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ON 2-ABSORBING FILTERS OF LATTICES

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Abstract

Let L be a lattice with 1. In this paper we study the concept of 2-absorbing filter which is a generalization of prime filter. A proper filter F of L is called a 2-absorbing filter (resp. a weakly 2-absorbing) if whenever $x_1 \lor x_2 \lor x_3 \in F$ (resp. $1 \neq x_1 \lor x_2 \lor x_3 \in F$), for $x_1, x_2, x_3 \in L$, then there are 2 of the x_i 's whose join is in F. A basic number of results concerning 2-absorbing filters and weakly of 2-absorbing filters are given in the case when L is distributive.

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1. INTRODUCTION

Recently, the study of the 2-absorbing property in the rings, modules, and semigroups has become quite popular. In many ways this program began with the paper in 2007, by Ayman Badawi, [2]. He introduced, for a commutative ring R, the notion of 2-absorbing ideals of R. A proper ideal I of R is called a 2-absorbing ideal if whenever $x_1x_2x_3 \in I$ for $x_1, x_2, x_3 \in R$, then there are 2 of the x_i 's whose product is in I. There have been several generalizations and extensions of this concept in the literature (see e.g. [1, 3, 5], and [10]).

In this paper, we are interested in investigating 2-absorbing filters to use other notions of 2-absorbing and associate which exist in the literature as laid

forth in [2]. Now we summarize the content of the paper. Among many results in this paper, in Section 2, it is shown (Theorem 2.2) that the only weakly 2absorbing filters of L that are not 2-absorbing can only be $\{1\}$ (so if L is an L-domain, then a filter is 2-absorbing if and only if it is weakly 2-absorbing), and F is a 2-absorbing filter of L if and only if whenever $F_1 \vee F_2 \vee F_3 \subseteq F$ for some filters F_1, F_2, F_3 of L, then $F_1 \lor F_2 \subseteq F$ or $F_1 \lor F_3 \subseteq F$ or $F_2 \lor F_3 \subseteq F$ (Theorem 2.5). It is shown (Theorem 2.8) that If F is a 2-absorbing filter of L, then either F is a prime filter or $F = \mathbf{p} \cap \mathbf{q} = \mathbf{p} \vee \mathbf{q}$, where \mathbf{p} , \mathbf{q} are the only distinct filters of L that are minimal over F. Let G be a 2-absorbing subfilter of a filter F of L. It is shown (Theorem 2.14 and Theorem 2.15) that either $Ass_L(G:LF)$ is a totally ordered set or $Ass_L(G :_L F)$ is the union of two totally ordered set. Payrovi and Babaei [10], using the technique of efficient covering of submodules (see [8]) proved the avoidance theorem for 2-absorbing submodules. They proved that if a submodule N of a module is contained in the union of a finite number of 2-absorbing submodules with some conditions, then N must be contained in one of them. Section 3 is devoted to prove that the 2-absorbing avoidance theorem. More precisely, let F, F_1, F_2, \ldots, F_n $(n \ge 2)$ be filters of L such that at most two of F_1, F_2, \ldots, F_n are not 2-absorbing. If $F \subseteq \bigcup_{i=1}^n F_i$ and $F_i \nsubseteq (F_j : L x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then $F \subseteq F_i$ for some i with $1 \leq i \leq n$ (Theorem 3.4).

Let us briefly review some definitions and tools that will be used later. A lattice is a poset (L, \leq) in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$ and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting X = L, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \lor b) \land c = (a \land c) \lor (b \land c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A lattice L with 1 is called L-domain if $a \lor b = 1$ $(a, b \in L)$, then a = 1 or b = 1. A proper filter F of L is called prime if $x \lor y \in F$, then $x \in F$ or $y \in F$. Let L be a lattice with 0 and 1. If $a \in L$, then a complement of a in L is an element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. The lattice L is complemented if every element of L has a complement in L [4]. First we need the following well-known lemma.

Lemma 1.1. Let L be a lattice.

(i) A non-empty subset F of L is a filter of L if and only if x ∨ z ∈ F and x ∧ y ∈ F for all x, y ∈ F, z ∈ L (so 0 ∈ F if and only if F = L). Moreover, since x = x ∨ (x ∧ y) and y = y ∨ (x ∧ y), F is a filter and x ∧ y ∈ F gives x, y ∈ F for all x, y ∈ L.

- (ii) If F_1, \ldots, F_n are filters of L and $a \in L$, then $\bigvee_{i=1}^n F_i = \{\bigvee_{i=1}^n a_i : a_i \in F_i\}$ and $a \lor F_i = \{a \lor a_i : a_i \in F_i\}$ are filters of L.
- (iii) If D is an arbitrary non-empty subset of L, then the set T(D) consisting of all elements of L of the form $(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \vee x$ (with $a_i \in D$ for all $1 \leq i \leq n$ and $x \in L$) is a filter of L containing D (so if $D = \{a\}$, then $T(\{a\}) = T(a) = \{a \lor t : t \in L\}$).
- (iv) If L is distributive, F,G are filters of L, and $x \in L$, then $(G :_L F) = \{x \in L : x \lor F \subseteq G\}$ and $(F :_L \{x\}) = (F :_L x) = \{a \in L : a \lor x \in F\}$ are filters of L.
- (v) If $\{F_i\}_{i\in\Delta}$ is a chain of filters of L, then $\cup_{i\in\Delta}F_i$ is a filter of L.

2. Some basic properties of 2-absorbing filters

In this section, we collect some properties concerning 2-absorbing filters of a lattice L. Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1 and 0.

Definition 2.1. A proper Filter F of L is called a 2-absorbing (resp. a weakly 2-absorbing) filter if whenever $a, b, c \in L$ and $a \lor b \lor c \in F$ (resp. $1 \neq a \lor b \lor c \in F$), then $a \lor b \in F$ or $a \lor c \in F$ or $b \lor c \in F$.

Clearly, every 2-absorbing filter of L is a weakly 2-absorbing. However, since $\{1\}$ is always weakly 2-absorbing "by definition", a weakly 2-absorbing filter need not be 2-absorbing.

Theorem 2.2. If F is a weakly 2-absorbing of L that is not 2-absorbing, then $F = \{1\}$. In particular, the only weakly 2-absorbing filters of L that are not 2-absorbing can only be $\{1\}$.

Proof. We suppose that $F \neq \{1\}$, and look for a contradiction. Let $x \lor y \lor z \in F$. If $x \lor y \lor z \neq 1$, then F weakly 2-absorbing gives $x \lor y \in F$ or $y \lor z \in F$ or $x \lor z \in F$; so F is 2-absorbing which is a contradiction. So assume that $x \lor y \lor z = 1$. Since $F \neq \{1\}$, there exists $b \in F$ with $b \neq 1$. Then $1 \neq b = b \land 1 = b \land (x \lor y \lor z) =$ $((b \land (x \lor y)) \lor ((b \land (x \lor z)) \lor ((b \land (y \lor z)) \in F, \text{ so } b \land (x \lor y) \in F \text{ or } b \land (x \lor z) \in F$ or $b \land (y \lor z) \in F$. Thus $x \lor y \in F$ or $x \lor z \in F$ or $y \lor z \in F$ by Lemma 1.1 (i), and so F is 2-absorbing, a contradiction. Thus $F = \{1\}$. The "in particular" statement is clear.

Remark 2.3. (i) If F, F_1, F_2 are filters of L with $F \subseteq F_1 \cup F_2$, then we show that either $F \subseteq F_1$ or $F \subseteq F_2$. Suppose that $F \subseteq F_1 \cup F_2$ such that $F \nsubseteq F_1$; we show that $F \subseteq F_2$. Let $a \in F$ be such that $a \notin F_1$. Let $x \in F \cap F_1$. Then F is a filter gives $a \land x \in F \subseteq F_1 \cup F_2$; so $a, x \in F_2$. Therefore $F \cap F_1 \subseteq F_2$. Thus $F = F \cap (F_1 \cup F_2) = (F \cap F_1) \cup (F \cap F_2) \subseteq F_2$. (ii) Assume that **m** is a maximal filter of a lattice L with 0 and let $a \lor b \in \mathbf{m}$ with $a, b \notin \mathbf{m}$ for some $a, b \in L$. Then $T(\mathbf{m} \cup \{a\}) = T(\mathbf{m} \cup \{b\}) = L$ since **m** is maximal. An inspection will show that $0 \in L$ implies that L = F which is a contradiction. Thus every maximal filter of L is prime [6].

(iii) If F is a filter of a L-domain L, then F is 2-absorbing if and only if it is weakly 2-absorbing.

Proposition 2.4. Let F_1, F_2, F be filters of L such that F is 2-absorbing.

(i) If $a, b \in L$ and $(a \lor b) \lor F_1 \subseteq F$, then $a \lor b \in F$ or $a \lor F_1 \subseteq F$ or $b \lor F_1 \subseteq F$. (ii) If $a \in L$ and $a \lor (F_1 \lor F_2) \subseteq F$, then $a \lor F_1 \subseteq F$ or $a \lor F_2 \subseteq F$ or $F_1 \lor F_2 \subseteq F$. **Proof.** (i) Let $a \lor b \notin F$ and $a \lor F_1 \not\subseteq F$. Then there is an element $c \in F_1$ such that $a \lor c \notin F$. Now $a \lor b \lor c \in F$ gives $b \lor c \in F$ since F is 2-absorbing. We have to show that $b \lor F_1 \subseteq F$. Let d be an arbitrary element of F_1 . Then $(d \land c) \lor (a \lor b) = (a \lor b \lor c) \land (a \lor b \lor d) \in F$ since F is a filter; so either $(d \land c) \lor a = (a \lor c) \land (a \lor d) \in F$ or $(d \land c) \lor b = (b \lor c) \land (b \lor d) \in F$. If $(d \land c) \lor a \in F$, then $a \lor c \in F$ by Lemma 1.1 (i) that is a contradiction. If $(d \land c) \lor b \in F$, then $b \lor d \in F$. Thus $b \lor F_1 \subseteq F$.

(ii) Let $a \vee F_1 \not\subseteq F$ and $a \vee F_2 \not\subseteq F_2$. We have to show that $F_1 \vee F_2 \subseteq F$. Suppose that $x \in F_1$ and $y \in F_2$. By hypothesis, there exist $z \in F_1 \setminus F$ and $w \in F_2 \setminus F$ such that $a \vee z \notin F$ and $a \vee w \notin F$. As $a \vee z \vee w \in a \vee (F_1 \vee F_2) \subseteq F$, we get $z \vee w \in F$. Now $z \wedge x \in F_1$ and $y \wedge w \in F_2$ gives $a \vee (z \wedge x) \vee (y \wedge w) \in F$; so $(z \wedge x) \vee (y \wedge w) \in F$ since F is 2-absorbing (see Lemma 1.1 (i)). It follows that $(z \wedge x) \vee y \in F$; hence $x \vee y \in F$ by Lemma 1.1 (i). Therefore, $F_1 \vee F_2 \subseteq F$.

Theorem 2.5. Let F be a proper filter of L. The following statements are equivalent:

- (i) F is a 2-absorbing filter of L.
- (ii) If $F_1 \vee F_2 \vee F_3 \subseteq F$ for some filters F_1, F_2, F_3 of L, then $F_1 \vee F_2 \subseteq F$ or $F_1 \vee F_3 \subseteq F$ or $F_2 \vee F_3 \subseteq F$.

Proof. (i) \Rightarrow (ii) Suppose that $F_1 \lor F_2 \lor F_3 \subseteq F$ for some filters F_1, F_2, F_3 of Land $F_1 \lor F_2 \nsubseteq F$. Then by Proposition 2.4 for all $a \in F_3$ either $a \lor F_1 \subseteq F$ or $a \lor F_2 \subseteq F$. If $a \lor F_1 \subseteq F$, for all $a \in F_3$ we are done. Similarly, if $a \lor F_2 \subseteq F$, for all $a \in F_3$ we are done. Assume that $a, b \in L$ are such that $a \lor F_1 \nsubseteq F$ and $b \lor F_2 \nsubseteq F$. It follows that $b \lor F_1 \subseteq F$ and $a \lor F_2 \subseteq F$. Since $(a \land b) \lor (F_1 \lor F_2) \subseteq F$, we get either $(a \land b) \lor F_1 \subseteq F$ or $(a \land b) \lor F_2 \subseteq F$ by Proposition 2.4. If $(a \land b) \lor F_1 \subseteq F$, then $z \lor (a \land b) = (z \lor a) \land (z \lor b) \in F$ for all $z \in F_1$ which implies that $a \lor z \in F$ by Lemma 1.1 (i); so $a \lor F_1 \subseteq F$ which is a contradiction. Similarly, if $(a \land b) \lor F_2 \subseteq F$, we get a contradiction. Thus either $F_1 \lor F_3 \subseteq F$ or $F_2 \lor F_3 \subseteq F$.

(ii) \Rightarrow (i) Let $a, b, c \in L$ with $a \lor b \lor c \in F$. Then by (ii), $T(a) \lor T(b) \lor T(c) \subseteq F$ gives $a \lor b \in T(a) \lor T(b) \subseteq F$ or $a \lor c \in T(a) \lor T(c) \subseteq F$ or $b \lor c \in T(b) \lor T(c) \subseteq F$. Thus F is 2-absorbing. We say that a subset $D \subseteq L$ is Join closed if $0 \in D$ and $a \lor b \in D$ for all $a, b \in D$. Clearly, if **p** is a prime filter of L, then $L \setminus \mathbf{p}$ is a join closed subset of L. The set of all prime filters of L is denoted by $\operatorname{Spec}(L)$. If F is a filter of L, then we set $\operatorname{var}(F) = \{\mathbf{p} \in \operatorname{Spec}(L) : F \subseteq \mathbf{p}\}$, and the set of all prime filters of L that are minimal over F is denoted by $\min(F)$.

- **Lemma 2.6.** (i) Assume that F is a filter of L and let S be a join closed set of L such that $S \cap F = \emptyset$. Then the set $\sum = \{K : F \subseteq K, K \cap S = \emptyset\}$ of filters under the relation of inclusion has at least one maximal element, and any such maximal element of \sum is a prime filter.
- (ii) If F is a filter of L, then $F = \cap_{\mathbf{p} \in var(F)} \mathbf{p}$.
- (iii) Let F, p be filters of L with p prime and F ⊆ p. Then there exists a minimal prime filter q of F with q ⊆ p.
- (iv) If F is a filter of L, then $F = \bigcap_{\mathbf{p} \in \min(F)} \mathbf{p}$.

Proof. (i) Since $F \in \sum, \sum \neq \emptyset$. Of course, the relation of inclusion, \subseteq , is a partial order on \sum . Now \sum is easily seen to be inductive under inclusion, so by Zorn's Lemma \sum has a mximal element \mathbf{q} with $\mathbf{q} \cap S = \emptyset$ and $F \subseteq \mathbf{q}$. It suffices to show that \mathbf{q} is prime. Now let $x, x' \in L \setminus \mathbf{q}$; we must show that $x \lor x' \notin \mathbf{q}$. Since $x \notin \mathbf{q}$, we have $F \subseteq \mathbf{q} \subsetneqq T(\mathbf{q} \cup \{x\})$. By the maximality of \mathbf{q} , we have $T(\mathbf{q} \cup \{x\}) \cap S \neq \emptyset$, and so there exist $s \in S$, $c \in L$ and $q \in \mathbf{q}$ such that $s = (q \land x) \lor c$. Similarly, $s' = (q' \land x') \lor c'$ for some $s' \in S$, $q' \in \mathbf{q}$ and $c' \in L$. Set $z = c \lor c'$. Then $s \lor s' = (q \land x) \lor (q' \land x') \lor z = [(q \land x) \lor x'] \land [(q \land x) \lor q'] \lor z =$ $[(x \lor x') \land (q \lor x')] \land [(q \land x) \lor q'] \lor z$. As $(q \land x) \lor q'$, $q \lor x' \in \mathbf{q}$, $S \cap \mathbf{q} = \emptyset$ and \mathbf{q} is a filter, we have $x \lor x' \notin \mathbf{q}$. Thus \mathbf{q} is a prime filter.

(ii) It is enough to show that $\bigcap_{\mathbf{p}\in var(F)}\mathbf{p} \subseteq F$. Let $a \in \bigcap_{\mathbf{p}\in var(F)}\mathbf{p}$. We suppose that $a \notin F$, and look for a contradiction. Set $S = \{0, a\}$. Then S is a join closed set of L with $S \cap F = \emptyset$. Hence, by (i), there exists a prime filter \mathbf{q} of L such that $F \subseteq \mathbf{q}$ and $\mathbf{q} \cap S = \emptyset$. It follows that $\mathbf{q} \in var(F)$, so that $a \in S \cap \mathbf{q}$, a contradiction.

(iii) Set $\Delta = \{\mathbf{q} \in \operatorname{Spec}(L) : F \subseteq \mathbf{q} \subseteq \mathbf{p}\}$. Then $\mathbf{p} \in \Delta$, and so $\Delta \neq \emptyset$. By an argument like that in (i) (take $S = L \setminus \mathbf{p}$), the set Δ of prime filters of L has a minimal member with respect to inclusion (by partially ordering Δ by reverse inclusion and using Zorn's Lemma) which is prime. (iv) follows from (iii) (since every prime filter in var(F) contains a minimal prime filter of F).

Compare the next Proposition with Theorem 2.1, p. 2 in [7].

Proposition 2.7. Let $F \subseteq \mathbf{p}$ be filters of L, where \mathbf{p} is a prime filter. Then the following conditions are equivalent:

- (i) **p** is a minimal prime filter of *F*.
- (ii) $L \setminus \mathbf{p}$ is a join closed set that is maximal with $(L \setminus \mathbf{p}) \cap F = \emptyset$.

(iii) For each $x \in \mathbf{p}$, there is a $y \notin \mathbf{p}$ such that $y \lor x \in F$.

Proof. (i) \Rightarrow (ii) Since $(L \setminus \mathbf{p}) \cap F = \emptyset$, the set Δ of all join closed sets, say H, with $H \cap F = \emptyset$ is not empty. Of course, the relation of inclusion, \subseteq , is a partial order on Δ . Now Δ is easily seen to be inductive under inclusion, so by Zorn's Lemma Δ has a mixmal element S. Again by Zorn's Lemma, there is a filter \mathbf{q} of L containing F that is maximal with respect to being disjoint from S which is prime by Lemma 2.6 (i). Note that \mathbf{q} is disjoint from $L \setminus \mathbf{p}$ which implies that $\mathbf{p} = \mathbf{q}$. Thus $S = L \setminus \mathbf{p}$.

(ii) \Rightarrow (iii) Assume that $1 \neq x \in \mathbf{p}$ and let $S = \{y \lor (\wedge_{j=1}^{i} x) : y \in L \setminus \mathbf{p}, i = 0, 1, ...\}$ (Note that $\wedge_{j=1}^{0} x$ is interpreted as 0, and clearly, $\wedge_{j=1}^{i} x = x$). Then S is a join closed set that properly contains $L \setminus \mathbf{p}$; so $F \cap S \neq \emptyset$ by maximality of $L \setminus \mathbf{p}$. Thus there exists $y \in L \setminus \mathbf{p}$ such that $x \lor y \in F$.

(iii) \Rightarrow (i) Let **q** be a prime filter such that $F \subsetneq \mathbf{q} \subseteq \mathbf{p}$. If $\mathbf{p} \neq \mathbf{q}$, then there is an element $x \in \mathbf{p}$ with $x \notin \mathbf{q}$; so $x \lor y \in F \subsetneq \mathbf{q}$ for some $y \notin \mathbf{p}$ which is a contradiction. Therefore $\mathbf{p} = \mathbf{q}$.

The following theorem is a lattice counterpart of Theorem 2.4 in [2] describing the structure of 2-absorbing ideals.

Theorem 2.8. (i) If F is a 2-absorbing filter of L, then there exist at most two prime filters of L that are minimal over F.

- (ii) If F is a 2-absorbing filter of L, then either F is a prime filter of L or F = p ∩ q = p ∨ q, where p, q are the only distinct filters of L that are minimal over F.
- (iii) If either F is a prime filter of L or F is an intersection of two prime filter of L, then F is 2-absorbing.

Proof. (i) Assume that that Δ is the set of prime filters of L which are minimal over F and let Δ has at least three elements. Let $\mathbf{p}, \mathbf{q} \in \Delta$ with $\mathbf{p} \neq \mathbf{q}$. Then there exist $x_1, x_2 \in L$ such that $x_1 \in \mathbf{p} \setminus \mathbf{q}$ and $x_2 \in \mathbf{q} \setminus \mathbf{p}$. First we show that $x_1 \vee x_2 \in F$. By Proposition 2.7, there exist $a \notin \mathbf{p}$ and $b \notin \mathbf{q}$ such that $a \vee x_1, b \vee x_2 \in F$. Since $x_1, x_2 \notin \mathbf{p} \cap \mathbf{q}$ and $a \vee x_1, b \vee x_2 \in F \subseteq \mathbf{p} \cap \mathbf{q}$, we conclude that $a \in \mathbf{q} \setminus \mathbf{p}$ and $b \in \mathbf{p} \setminus \mathbf{q}$; so $a, b \notin \mathbf{p} \cap \mathbf{q}$. Since $a \vee x_1, b \vee x_2 \in F$, we have $(a \wedge b) \vee (x_1 \vee x_2) = [(a \vee x_1) \vee x_2] \wedge [(b \vee x_2) \vee x_1] \in F$ since F is a filter. By Lemma 1.1 (i), $a \wedge b \notin \mathbf{p}$ and $a \wedge b \notin \mathbf{q}$. Since $(a \wedge b) \vee x_1 \notin \mathbf{q}$ and $(a \wedge b) \vee x_2 \notin \mathbf{p}$, F is a 2-absorbing filter gives $x_1 \vee x_2 \in F$. Now suppose there is a $\mathbf{r} \in \Delta$ such that \mathbf{r} is neither \mathbf{p} nor \mathbf{q} . Then we can choose $z_1 \in \mathbf{p} \setminus (\mathbf{q} \cup \mathbf{r}), z_2 \in \mathbf{q} \setminus (\mathbf{p} \cup \mathbf{r}),$ and $z_3 \in \mathbf{r} \setminus (\mathbf{p} \cup \mathbf{q})$. By an argument like that as above, we have $z_1 \vee z_2 \in F$. Since $F \subseteq \mathbf{p} \cap \mathbf{q} \cap \mathbf{r}$ and $z_1 \vee z_2 \in F$, we get either $z_1 \in \mathbf{r}$ or $z_2 \in \mathbf{r}$ that is a contradiction, as required. (ii) By (i) and Lemma 2.6 (iv), we conclude that either F is a prime filter or $F = \mathbf{p} \cap \mathbf{q}$, where \mathbf{p} , \mathbf{q} are the only distinct filters of L that are minimal over F. An inspection will show that $\mathbf{p} \cap \mathbf{q} = \mathbf{p} \vee \mathbf{q}$.

(iii) The first assertion is clear. Let \mathbf{p} and \mathbf{p} be two prime filters of L; we have to show that $F = \mathbf{p} \cap \mathbf{q}$ is a 2-absorbing filter of L. Let $a, b, c \in L$ such that $a \lor b \lor c \in \mathbf{p} \cap \mathbf{q}$. Therefore $a \lor b \lor c \in \mathbf{p}$ and $a \lor b \lor c \in \mathbf{q}$. If $a \in \mathbf{p} \cap \mathbf{q}$, then $a \lor b \in \mathbf{p} \cap \mathbf{q}$. If $a \in \mathbf{p} \cap \mathbf{q}$, then $a \lor b \in \mathbf{p} \cap \mathbf{q}$ since \mathbf{p} and \mathbf{q} are filters of L. The other cases we do the same.

The collection of ideals of Z, the ring of integers, form a lattice under set inclusion which we shall denote by L(Z) with respect to the following definitions: $mZ \lor nZ = (m, n)Z$ and $mZ \land nZ = [m, n]Z$ for all ideals mZ and nZ of Z, where (m, n) and [m, n] are greatest common divisor and least common multiple of m, n, respectively. Note that L(Z) is a distributive complete lattice with least element the zero ideal and the greatest element Z.

Theorem 2.9. The following hold:

- (i) If p is a prime number and k is a positive integer, then the set $F_{p^k} = \{mZ \in L(Z) : p^k \nmid m\}$ is a prime filter of L(Z).
- (ii) $L(Z) \setminus \{0\}$ is the only maximal filter of L(Z).
- (iii) Every prime filter of L(Z) is of the form either F_{p^k} for some prime number p and positive integer k or $L(Z) \setminus \{0\}$.
- (iv) Every 2-absorbing filter of L(Z) is of the form $L(Z) \setminus \{0\}$ or F_{p^m} or $F_{p^m} \cap F_{q^n}$ for some positive integers m, n and prime numbers p, q with $p \neq q$.

Proof. (i) Let $mZ, nZ \in F_{p^k}$ and $sZ \in L(Z)$. Now $p^k \nmid m$ and $p^k \nmid n$ gives $p^k \nmid [m,n]$; so $[m,n]Z \in F_{p^k}$. As $p^k \nmid m$, we get $p^k \nmid (m,s)$ which implies that $(m,s)Z \in F_{p^k}$. Thus F_{p^k} is a filter of L(Z). Let $mZ \lor nZ = (m,n)Z \in F_{p^k}$ with $mZ \notin F_{p^k}$. Then $p^k \nmid (m,n)$ and $p^k \mid m$ gives $p^k \nmid n$; so $nZ \in F_{p^k}$. Thus F_{p^k} is prime.

(ii) is clear.

(iii) Let F be a prime filter of L(Z). First we show that there exist at most one prime number p and positive integer k such that for every $mZ \in F$ implies that $p^k \nmid m$. Otherwise, there are distinct prime numbers p, q and positive integers k, s such that for every $mZ \in F$ implies that $p^k \nmid m$ and $q^s \nmid m$. Then $p^k Z \lor q^s Z = Z \in F$ gives either $p^k Z \in F$ or $q^s Z \in F$ which is a contradiction. If there exists p^k such that for every $mZ \in F$ implies that $p^k \nmid m$. Let t be least positive integer such that for every $mZ \in F$ implies that $p^t \nmid m$; we show that $F = F_{p^t}$. It suffices to show that for every mZ with $p^t \nmid m, mZ \in F$. There are distinct prime numbers q_1, \ldots, q_n such that $m = p^l q_1^{s_1} \cdots q_n^{s_n}$, where $0 \leq l < t$, $p \neq q_j$ with $1 \leq j \leq n$, and s_j is a positive integer for $1 \leq j \leq n$. As l < t, there exist $m'Z \in F$ such that $p^l \mid m'$, so $m'Z \subseteq p^l Z$. Thus $p^l Z \in F$ since F is a filter. Moreover, $p^l Z \vee q_i^{s_i} Z = Z \in F$ gives $q_i^{s_i} Z \in F$ with $1 \leq i \leq n$. Thus $mZ = p^l Z \wedge (\wedge_{i=1}^n q_i^{s_i} Z) \in F$. Suppose that there is not such p^k ; we show that $F = L(Z) \setminus \{0\}$. Let m be a non-zero integer. It is enough to show that $mZ \in F$. We can write $m = p_1^{s_1} \cdots p_n^{s_n}$, where $p_i \neq p_j$ with $i \neq j$ and for each i, s_i is a positive integer. Then for each i, there exists $m_i Z \in F$ such that $p_i^{s_i} \mid m_i$, so $m_i Z \subseteq p_i^{s_i} Z \in F$ since F is a filter. Thus $mZ = \wedge_{i=1}^n p_i^{s_i} Z \in F$.

(iv) This follows from (i), (ii), (iii), and Theorem 2.8.

Remark 2.10 shows that prime filters which are maximals are abundant.

Remark 2.10. (i) Assume that F is a prime filter of a complemented lattice L with 0 and 1 and let F' be a filter of L such that $F \subsetneq F' \subseteq L$. Then there exist $x \in F' \setminus F$ and $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1 \in F$. Then F is prime gives $y \in F \subseteq F'$, and so $x \wedge y = 0 \in F'$; hence F' = L. Thus F is maximal.

(ii) Let $D = \{1, ..., n\}$. Then the set $L = \{X : X \subseteq D\}$ forms a complemented distributive lattice under set inclusion with greatest element D and least element \emptyset (note that if $x, y \in L$, then $x \lor y = x \cup y$ and $x \land y = x \cap y$). Then every prime filter of L is a maximal filter by (i).

Corollary 2.11. The following statements are equivalent:

- (i) Every prime filter of L is maximal;
- (ii) If F is a 2-absorbing filter of L, then either F is maximal or $F = \mathbf{p_1} \cap \mathbf{p_2} = \mathbf{p_1} \vee \mathbf{p_2}$, where $\mathbf{p_1}, \mathbf{p_2}$ are some maximal filters of L.

Proof. (i) \Rightarrow (ii) follows from Theorem 2.8. To see that (ii) \Rightarrow (i), assume that F is a prime filter of L. By (ii), if F is maximal, then we are done. So we assume that $F = \mathbf{p_1} \lor \mathbf{p_2}$, where $\mathbf{p_1}, \mathbf{p_2}$ are some maximal filters of L. Then either $\mathbf{p_1} \subseteq F$ or $\mathbf{p_2} \subseteq F$; hence either $F = \mathbf{p_1}$ or $F = \mathbf{p_1}$ (otherwise, there exist $a \in \mathbf{p_1} \setminus F$ and $b \in \mathbf{p_2} \setminus F$ with $a \lor b \notin F$ since F is a prime filter, and this contradicts the statements of (ii)).

Proposition 2.12. If G is a 2-absorbing subfilter of a filter F of L, then $(G :_L F)$ is a 2-absorbing filter of L.

Proof. Let $a, b, c \in L$, $a \lor b \lor c \in (G :_L F)$, $a \lor c \notin (G :_L F)$, and $b \lor c \notin (G :_L F)$. We must to show that $a \lor b \in (G :_L F)$. There exist $x_1, x_2 \in L$ such that $a \lor c \lor x_1, b \lor c \lor x_2 \notin G$ but $(a \lor b) \lor [(c \lor x_1) \land (c \lor x_2)] = (a \lor b \lor c) \lor (x_1 \land x_2) \in G$ since G is a filter. Now G is a 2-absorbing filter gives $a \lor [(c \lor x_1) \land (c \lor x_2)] = (a \lor c \lor x_1) \land (a \lor c \lor x_2) \in G$ or $b \lor [(c \lor x_1) \land (c \lor x_2) = (b \lor c \lor x_1) \land (b \lor c \lor x_2) \in G$ or $a \lor b \in G$. If $a \lor b \in G$, we are done. If $a \lor [(c \lor x_1) \land (c \lor x_2)] \in G$, then by Lemma 1.1 (i), $a \lor c \lor x_1 \in G$ which is a contradiction. Similarly, $b \lor [(c \lor x_1) \land (c \lor x_2)] \notin G$. This completes the proof. **Proposition 2.13.** If G is a 2-absorbing subfilter of a filter F of L, then $(G :_L F)$ is a prime filter if and only if $(G :_L x)$ is a prime filter for all $x \in F \setminus G$.

Proof. Let $a, b \in L$, $x \in F \setminus G$, and $a \lor b \in (G : L x)$. Then $a \lor b \lor x \in G$ gives $a \lor x \in G$ or $b \lor x \in G$ or $a \lor b \in G$. If $a \lor x \in G$ or $b \lor x \in G$ we are done. If $a \lor b \in G$, then $(a \lor b) \lor F \subseteq G$ since G is a filter; so $a \lor b \in (G : L F)$. By assumption, $a \in (G : L F)$ or $b \in (G : L F)$; hence $a \in (G : L x)$ or $b \in (G : L x)$. Thus (G : L x) is a prime filter of L. Conversely, suppose that $a \lor b \in (G : L F)$ for some $a, b \in L$ with $a, b \notin (G : L F)$. It follows that $a \lor x \notin G$ and $b \lor y \notin G$ for some $x, y \in F \setminus G$ (so $x \land y \notin G$ by Lemma 1.1 (i)). As $a \lor b \lor (x \land y) = (a \lor b \lor x) \land (a \lor b \lor y) \in G$, we have $a \lor b \in (G : L (x \land y))$; hence $a \lor (x \land y) = (a \lor x) \land (a \lor y) \in G$ or $b \lor (x \land y) = (b \lor x) \land (b \lor y) \in G$ since (G : L F) which implies that (G : L F) is a prime filter of L.

Let G be a proper subfilter of a filter F of L. We say that $\mathbf{p} \in \text{Spec}(L)$ is an associated prime filter of F with respect to G if there is an element $x \in F \setminus G$ such that $(G :_L x) = \mathbf{p}$. The set of associated prime filters of F with respect to G is denoted $\text{Ass}_L(G :_L F)$.

Compare the next Theorem with Theorem 2.6 in [10].

Theorem 2.14. Let G be a 2-absorbing subfilter of a filter F of L. If $(G :_L F)$ is a prime filter of L, then $Ass_L(G :_L F)$ is a totally ordered set.

Proof. Let $\mathbf{p}, \mathbf{q} \in \operatorname{Ass}_L(G : :_L F)$. Then there are elements $x, y \in F \setminus G$ such that $(G :_L x) = \mathbf{p}$ and $(G :_L y) = \mathbf{q}$. Suppose that $\mathbf{q} \notin \mathbf{p}$. We have to show that $(G :_L x) \subseteq (G :_L y)$. Let $z \in (G :_L x)$ (so $z \lor x \in G$). There exists $w \in (G :_L y)$ such that $w \notin (G :_L x)$; so $w \lor y \in G$ and $w \lor x \notin G$. Clearly, $x \land y \notin G$. If $z \lor (x \land y) = (z \lor x) \land (z \lor y) \in G$, then $z \lor y \in G$ by Lemma 1.1 (i) and so $z \in (G :_L y)$. Now assume that $z \lor (x \land y) \notin G$, so $(z \lor w) \lor (x \land y) = (z \lor w \lor x) \land (z \lor w \lor y) \in G$ since G is a filter; hence $z \lor w \in (G :_L (x \land y))$. By Proposition 2.13 and Lemma 1.1 (i), $(G :_L (x \land y))$ is a prime filter gives $z \lor (x \land y) = (z \lor x) \land (z \lor y) \in G$ and $w \lor (x \land y) = (w \lor x) \land (w \lor y) \notin G$. Thus $z \lor y \in G$ and so $z \in (G :_L y)$.

Compare the next Theorem with Theorem 2.7 in [10].

Theorem 2.15. Let G be a 2-absorbing subfilter of a filter F of L such that $(G:_L F) = \mathbf{p} \cap \mathbf{q}$ for some prime filters \mathbf{p}, \mathbf{q} of L.

- (i) If $x \in F \setminus G$ and $\mathbf{p} \subseteq (G :_L x)$, then $(G :_L x)$ is a prime filter of L.
- (ii) If $x, y \in F \setminus G$ and $\mathbf{p} \subseteq (G :_L x) \cap (G :_L y)$, then either $(G :_L x) \subseteq (G :_L y)$ or $(G :_L y) \subseteq (G :_L x)$. Therefore $\operatorname{Ass}_L(G :_L F)$ is the union of two totally ordered sets.

Proof. (i) Let $a, b \in L$ and $a \lor b \in (G :_L x)$. Then $a \lor b \lor x \in G$ gives $a \lor x \in G$ or $b \lor x \in G$ or $a \lor b \in G$. If $a \lor x \in G$ or $b \lor x \in G$ we are done. If $a \lor b \in G$, then $(a \lor b) \lor F \subseteq G$ since G is a filter; so $a \lor b \in (G :_L F) \subseteq \mathbf{p}$. thus either $a \in \mathbf{p} \subseteq (G :_L x)$ or $b \in \mathbf{p} \subseteq (G :_L x)$.

(ii) Suppose that $(G :_L y) \notin (G :_L x)$. We have to show that $(G :_L x) \subseteq (G :_L y)$. Let $z \in (G :_L x)$ (so $z \lor x \in G$). There exists $w \in (G :_L y)$ such that $w \notin (G :_L x)$; so $w \lor y \in G$ and $w \lor x \notin G$. Clearly, $x \land y \notin G$. If $z \lor (x \land y) = (z \lor x) \land (z \lor y) \in G$, then $z \lor y \in G$ by Lemma 1.1 (i) and so $z \in (G :_L y)$. Now assume that $z \lor (x \land y) \notin G$, so $(z \lor w) \lor (x \land y) = (z \lor w \lor x) \land (z \lor w \lor y) \in G$ since G is a filter; hence $z \lor w \in G$ since $w \lor (x \land y) = (w \lor x) \land (w \lor y) \notin G$ and $z \lor (x \land y) \notin G$. Thus $z \lor w \in (G :_L F) \subseteq \mathbf{p}$. If $w \in \mathbf{p} \subseteq (G :_L x)$, then $w \lor x \in G$ that is a contradiction; hence $z \in \mathbf{p} \subseteq (G :_L y)$.

Theorem 2.16. If G is a 2-absorbing subfilter of a filter F of L, then $(G :_L F)$ is a prime filter if and only if $(G :_L H)$ is a prime filter of L for all subfilters H of F containing G.

Proof. By Proposition 2.13 and Theorem 2.14, the set $\{(G :_L x) : x \in H \setminus G\}$ is a totally ordered set of prime filters of L; so $(G :_L H) = \bigcap_{x \in H} (G :_L x)$ is a prime filter of L. Conversely, suppose that $x \lor y \in (G :_L F)$ with $x, y \notin (G :_L F)$. Then there exist $a, b \in F \setminus G$ (so $a \land b \notin G$) such that $x \lor a, y \lor b \notin G$, so $x \lor y \in (G :_L$ $(a \land b)$). Now $(G :_L (a \land b))$ is a prime filter gives $x \lor (a \land b) = (x \lor a) \land (x \lor b) \in G$ or $y \lor (a \land b) = (y \lor a) \land (y \lor b) \in G$ which is a contradiction. Thus $(G :_L F)$ is prime.

3. 2-Absorbing Avoidance Theorem

Let F, F_1, F_2, \ldots, F_n be filters of L. We call a covering $F \subseteq \bigcup_{i=1}^n F_i$ efficient if no F_i is superfluous. Analogously, we say that $F = \bigcup_{i=1}^n F_i$ is an efficient union if none of the F_i may be excluded. Any cover or union consisting of filters of Lcan be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

Theorem 3.1. If G is a 2-absorbing subfilter of a filter F of L and $x \in F \setminus G$, then either $(G :_L x)$ is a prime filter of L or there exists an element $a \in L$ such that $(G :_L a \lor x)$ is a prime filter of L.

Proof. By Proposition 2.12 and Theorem 2.8 (iii), $(G :_L F)$ is a prime filter of L or $(G :_L F)$ is an intersection of two prime filter of L. We split the proof into two cases:

Case 1. $(G :_L F) = \mathbf{p}$, where \mathbf{p} is a prime filter of L. We show that $(G :_L x)$ is a prime filter of L. Clearly, $\mathbf{p} \subseteq G :_L x$. Suppose that $a, b \in L$ and

 $a \lor b \in G :_L x$). Thus $a \lor b \lor x \in G$; hence $a \lor x \in G$ or $b \lor x \in G$ or $a \lor b \in G$. If either $a \lor x \in G$ or $b \lor x \in G$, we are done. So we may assume that $a \lor b \in G$. As G is a filter, $(a \lor b) \lor F \subseteq G$; thus $a \lor b \in \mathbf{p}$ and so $a \in \mathbf{p}$ or $b \in \mathbf{p}$. Therefore, $a \in G :_L x$) or $b \in G :_L x$ and the assertion follows.

Case 2. $(G :_L F) = \mathbf{p} \cap \mathbf{q}$, where \mathbf{p} and \mathbf{p} are distinct prime filters of L. If $\mathbf{p} \subseteq (G :_L x)$, then the result follows by an argument like that in the Case 1. So we may assume that $\mathbf{p} \not\subseteq (G :_L x)$. There is an element $a \in \mathbf{p}$ such that $a \lor x \notin G$. By Theorem 2.8 (ii), $\mathbf{p} \lor \mathbf{q} \subseteq (G :_L x)$; so $\mathbf{q} \subseteq G :_L a \lor x$) and the result follows by a similar proof to that of Case 1.

Compare the next lemma with Lemma 1 in [7].

Lemma 3.2. Let F and F_i (i = 1, 2, ..., n) be filters such that $F \subseteq \bigcup_{i=1}^n F_i$ is an efficient covering of filters of L, where $n \ge 3$. Then The intersection of any n-1 of the filters $F \cap F_i$ coincides with $H = \bigcap_{i=1}^n (F \cap F_i)$.

Proof. It suffices to show that the intersection of any n-1 of the filters $F \cap F_i$ is contained in H. Since $F \subseteq \bigcup_{i=1}^n F_i$ is an efficient covering, we have $F = \bigcup_{i=1}^n (F \cap F_i)$ is an efficient union consisting of subfilters of F, so F is not contained in the union of any n-1 of the filters $F \cap F_i$; hence there exists an element $c_n \in F_n$ which is not in $\bigcup_{i=1}^{n-1} (F \cap F_i)$. If $x \in \bigcap_{i=1}^{n-1} (F \cap F_i)$, then the element $x \wedge c_n$ in Fcan not be in F_i for $1 \leq i \leq n-1$; thus $x \wedge c_n \in F_n$. By Lemma 1.1 (i), $x \in F_n$ and so $x \in H$, as needed.

Proposition 3.3. Let F and F_i (i = 1, 2, ..., n) be filters such that $F \subseteq \bigcup_{i=1}^n F_i$ is an efficient covering of filters of L, where $n \ge 3$. If $F_i \nsubseteq (F_j :_L x)$ for all $x \in L \setminus F_j$ whenever $i \ne j$, then no F_i for $1 \le i \le n$ is a 2-absorbing filter of L.

Proof. Assume to the contrary, F_k is a 2-absorbing filter of L for some $k = 1, \ldots, n$. By Lemma 3.2, $\bigcap_{i \neq j} (F_i \cap F) \subseteq F \cap F_k$. Clearly, $F \notin F_k$, so there is an element $b \in F$ with $b \notin F_k$. Now Theorem 3.1 gives either $(F_k :_L b)$ is a prime filter or there exists $a \in L$ such that $(F_k :_L (a \lor b))$ is a prime filter of L. Suppose first that $(F_k :_L b)$ is a prime filter. By assumption, there is $a_i \in F_i \setminus (F_k :_L b)$ for all $i \neq k$; so $(\lor_{i \neq j} a_i) \lor b \in \bigcap_{i \neq j} (F \cap F_i) \setminus (F \cap F_k)$ since $(\lor_{i \neq j} a_i) \lor b \in F \cap F_k$ implies that $(\lor_{i \neq j} a_i) \in (F_k :_L b)$ and so there is $a_i \in (F_k :_L b)$ for some $i \neq k$ that is a contradiction. If $(F_k :_L (a \lor b))$ is a prime filter of L for some $a \in L$, then there exists $c_i \in F_i \setminus (F_k :_L (a \lor b))$ for all $i \neq k$. Therefore $(\lor_{i \neq j} c_i) \lor (a \lor b \in \bigcap_{i \neq j} (F \cap F_i) \setminus (F \cap F_k)$ which is a contradiction. Thus F_k is not a 2-absorbing filter, as required.

The following theorem is a lattice counterpart of Theorem 3.2 in [10] describing the structure of 2-absorbing submodules. **Theorem 3.4** (2-Absorbing Avoidance Theorem). Let F, F_1, F_2, \ldots, F_n $(n \ge 2)$ be filters of L such that at most two of F_1, F_2, \ldots, F_n are not 2-absorbing. If $F \subseteq \bigcup_{i=1}^n F_i$ and $F_i \nsubseteq (F_j : L x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then $F \subseteq F_i$ for some i with $1 \le i \le n$.

Proof. By Remark 2.3 (i), we may assume that $n \ge 3$. Let $F \nsubseteq F_i$ for all i with $1 \le i \le n$. Then $F \subseteq \bigcup_{i=1}^n F_i$ is an efficient covering of filters of L. Then by Proposition 3.3, no F_i is 2-absorbing that contradicts the assumption. Therefore $F \subseteq F_i$ for some i with $1 \le i \le n$.

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