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SYMMETRIC BI-DERIVATIONS IN ALMOST DISTRIBUTIVE LATTICES

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Abstract

In this paper, we introduce the concept of symmetric bi-derivation in an Almost Distributive Lattice (ADL) and derive some important properties of symmetric bi-derivations in ADLs.

Keywords: Almost Distributive Lattice (ADL), derivations, isotone derivations and symmetric bi-derivations.

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1. Introduction

In 1980, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [13]. This class of ADLs include most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other.

The concept of derivation in an ADL was introduced in our earlier paper [11]. The notion of derivation in Lattices was first given in Szasz [14] in 1974. Posner [9] introduced derivations in ring theory and later several authors worked on it ([2, 5]). Several authors worked on derivations in Lattices ([1, 3, 4, 6, 7, 8, 14, 15, 16] and [17]). We have introduced the concept of f-derivations in an ADL in our paper [12]. The concept of symmetric bi-derivations in lattices was introduced by Ceven [4] in 2009.

In this paper, we introduce the concept of symmetric bi-derivation in an ADL and invistigate some important properties. Also, we define the trace d of a

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symmetric bi-derivation D on an ADL L and prove some important properties based on it. We define a fixed set $Fix_d(L)$ and prove that it is a weak ideal if d is the trace of a join preserving symmetric bi-derivation D on an associative ADL L. Also, we introduce the concept of an isotone symmetric bi-derivation in an ADL and we establish a set of conditions which are sufficient for a symmetric bi-derivation on an ADL with a maximal element to become an isotone symmetric bi-derivation. We prove that if an ADL L has a maximal element, then the trace of a join preserving symmetric bi-derivation on L is a homomorphism. Finally, we prove that the set of all principal symmetric bi-derivations on an ADL L forms an ADL.

2. Preliminaries

In this section, we recollect certain basic concepts and important results on Almost Distributive Lattices.

Definition 2.1 [10]**.** An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called an Almost Distributive Lattice, if it satisfies the following axioms:

L_1 : $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ $(RD\wedge)$
L_2 : $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $(LD\wedge)$
L_3 : $(a \vee b) \wedge b = b$
L_4 : $(a \vee b) \wedge a = a$
L_5 : $a \vee (a \wedge b) = a$.

Definition 2.2 [10]**.** Let L be any non-empty set. Define, for any $x, y \in L$, $x \vee y = x$ and $x \wedge y = y$. Then $(L, \vee, \wedge)$ is an ADL and it is called a discrete ADL.

Throughout this paper L stands for an ADL $(L, \vee, \wedge)$ unless otherwise specified.

Lemma 2.3 [10]**.** *For any $a, b \in L$, we have*

(i) $a \wedge a = a$,
(ii) $a \vee a = a$,
(iii) $(a \wedge b) \vee b = b$,
(iv) $a \wedge (a \vee b) = a$,
(v) $a \vee (b \wedge a) = a$,
(vi) $a \vee b = a$ *if and only if* $a \wedge b = b$,
(vii) $a \vee b = b$ *if and only if* $a \wedge b = a$.

Definition 2.4 [10]**.** For any $a, b \in L$, we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$ or, equivalently, $a \vee b = b$.

Theorem 2.5 [10]**.** *For any $a, b, c \in L$, we have the following*

(i) *The relation $\leq$ is a partial ordering on L,*
(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad (LD\vee)$,
(iii) $(a \vee b) \vee a = a \vee b = a \vee (b \vee a)$,
(iv) $(a \vee b) \wedge c = (b \vee a) \wedge c$,
(v) *The operation $\wedge$ is associative in L,*
(vi) $a \wedge b \wedge c = b \wedge a \wedge c$.

Theorem 2.6 [10]**.** *For any $a, b \in L$, the following are equivalent.*

(i) $(a \wedge b) \vee a = a$,
(ii) $a \wedge (b \vee a) = a$,
(iii) $(b \wedge a) \vee b = b$,
(iv) $b \wedge (a \vee b) = b$,
(v) $a \wedge b = b \wedge a$,
(vi) $a \vee b = b \vee a$,
(vii) *The supremum of a and b exists in L and equals to $a \vee b$,*
(viii) *there exists $x \in L$ such that $a \leq x$ and $b \leq x$,*
(ix) *the infimum of a and b exists in L and equals to $a \wedge b$.*

Definition 2.7 [10]**.** L is said to be associative, if the operation $\vee$ in L is associative.

Theorem 2.8 [10]**.** *The following are equivalent.*

(i) *L is a distributive lattice,*
(ii) *the poset $(L, \leq)$ is directed above,*
(iii) *$a \wedge (b \vee a) = a$, for all $a, b \in L$,*
(iv) *the operation $\vee$ is commutative in L,*
(v) *the operation $\wedge$ is commutative in L,*
(vi) *the relation $\theta := \{(a, b) \in L \times L \mid a \wedge b = b\}$ is anti-symmetric,*
(vii) *the relation θ defined in (vi) is a partial order on L.*

Lemma 2.9 [10]**.** *For any $a, b, c, d \in L$,we have the following*

(i) *$a \wedge b \leq b$ and $a \leq a \vee b$,*
(ii) *$a \wedge b = b \wedge a$ whenever $a \leq b$,*
(iii) $[a \vee (b \vee c)] \wedge d = [(a \vee b) \vee c] \wedge d$,
(iv) *$a \leq b$ implies $a \wedge c \leq b \wedge c$, $c \wedge a \leq c \wedge b$ and $c \vee a \leq c \vee b$.*

Definition 2.10 [10]**.** An element $0 \in L$ is called zero element of L, if $0 \wedge a = 0$ for all $a \in L$.

Lemma 2.11 [10]. *If L has 0, then for any $a, b \in L$, we have the following*

(i) $a \vee 0 = a$, (ii) $0 \vee a = a$, (iii) $a \wedge 0 = 0$
and
(iv) $a \wedge b = 0$ *if and only if* $b \wedge a = 0$.

Definition 2.12 [13]. Let L be a non-empty set and $x_0 \in L$. Define, for $x, y \in L$,
$x \wedge y = y$ if $x \neq x_0$
$\quad = x$ if $x = x_0$ and
$x \vee y = x$ if $x \neq x_0$
$\quad = y$ if $x = x_0$, then $(L, \vee, \wedge, x_0)$ is an ADL with x_0 as zero element. This is called discrete ADL with zero.

Definition 2.13 [10]. An element $x \in L$ is called maximal if, for any $y \in L$, $x \leq y$ implies $x = y$.

We immediately have the following.

Lemma 2.14 [10]. *For any $m \in L$, the following are equivalent:*

(1) *m is maximal,*
(2) *$m \vee x = m$ for all $x \in L$,*
(3) *$m \wedge x = x$ for all $x \in L$.*

Definition 2.15 [10]. A nonempty subset I of L is said to be an ideal if and only if it satisfies the following:

(1) $a, b \in I \Rightarrow a \vee b \in I$,
(2) $a \in I$, $x \in L \Rightarrow a \wedge x \in I$.

Definition 2.16. A nonempty subset I of L is said to be a weak ideal if and only if it satisfies the following:

(1) $a, b \in I \Rightarrow a \vee b \in I$,
(2) $a \in I$, $x \in L$ and $x \leq a \Rightarrow x \in I$.

Observe that every ideal of L is a weak ideal, but not conversely.

Definition 2.17 [10]. If L_1, L_2 are ADLs, then a function $f : L_1 \to L_2$ is said to be a homomorphism if it satisfies the following:

(1) $f(x \wedge y) = fx \wedge fy$,
(2) $f(x \vee y) = fx \vee fy$ for all $x, y \in L$.

Definition 2.18. A function $d : L \to L$ is called an isotone, if $dx \leq dy$ for any $x, y \in L$ with $x \leq y$.

3. Symmetric bi-derivations in ADLs

We begin this section with the following definition of a derivation in an ADL.

Definition 3.1 [11]**.** A function $d : L \to L$ is called a derivation on L, if $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$ for all $x, y \in L$.

Definition 3.2.
(i) A mapping $D : L \times L \to L$ is called symmetric if $D(x, y) = D(y, x)$ for all $x, y \in L$,
(ii) D is called an isotone map if, for any $x, y, z \in L$ with $x \leq y$, $D(x, z) \leq D(y, z)$.

The following definition introduces the notion of a symmetric bi-derivation on ADLs.

Definition 3.3. A symmetric function $D : L \times L \to L$ is called a symmetric bi-derivation on L, if $D(x \wedge y, z) = [y \wedge D(x, z)] \vee [x \wedge D(y, z)]$.

Observe that a symmetric bi-derivation D on L also satisfies $D(x, y \wedge z) = [z \wedge D(x, y)] \vee [y \wedge D(x, z)]$ for all $x, y, z \in L$.

Example 3.4. Let L be an ADL and $a \in L$. If we define a mapping $D_a : L \times L \to L$ by $D_a(x, y) = x \wedge y \wedge a$ for all $x, y \in L$, then D_a is a symmetric bi-derivation on L.

The symmetric bi-derivation D_a given in the above example, is called as the principal symmetric bi-derivation on L induced by $a \in L$.

Example 3.5. Let L be an ADL with 0 and $0 \neq a \in L$. If we define a mapping $D : L \times L \to L$ by $D(x, y) = (x \vee y) \wedge a$ for all $x, y \in L$, then D is a symmetric map, but not a symmetric bi-derivation on L.

Example 3.6. In a discrete ADL $L = \{0, a, b\}$, if we define $D : L \times L \to L$ by $D(x, y) = a$ when $(x, y) \in \{(0, 0), (0, a), (a, 0)\}$ and $D(x, y) = 0$ otherwise, then D is a symmetric map but not a symmetric bi-derivation on L.

Example 3.7. Let L be a discrete ADL. Then any constant map $D : L \times L \to L$ is a symmetric bi-derivation on L.

Definition 3.8. If D is a symmetric bi-derivation on L, then the mapping $d : L \to L$ defined by $d(x) = D(x, x)$ for all $x \in L$ is called the trace of D.

Theorem 3.9. *Let D be a symmetric bi-derivation on L and d be the trace of D. Then the following hold:*
1. $D(x, y) = x \wedge D(x, y)$ *for all* $x, y \in L$,

2. *$dx = x \wedge dx$ for any $x \in L$,*
3. *$D(x, dx) = dx$ for any $x \in L$,*
4. *$d^2x = dx$ for any $x \in L$.*

Proof. (1) Let $x, y \in L$. Then $D(x, y) = D(x \wedge x, y) = [x \wedge D(x, y)] \vee [x \wedge D(x, y)] = x \wedge D(x, y)$.

(2) Let $x \in L$. Then replace y by x in (1) above, we get $D(x, x) = x \wedge D(x, x)$. That is $dx = x \wedge dx$.

(3) Let $x \in L$. Then $D(x, dx) = D(x, x \wedge dx) = [dx \wedge D(x, x)] \vee [x \wedge D(x, dx)] = dx \vee D(x, dx) = dx$.

(4) Let $x \in L$. Then $d^2x = D(dx, dx) = D(x \wedge dx, dx) = [dx \wedge D(x, dx)] \vee [x \wedge D(dx, dx)] = dx \vee [x \wedge d^2x] = (dx \vee x) \wedge (dx \vee d^2x) = (dx \vee x) \wedge dx = dx$. ∎

Corollary 3.10. *Suppose L has 0 and D is a symmetric bi-derivation on L. Then $D(0, x) = 0$ for all $x \in L$.*

Definition 3.11. A symmetric map $D : L \times L \to L$ is called a joini preserving map if $D(x \vee y, z) = D(x, z) \vee D(y, z)$ for all $x, y, z \in L$.

Let us recall that in an ADL, it is not known whether $\vee$ is assosiative or not. If $\vee$ is assosiative in an ADL, then it is called an assosiative ADL. Now we prove the following.

Lemma 3.12. *Let d be the trace of a join preserving symmetric bi-derivation D on an associative ADL L. Then $d(x \vee y) = dx \vee D(x, y) \vee dy$ for all $x, y \in L$ and d is an isotone map on L.*

Proof. Let $x, y \in L$. Then $d(x \vee y) = D(x \vee y, x \vee y) = D(x, x) \vee D(x, y) \vee D(y, x) \vee D(y, y) = dx \vee D(x, y) \vee dy$. If $x \leq y$, then $dy = d(x \vee y) = dx \vee D(x, y) \vee dy$. Thus $dx \leq dy$. Hence d is an isotone map on L. ∎

Lemma 3.13. *Let d be the trace of a symmetric bi-derivation D on an associative ADL L. Then $d(x \wedge y) = (y \wedge dx) \vee D(x, y) \vee (x \wedge dy)$ for all $x, y \in L$.*

Proof. Let $x, y \in L$. Then $d(x \wedge y) = D(x \wedge y, x \wedge y) = [y \wedge D(x, x \wedge y)] \vee [x \wedge D(y, x \wedge y)] = [y \wedge [[y \wedge D(x, x)] \vee [x \wedge D(x, y)]]] \vee [x \wedge [[y \wedge D(y, x)] \vee [x \wedge D(y, y)]]] = (y \wedge dx) \vee D(x, y) \vee D(y, x) \vee (x \wedge dy) = (y \wedge dx) \vee D(x, y) \vee (x \wedge dy)$. ∎

Corollary 3.14. *If d is the trace of a symmetric bi-derivation D on L, then $y \wedge dx \leq d(x \wedge y)$ for all $x, y \in L$.*

Corollary 3.15. *Suppose m is a maximal element of L and d is the trace of a symmetric bi-derivation on L. Then, for any $x \in L$, we have,*

1. *$x \geq dm$ implies $dx \geq dm$,*
2. *$x \leq dm$ implies $dx = x$.*

Proof. (1) If $x \geq dm$, then $dm = (x \wedge dm) \leq d(m \wedge x)$ by above Corollary. Thus $dx \geq dm$.

(2) If $x \leq dm$, then $x = x \wedge dm \leq d(m \wedge x) = dx$. Hence $dx = x \wedge dx = x$. ■

Theorem 3.16. *Let d be the trace of a join preserving symmetric bi-derivation D on an associative ADL L. Then $Fix_d(L) = \{x \in L/dx = x\}$ is a weak ideal of L.*

Proof. Let $x \in L$, $y \in Fix_d(L)$ and $x \leq y$. Then, by Lemma 3.13, $dx = d(y \wedge x) = (x \wedge dy) \vee D(x, y) \vee (y \wedge dx) = (x \wedge y) \vee D(x, y) \vee (y \wedge x \wedge dx) = x \vee D(x, y) \vee (x \wedge dx) = x \vee (x \wedge dx) = x$. Thus $x \in Fix_d(L)$. Now, let $x, y \in Fix_d(L)$. Then, by Lemma 3.12, $d(x \vee y) = dx \vee D(x, y) \vee dy = x \vee D(x, y) \vee y = x \vee y$. Thus $x \vee y \in Fix_d(L)$. Hence $Fix_d(L)$ is a weak ideal of L. ■

Theorem 3.17. *Let m be a maximal element of L and d be the trace of a symmetric bi-derivation D on L. Then the following are equivalent.*

1. *d is an isotone map on L,*
2. *$dx = x \wedge dm$ for all $x \in L$,*
3. *$d(x \wedge y) = dx \wedge dy$ for all $x, y \in L$,*
4. *$d(x \vee y) = dx \vee dy$ for all $x, y \in L$.*

Proof. (1) $\Rightarrow$ (2) : Let $x \in L$. By Corollary 3.14, $x \wedge dm \leq d(m \wedge x) = dx$. On the other hand, since d is isotone, $d(x \wedge m) \leq dm$. Thus, by Corollary 3.14, $m \wedge dx \leq d(x \wedge m) \leq dm$. Now, $dx = x \wedge dx = m \wedge x \wedge dx = x \wedge m \wedge dx \leq x \wedge dm$. Hence $dx = x \wedge dm$.

(2) $\Rightarrow$ (3) : Let $x, y \in L$. Then $d(x \wedge y) = x \wedge y \wedge dm = x \wedge dm \wedge y \wedge dm = dx \wedge dy$.

(2) $\Rightarrow$ (4) : Let $x, y \in L$. Then $d(x \vee y) = (x \vee y) \wedge dm = (x \wedge dm) \vee (y \wedge dm) = dx \vee dy$.

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are trivial. ■

Theorem 3.18. *Suppose L has a maximal element m. Then the trace of every join preserving symmetric bi-derivation on L is a homomorphism.*

Proof. Let d be the trace of a join preserving symmetric bi-derivation D on L and $x, y \in L$ with $x \leq y$. Then $d(x \vee y) = D(x \vee y, x \vee y) = D(x, x \vee y) \vee D(y, x \vee y) = [D(x, x) \vee D(x, y)] \vee D(y, x \vee y) = [dx \vee D(x, y)] \vee D(y, x \vee y)$. Now, $dx \leq dx \vee D(x, y) \leq d(x \vee y) = dy$. Hence d is an isotone map on L. Therefore, by Theorem 3.17, d is a homomorphism. ■

Finally we conclude this section with the following Theorem.

Theorem 3.19. *Let $\mathscr{D}(L)$ be the set of all principal symmetric bi-derivations on L. Then $(\mathscr{D}(L), \vee, \wedge)$ is an ADL.*

Proof. For $a, b, x, y \in L$, define $(D_a \vee D_b)(x, y) = D_a(x, y) \vee D_b(x, y)$ and $(D_a \wedge D_b)(x, y) = D_a(x, y) \wedge D_b(x, y)$.

Now, $(D_a \vee D_b)(x, y) = D_a(x, y) \vee D_b(x, y) = (x \wedge y \wedge a) \vee (x \wedge y \wedge b) = (x \wedge y) \wedge (a \vee b) = D_{a \vee b}(x, y)$. Thus $D_a \vee D_b = D_{a \vee b} \in \mathscr{D}(L)$. Now $(D_a \wedge D_b)(x, y) = D_a(x, y) \wedge D_b(x, y) = (x \wedge y \wedge a) \wedge (x \wedge y \wedge b) = (x \wedge y) \wedge (a \wedge b) = D_{a \wedge b}(x, y)$. Thus $D_a \wedge D_b = D_{a \wedge b} \in \mathscr{D}(L)$. Therefore, $\mathscr{D}(L)$ is closed under $\vee$ and $\wedge$ and clearly it satisfies the properties of an ADL. ∎

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