedge}, \underline{\vee})$ is not always a hyperlattice.

Example 20. Let $L = \{0, a, b, c, 1\}$ and define $\wedge$ and $\vee$ by the following Cayley tables

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	b	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	L	L\{0 }	L\{0 }	L\{0 }	L\{0 }
a	L\{0 }	L	{0, 1 }	L\{a }	L\{a }
b	L\{0 }	{0, 1 }	L	L\{b }	L\{b }
c	L\{0 }	L\{a }	L\{b }	L	L\{c }
1	L\{0 }	L\{a }	L\{b }	L\{c }	L

It is easy to verify that $\mathcal{L} = (L, \vee, \wedge, 0, 1)$ is a non-distributive bounded hyperlattice. Consider ρ the fuzzy relation defined on $\mathcal{L}$ by the following matrix:

$$\rho = \left[\begin{array}{c|ccccc} & 0 & a & b & c & 1 \\ \hline 0 & 1 & 1 & 0,5 & 0,5 & 0,5 \\ a & 1 & 1 & 0,5 & 0,5 & 0,5 \\ b & 0,5 & 0,5 & 1 & 1 & 1 \\ c & 0,5 & 0,5 & 1 & 1 & 1 \\ 1 & 0,5 & 0,5 & 1 & 1 & 1 \end{array}\right]$$

One can easily verified that ρ is a fuzzy congruence relation on $\mathcal{L}$. $L/\rho = \{\rho^0, \rho^1\}$ and it is not a hyperlattice because, $\rho^0 \in \rho^0 \veebar \rho^1$ (since $a \in 0 \vee 1$ and $\rho^0 = \rho^a$) and $\rho^0 \barwedge \rho^1 = \rho^{0\wedge 1} = \rho^0 \neq \rho^1$.

Example 21. Let $L = \{0, a, b, 1\}$ and define $\wedge$ and $\vee$ by the following Cayley tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	{0}	{a}	{b}	{1}
a	{a}	{0, a}	{1}	{b, 1}
b	{b}	{1}	{0, b}	{a, 1}
1	{1}	{b, 1}	{a, 1}	L

$\mathcal{L} = (L, \vee, \wedge, 0, 1)$ is a bounded hyperlattice. Consider ρ the fuzzy relation defined on $\mathcal{L}$ by the following matrix :

$$\rho = \left[\begin{array}{c|cccc} & 0 & a & b & 1 \\ \hline 0 & 1 & 1 & 0,2 & 0,2 \\ a & 1 & 1 & 0,2 & 0,2 \\ b & 0,2 & 0,2 & 1 & 1 \\ 1 & 0,2 & 0,2 & 1 & 1 \end{array}\right]$$

One can easily verified that ρ is a fuzzy congruence relation on $\mathcal{L}$. We have the following Cayley table on $L/\rho = \{\rho^0, \rho^1\}$,

$\veebar$	ρ^0	ρ^1
ρ^0	$\{\rho^0\}$	$\{\rho^1\}$
ρ^1	$\{\rho^1\}$	L/ρ.

The quotient hyperstructure $(L/\rho; \barwedge, \veebar)$ is a hyperlattice.

Let us now prove that the quotient hyperstructure $\mathcal{L}/\rho := (L/\rho; \barwedge, \veebar)$ is always a weak-hyperlattice.

Proposition 22. *Let ρ be a fuzzy congruence relation on a hyperlattice $\mathcal{L}$. Then $\mathcal{L}/\rho$ is a weak-hyperlattice.*

Proof. Let a, b, $c \in L$. We have

(i) $\rho^a \veebar \rho^a = \{\rho^x;\, x \in a \vee a\}$, then, $\rho^a \in \rho^a \veebar \rho^a$, since $a \in a \vee a$.

(ii) $\rho^a \veebar \rho^b = \rho^b \veebar \rho^a$.

(iii)
$$\begin{aligned} \rho^x \in \rho^a \veebar (\rho^b \veebar \rho^c) &\Leftrightarrow \exists y \in b \vee c,\ \rho^x \in \rho^a \veebar \rho^y \\ &\Leftrightarrow \exists y \in b \vee c,\ \exists z \in a \vee y,\ \rho^x = \rho^z \\ &\Leftrightarrow \exists z \in a \vee (b \vee c),\ \rho^x = \rho^z \\ &\Leftrightarrow \exists z \in (a \vee b) \vee c,\ \rho^x = \rho^z \text{ since } \mathcal{L} \text{ is a hyperlattice} \\ &\Leftrightarrow \exists z \in L,\ \exists y \in a \vee b,\ z \in y \vee c \text{ and } \rho^x = \rho^z \\ &\Leftrightarrow \rho^x \in (\rho^a \veebar \rho^b) \veebar \rho^c. \end{aligned}$$

Hence, $\rho^a \veebar (\rho^b \veebar \rho^c) = (\rho^a \veebar \rho^b) \veebar \rho^c$.

(iv) $\rho^a \barwedge (\rho^a \veebar \rho^b) = \{\rho^x;\ \exists y \in a \vee b$ and $\rho^x = \rho^a \barwedge \rho^y\} = \{\rho^x;\ \exists y \in a \vee b$ and $\rho^x = \rho^{a\wedge y}\}$ and

$\rho^a \veebar (\rho^a \barwedge \rho^b) = \rho^a \veebar \rho^{a\wedge b} = \{\rho^x;\ x \in a \vee (a \wedge b)\}$. Then, $\rho^a \in \rho^a \barwedge (\rho^a \veebar \rho^b)$ and $\rho^a \in \rho^a \veebar (\rho^a \barwedge \rho^b)$, because $a \in a \wedge (a \vee b)$ and $a \in a \vee (a \wedge b)$.

Thus, the quotient hyperstructure $(L/\rho; \barwedge, \veebar)$ is a weak-hyperlattice. ■

In the case of lattice $\mathcal{L}$ with bottom element 0, for any fuzzy congruence relation ρ on $\mathcal{L}$, ρ^0 is always a fuzzy ideal of $\mathcal{L}$, see [20]. But in the framework of hyperlattices it is not true as the following example show.

Example 23. (i) Let $\mathcal{L} = (L; \vee, \wedge)$ be the hyperlattice and ρ the fuzzy congruence relation on $\mathcal{L}$ of Example 21. ρ^0 is a fuzzy ideal of $\mathcal{L}$.

(ii) Let $\mathcal{L} = (L; \vee, \wedge)$ be the hyperlattice and ρ the fuzzy congruence relation on $\mathcal{L}$ of Example 20. ρ^0 is not a fuzzy ideal of $\mathcal{L}$, because $\inf\limits_{s\in 0\vee 0} \rho^0(s) < (\rho^0(0) \wedge \rho^0(0))$, since $\inf\limits_{s\in 0\vee 0} \rho^0(s) = 0,5$ and $\rho^0(0) = 1$.

In the following theorem, we prove the necessarily and sufficient condition for ρ^0 to be a fuzzy ideal.

Proposition 24. *Let $\mathcal{L} = (L; \vee, \wedge)$ be a hyperlattice with bottom element 0 and ρ a fuzzy congruence relation on $\mathcal{L}$. ρ^0 is a fuzzy ideal of $\mathcal{L}$ if and only if, $\rho^0 \veebar \rho^0 = \{\rho^0\}$ (i.e., $\forall s \in 0 \vee 0$, $\rho^0 = \rho^s$).*

Proof. Suppose $\rho^0 \veebar \rho^0 = \{\rho^0\}$, i.e., $\forall z \in L$, $\forall s \in 0 \vee 0$, $\rho(0, z) = \rho(s, z)$. Since ρ is a fuzzy congruence relation on $\mathcal{L}$, we have for all x, $y \in L$,

$$\begin{aligned}
\widehat{\rho}(0 \vee 0, x \vee y) \geq \rho(0, x) \wedge \rho(0, y) &\Rightarrow \bigwedge_{z\in x\vee y} \bigvee_{s\in 0\vee 0} \rho(s, z) \geq \rho(0, x) \wedge \rho(0, y) \\
&\Rightarrow \bigwedge_{z\in x\vee y} \rho(0, z) \geq \rho(0, x) \wedge \rho(0, y) \\
&\Leftrightarrow \inf_{z\in x\vee y} \rho^0(z) \geq \rho^0(x) \wedge \rho^0(y)
\end{aligned}$$

and we have $\rho(0 \wedge x, x \wedge y) \geq \rho(0, y) \wedge \rho(x, x)$, then $\rho(0, x \wedge y) \geq \rho(0, y)$. Therefore, if $x \leq y$, then $\rho(0, x) \geq \rho(0, y)$.

Hence ρ^0 is a fuzzy ideal of $\mathcal{L}$.

Conversely, suppose that ρ^0 is a fuzzy ideal of $\mathcal{L}$. Then, for all x, $y \in L$, we have $\inf\limits_{z\in x\vee y} \rho^0(z) \geq \rho^0(x) \wedge \rho^0(y)$. Therefore, $\inf\limits_{s\in 0\vee 0} \rho^0(s) \geq \rho^0(0)$, then, $\forall s \in 0 \vee 0$, $\rho^0(s) \geq \rho(0, 0)$. Thus, by Proposition 18, $\forall s \in 0 \vee 0$, $\rho^0 = \rho^s$. ■

Corollary 25. *For all fuzzy congruence relation ρ on a distributive hyperlattice $\mathcal{L}$ with bottom element 0, ρ^0 is a fuzzy ideal of $\mathcal{L}$.*

Proof. It is true because in a distributive hyperlattice $\mathcal{L}$ with bottom element 0, $\rho^0 \veebar \rho^0 = \{\rho^0\}$ and applying the previous Proposition 24. ■

Here we will prove that any fuzzy congruence ρ of a bounded distributive hyperlattice $\mathcal{L}$ can induce a fuzzy ideal of $\mathcal{L}/\rho$. First we have the following Lemma.

Lemma 26. *Let $\mathcal{L}$ be a hyperlattice and ρ a fuzzy congruence on $\mathcal{L}$. For all x, $y \in L$, $\rho^x \leq \rho^y \Rightarrow \rho^x = \rho^{x\wedge y}$.*

Proof. Let x, $y \in L$. If $\rho^x \leq \rho^y$, then

$$\begin{aligned}\rho(x, x\wedge y) &= \rho(x\wedge x, x\wedge y), \quad \text{because } \ x\wedge x = x\\ &\geq \rho(x,x)\wedge\rho(x,y), \quad \text{by transitivity of } \ \rho\\ &= \rho^x(x)\wedge\rho^y(x)\\ &= \rho^x(x), \quad \text{since } \ \rho^x(x)\leq\rho^y(x).\end{aligned}$$

Therefore $\rho(x, x\wedge y) = \rho(x,x)$. Thus $\rho^x = \rho^{x\wedge y}$. ■

Remark 27. The converse of Lemma 26 does not hold, since in Example 20 we have $\rho^0 = \rho^{0\wedge 1}$, but ρ^0 and ρ^1 are not comparable.

Proposition 28. *Let $\mathcal{L}$ be a bounded distributive hyperlattice and ρ a fuzzy congruence on $\mathcal{L}$. Let φ be the fuzzy subset of $\mathcal{L}/\rho$ defined by $\varphi(\rho^x) = \rho^x(0) = \rho(x,0)$, for all $x \in L$.*

Then φ is a fuzzy ideal of $\mathcal{L}/\rho$.

Proof. Let x, $y \in L$. For any $a \in x \vee y$,

$$\begin{aligned}\varphi(\rho^a) &= \rho(a,0)\\ &\geq \widehat{\rho}(x\vee y, 0), \quad \text{because } \ a\in x\vee y\\ &\geq \widehat{\rho}(x\vee y, 0\vee 0), \quad \text{since } \ 0\vee 0=\{0\}\\ &\geq \rho(x,0)\wedge\rho(y,0)\\ &= \varphi(\rho^x)\wedge\varphi(\rho^y).\end{aligned}$$

Hence $\inf\limits_{a\in x\vee y}\varphi(\rho^a) \geq \varphi(\rho^x)\wedge\varphi(\rho^y)$. If $\rho^x \leq \rho^y$, then

$$\begin{aligned}\rho(x,0) &= \rho(x\wedge y, 0), \quad \text{because } \ \rho^x = \rho^{x\wedge y} \ \text{ by Lemma 26}\\ &= \rho(x\wedge y, 0\wedge x)\\ &\geq \rho(x,x)\wedge\rho(y,0), \quad \text{by transitivity of } \ \rho\\ &= \rho(y,0), \quad \text{since } \ \rho(x,x)\geq\rho(y,0).\end{aligned}$$

Therefore $\varphi(\rho^x) \geq \varphi(\rho^y)$ and we conclude that φ is a fuzzy ideal of $\mathcal{L}/\rho$. ■

Now in the following will establish the connection between fuzzy congruences and homomorphisms of hyperlattices.

3.2. Fuzzy congruence relations and homomorphisms of hyperlattices

We have shown that, from any fuzzy congruence relation ρ on a hyperlattice $\mathcal{L}$, we can construct a set of fuzzy ρ-equivalence class. This induces a natural homomorphism from $\mathcal{L}$ to $\mathcal{L}/\rho$ as the following Proposition shows.

Proposition 29. *Let ρ be a fuzzy congruence relation on a hyperlattice $\mathcal{L}$. Then the map $\widetilde{\rho} : \mathcal{L} \to \mathcal{L}/\rho$ defined by $\widetilde{\rho}(x) = \rho^x$, for all $x \in L$, is a surjective homomorphism.*

Proof. $\widetilde{\rho}$ is well defined (see Proposition 19). We have for all x, $y \in L$,

$$\begin{aligned} \widetilde{\rho}(x \vee y) &= \rho^{x\vee y} \\ &= \rho^x \veebar \rho^y \\ &= \widetilde{\rho}(x) \veebar \widetilde{\rho}(y) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{\rho}(x \wedge y) &= \rho^{x\wedge y} \\ &= \rho^x \barwedge \rho^y \\ &= \widetilde{\rho}(x) \barwedge \widetilde{\rho}(y). \end{aligned}$$

Hence, $\widetilde{\rho}$ is a surjective homomorphism. ∎

Remark 30. Given $f : \mathcal{L}_1 \to \mathcal{L}_2$ a homomorphism of (weak)hyperlattices, the relation, $\ker(f) = \{(a, b);\ f(a) = f(b),\ a, b \in L_1\}$ is a congruence relation on $\mathcal{L}_1$. Therefore, the characteristic function $\chi_{\ker(f)}$ is a fuzzy congruence on $\mathcal{L}_1$.

Theorem 31. *Let $f : \mathcal{L}_1 \to \mathcal{L}_2$ be a homomorphism of (weak)hyperlattices. Then, there is a one-to-one homomorphism $g : \mathcal{L}_1/\chi_{\ker(f)} \to \mathcal{L}_2$ such that $f = g \circ \widetilde{\chi}_{\ker(f)}$.*

Proof. Define $g : \mathcal{L}_1/\chi_{\ker(f)} \to \mathcal{L}_2$ by $g((\chi_{\ker(f)})^a) = f(a)$, for all $a \in L_1$. Let a, $b \in L_1$ such that $(\chi_{\ker(f)})^a = (\chi_{\ker(f)})^b$, then $\chi_{\ker(f)}(a, b) = 1$. Therefore, $(a, b) \in \ker(f)$. Thus, we have $g((\chi_{\ker(f)})^a) = f(a) = f(b) = g((\chi_{\ker(f)})^b)$. This means that g is well-defined.

Let a, $b \in L_1$, we have

If $g((\chi_{\ker(f)})^a) = g((\chi_{\ker(f)})^b)$ (i.e., $f(a) = f(b)$), then $\chi_{\ker(f)}(a, b) = 1$. Therefore, $(\chi_{\ker(f)})^a = (\chi_{\ker(f)})^b$. Thus g is one-to-one.

$$\begin{aligned} g((\chi_{\ker(f)})^a \veebar (\chi_{\ker(f)})^b) &= g((\chi_{\ker(f)})^{a\vee b}) \\ &= f(a \vee b) \\ &= f(a) \vee f(b) \\ &= g((\chi_{\ker(f)})^a) \vee g((\chi_{\ker(f)})^b) \end{aligned}$$

and

$$\begin{aligned} g((\chi_{\ker(f)})^a \ \overline{\wedge}\ (\chi_{\ker(f)})^b) &= g((\chi_{\ker(f)})^{a\wedge b}) \\ &= f(a\wedge b) \\ &= f(a)\wedge f(b) \\ &= g((\chi_{\ker(f)})^a)\wedge g((\chi_{\ker(f)})^b)\,. \end{aligned}$$

Then, g is a one-to-one homomorphism.

Let $a \in L_1$, we have

$$\begin{aligned} (g\circ \widetilde{\chi}_{\ker(f)})(a) &= g(\widetilde{\chi}_{\ker(f)}(a)) \\ &= g((\chi_{\ker(f)})^a) \\ &= f(a)\,. \end{aligned}$$

Hence, $f = g\circ \widetilde{\chi}_{\ker(f)}$. ■

Next theorem is an extension of homomorphism theorem induced by fuzzy congruences.

Theorem 32. *Let ρ and δ be two fuzzy congruence relations on a hyperlattice $\mathcal{L}$, such that $\rho \subseteq \delta$ and $\sup\limits_{y,\,z\in L} \rho(y,z) = 1$. Then, there is a unique homomorphism $f : \mathcal{L}/\rho \to \mathcal{L}/\delta$ such that $f\circ\widetilde{\rho} = \widetilde{\delta}$ and $(\mathcal{L}/\rho)\,/\chi_{\ker(f)}$ is isomorphic to $\mathcal{L}/\delta$.*

Proof. Define $f : \mathcal{L}/\rho \to \mathcal{L}/\delta$ as follows: $f(\rho^a) = \delta^a$, for all $a \in L$. Also, define $g : (\mathcal{L}/\rho)\,/\chi_{\ker(f)} \to \mathcal{L}/\delta$ by, $g((\chi_{\ker(f)})^{\rho^a}) = \delta^a$, for all $a \in L$.

Let a, $b \in L$ such that $\rho^a = \rho^b$. Then, $1 = \rho(a,b) \le \delta(a,b)$. Therefore, $\delta(a,b) = 1$. Thus, by Proposition 18, $\delta^a = \delta^b$, i.e., $f(\rho^a) = f(\rho^b)$. Hence, f is well-defined.

It can be easily proved that f is a unique homomorphism such that $f\circ\widetilde{\rho} = \widetilde{\delta}$ and hence, we omit the details.

From Theorem 31, g is a one-to-one homomorphism and it is surjective by construction. Hence, $(\mathcal{L}/\rho)\,/\chi_{\ker(f)} \cong \mathcal{L}/\delta$. ■

Conclusion and future research

The study of connections between fuzzy congruences, fuzzy ideals and homomorphism of hyperlattices lead us to the classical relationships between these concepts. Particularly, the quotient set of a hyperlattice $\mathcal{L}$ by a fuzzy congruence relation is a weak-hyperlattice and if $\mathcal{L}$ is a bounded distributive hyperlattice then for any fuzzy congruence relation on $\mathcal{L}$: the fuzzy equivalence class of zero is a fuzzy ideal.

As future work, we will study conditions on a hyperlattice which guarantee the possibility of defining fuzzy congruences from a fuzzy ideal, in the same way that distributivity allows to do so in the theory of lattices [20].

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