*Discussiones Mathematicae*
*General Algebra and Applications 36 (2016) 71–84*
*doi:10.7151/dmgaa.1252*

# ON CENTRALIZER OF SEMIPRIME INVERSE SEMIRING

S. Sara, M. Aslam and M.A. Javed

*Department of Mathematics*
*GC University, Lahore, Pakistan*

**e-mail:** saro_c18@yahoo.com
aslam298@hotmail.com
anjum2512@yahoo.com

**Abstract**

Let $S$ be 2-torsion free semiprime inverse semiring satisfying $A_2$ condition of Bandlet and Petrich [1]. We investigate, when an additive mapping $T$ on $S$ becomes centralizer.

**Keywords:** inverse semiring, semiprime inverse semiring, commutators, left(right) centralizer.

**2010 Mathematics Subject Classification:** 16Y60, 16W25.

## 1. Introduction and preliminaries

Throughout this paper, $S$ we will represent inverse semiring which satisfies $A_2$ condition of Bandlet and Petrich [1]. $S$ is prime if $aSb = (0)$ implies either $a = 0$ or $b = 0$ and $S$ is semiprime if $aSa = (0)$ implies $a = 0$. $S$ is n-torsion free if $nx = 0$, $x \in S$ implies $x = 0$. Following Zalar [12], we canonically define left(right) centralizer of $S$ as an additive mapping $T : S \to S$ such that $T(xy) = T(x)y$ $(xT(y))$, $\forall x, y \in S$ and $T$ is called centralizer if it is both right and left centralizer.

Bresar and Zalar [2] have proved that an additive mapping $T$ on 2-torsion free prime ring $R$ which satisfies weaker condition $T(x^2) = T(x)x$ is a left centralizer. Later, Zalar [12] generalized this result for semiprime rings. Motivated by the work of Zalar [12], Vukman [10] proved that an additive mapping on 2-torsion free semiprime ring satisfying $T(xyz) = xT(y)x$ is a centralizer. In this paper, our objective is to explore the result of Vukman [10] in the setting of inverse semirings as follows: Let $S$ be 2-torsion free semiprime inverse semiring and let

$T : S \to S$ be additive mapping such that $T(xyx) + xT(y)\acute{x} = 0$ holds $\forall x, y \in S$ then $T$ is a centralizer.

To prove this result we will first generalize Proposition 1.4 of [12] in the framework of inverse semirings.

By semiring we mean a nonempty set $S$ with two binary operations '+' and '.' such that $(S, +)$ and $(S, .)$ are semigroups where +is commutative with absorbing zero 0, i.e., $a+0 = 0+a = a$, $a.0 = 0.a = a$ $\forall a \in S$ and $a.(b+c) = a.b+a.c$, $(b+c)a = b.a+c.a$ holds $\forall$ $a, b, c \in S$. Introduced by Karvellas [6], a semiring $S$ is an inverse semiring if for every $a \in S$ there exist a unique element $\acute{a} \in S$ such that $a + \acute{a} + a = a$ and $\acute{a} + a + \acute{a} = \acute{a}$, where $\acute{a}$ is called pseudo inverse of $a$. Karvellas [6] proved that for all $a, b \in S$, $(a.b)' = \acute{a}.b = a.\acute{b}$ and $\acute{a}\acute{b} = ab$.

In this paper, inverse semirings satisfying the condition that for all $a \in S$, $a + \acute{a}$ is in center $Z(S)$ of $S$ are considered (see [4] for more details). Commutative inverse semirings and distributive lattices are natural examples of inverse semirings satisfying $A_2$. In a distributive lattice pseudo inverse of every element is itself. Also if $R$ is commutative ring and $I(R)$ is semiring of all two sided ideals of $R$ with respect to ordinary addition and product of ideals and $T$ is subsemiring of $I(R)$ then set $S_1 = \{(a, I) : a \in R, I \in T\}$. Define on $S_1$ addition $\oplus$ and multiplication $\odot$ by $(a, I) \oplus (b, J) = (a + b, I + J)$ and $(a, I) \odot (b, J) = (ab, IJ)$. It is easy to see $S_1$ is an inverse semiring with $A_2$ condition where $(a, I)' = (\acute{a}, I)$.

By [4], commutator [.,.] in inverse semirings defines as $[x, y] = xy+\acute{y}x$. We will make use of commutator identities $[x, y+z] = [x, y]+[x, z]$, $[xy, z] = [x, z]y+x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$ (see [4] for their proofs).

The following Lemmas are useful in establishing main result.

**Lemma 1.1.** *For $a, b \in S$, $a + b = 0$ implies $a = \acute{b}$.*

***Proof.*** Let $a + b = 0$ which implies $a + b + \acute{a} + \acute{b} = 0$ or $a + b + \acute{a} + \acute{b} + a = a$ or $a + b + \acute{b} = a$ and by hypothesis, we get $a = \acute{b}$.

However, converse of Lemma 1.1. is not true for instance, in distributive lattice $D$, for $a \in D$ we have $a = \acute{a}$ but $a + a = a$.

**Lemma 1.2.** *If $x, y, z \in S$ then following identities are valid:*

(1) $[xy, x] = x[y, x]$, $[x, yx] = [x, y]x$, $[x, xy] = x[x, y]$, $[yx, x] = [y, x]x$

(2) $y[x, z] = [x, yz] + [x, y]\acute{z}$, $[x, y]z = \acute{y}[x, z] + [x, yz]$

(3) $x[y, z] = [xy, z] + [x, z]\acute{y}$, $[x, z]y = [xy, z] + \acute{x}[y, z]$.

***Proof.*** (1) $[xy, x] = xyx + \acute{x}xy = x(yx + \acute{x}y) = x[y, x]$.
(2) $y[x, z] = (y+\acute{y}+y)(xz+\acute{z}x) = (y+\acute{y})xz+(y+\acute{y})\acute{z}x+yxz+y\acute{z}x = x(y+\acute{y})z+(y + \acute{y})\acute{z}x + yxz + y\acute{z}x = xyz + x\acute{y}z + y\acute{z}x + yxz = xyz + y\acute{z}x + x\acute{y}z + yxz = [x, yz] + [x, y]\acute{z}$.

Proof of the other identities can be obtained using similar techniques.

In the following, we extend Lemma 1.1 of Zalar [12] in a canonical fashion.

**Lemma 1.3.** *Let $S$ be a semiprime inverse semiring such that for $a, b \in S$, $axb = 0, \forall x \in S$ then $ab = ba = 0$.*

**Definition 1.4.** A mapping $f : S \times S \to S$ is biadditive if $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ and $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2), \ \forall x, y, x_1, x_2, y_1, y_2 \in S$.

**Example.** Define mappings $f, g : S_1 \times S_1 \to S_1$ by $f((a, I), (b, J)) = (ab, IJ)$ and $g((a, I), (b, J)) = ([a, b], IJ)$. Then $f$ and $g$ are biadditive.

Also, if $(D, \wedge, \vee)$ is a distributive lattice then $h : D \times D \to D$ defined by $h(a, b) = a, \forall a, b \in D$ is a biadditive mapping.

**Lemma 1.5.** *Let $S$ be semiprime inverse semiring and $f, g : S \times S \to S$ are biadditive mappings such that $f(x, y)wg(x, y) = 0, \forall\, x, y, w \in S$, then $f(x, y)wg(s, t) = 0, \forall\, x, y, s, t, w \in S$.*

***Proof.*** Replace $x$ with $x + s$ in $f(x, y)wg(x, y) = 0$, we get $f(s, y)wg(x, y) + f(x, y)wg(s, y) = 0$. By Lemma 1.1, we have $f(x, y)wg(s, y) = f(s, y)\acute{w}g(x, y)$. This implies
$(f(x, y)wg(s, y))z(f(x, y)wg(s, y)) = (f(s, y)\acute{w}g(x, y))z(f(x, y)wg(s, y)) = 0$
and semiprimeness of $S$ implies that $f(x, y)wg(s, y) = 0$. Now replacing $y$ with $y + t$ in last equation and using similar approach we get the required result.

**Lemma 1.6.** *Let $S$ be a semiprime inverse semiring and $a \in S$ some fixed element. If $a[x, y] = 0$ for all $x, y \in S$, then there exists an ideal $I$ of $S$ such that $a \in I \subset Z(S)$ holds.*

***Proof.*** By Lemma 1.2, we have $[z, a]x[z, a] = zax[z, a] + \acute{a}zx[z, a] = za([z, xa] + [z, x]\acute{a}) + \acute{a}([z, zxa] + [z, zx]\acute{a}) = za[z, xa] + za[z, x]\acute{a} + \acute{a}[z, zxa] + a[z, zx]a = 0$.

Using semiprimeness of $S$ and then Lemma 1.1, we get $a \in Z(S)$. By Lemma 1.2, we have $zaw[x, y] = za([x, wy] + [x, w]\acute{y}) = 0, \forall x, y, z, w \in S$. By similar argument, we can show that $zaw \in Z(S)$ and hence $SaS \subset Z(S)$. Now it is easy to see that ideal generated by $a$ is central.

**Lemma 1.7.** *Let $S$ be semiprime inverse semiring and $a, b, c \in S$ such that*

$$axb + bxc = 0 \tag{1}$$

*holds for all $x \in S$ then $(a + c)xb = 0$ for all $x \in S$.*

***Proof.*** Replace $x$ with $xby$ in (1), we get

$$axbyb + bxbyc = 0, \ x, y \in S. \tag{2}$$

Post multiplying (1) by $yb$ gives

$$axbyb + bxcyb = 0, \ x, y \in S. \tag{3}$$

Applying Lemma 1.1 on (2) and using it in (3), we have

$$bx(\acute{b}yc + cyb) = 0, \ x, y \in S. \tag{4}$$

Replace $x$ with $ycx$ in (4), we get

$$bycx(\acute{b}yc + cyb) = 0, \ x, y \in S. \tag{5}$$

Pre multiplying (4) by $cy$ gives

$$cybx(\acute{b}yc + cyb) = 0 \ , x, y \in S. \tag{6}$$

Adding pseudo inverse of (5) and (6) we get

$$(\acute{b}yc + cyb)x(\acute{b}yc + cyb) = 0, \ x, y \in S.$$

Using semiprimeness of $S$ and Lemma 1.1, we get $byc = cyb, y \in S$. By using last relation in (1) we get the required result.

## 2. Main results

**Theorem 2.1.** *Let $S$ be a 2-torsion free semiprime inverse semiring and $T : S \to S$ be an additive mapping which satisfies $T(x^2) + T(x)\acute{x} = 0, \ \forall x \in S$. Then $T$ is a left centralizer.*

***Proof.*** Take,

$$T(x^2) + T(x)\acute{x} = 0, \ x \in S. \tag{7}$$

Linearization of (7) gives

$$T(xy + yx) + T(x)\acute{y} + T(y)\acute{x} = 0, \ x, y \in S. \tag{8}$$

Replace $y$ with $xy + yx$ in (8), we get

$$T(x^2y + yx^2) + 2T(xyx) + T(xy)\acute{x} + T(yx)\acute{x} + T(x)y\acute{x} + T(x)x\acute{y} = 0. \tag{9}$$

Using Lemma 1.1 in (8) and using it in (9), we have

$$T(x^2y + yx^2) + 2T(xyx) + T(x)y\acute{x} + T(y)\acute{x}^2 + T(x)y\acute{x} + T(x)x\acute{y} = 0. \tag{10}$$

Using Lemma 1.1 in (7) and using it in (10) we get

(11) $$T(x^2y+yx^2)+2T(xyx)+T(x)y\acute{x}+T(y)\acute{x}^2+T(x)y\acute{x}+T(x^2)\acute{y}=0.$$

Replace $x$ with $x^2$ in (8) we get

(12) $$T(x^2y+yx^2)+T(x^2)\acute{y}+T(y)\acute{x}^2=0.$$

Using (12) in (11), we get

$$2T(xyx)+2T(x)y\acute{x}=0.$$

As $S$ is 2-torsion free, so we have

(13) $$T(xyx)+T(x)y\acute{x}=0.$$

Linearization (by $x=x+z$) of (13) gives

(14) $$T(xyz+zyx)+T(x)y\acute{z}+T(z)y\acute{x}=0.$$

Replace $x$ with $xy$, $z$ with $yx$ and $y$ with $z$ in (14), we get

(15) $$T(xyzyx+yxzxy)+T(xy)zy\acute{x}+T(yx)zx\acute{y}=0.$$

Replace $y$ with $yzy$ in (13), we get

(16) $$T(xyzyx)+T(x)yzy\acute{x}=0.$$

Replace $x$ with $y$ and $y$ with $xzx$ in (13), we get

(17) $$T(yxzxy)+T(y)xzx\acute{y}=0.$$

By adding (16) and (17), we get

(18) $$T(xyzyx+yxzxy)+T(x)yzy\acute{x}+T(y)xzx\acute{y}=0.$$

Using Lemma 1.1 in (15) and using the result in (18), we get

(19) $$T(xy)zyx+T(yx)zxy+T(x)yzy\acute{x}+T(y)xzx\acute{y}=0.$$

Now if we define biadditive function $f:S\times S\to S$ by $f(x,y)=T(xy)+T(x)\acute{y}$, then (19) can be written as

(20) $$f(x,y)zyx+f(y,x)zxy=0.$$

From (8) and Lemma 1.1, we have

$$(f(x,y))\acute{}=f(y,x).$$

Thus (20) can be rewritten as

$$f(x,y)zyx + f(x,y)zx\acute{y} = 0, \quad \text{or}$$

$$f(x,y)z[x,y] = 0, x,y,z \in S.$$

Using Lemma 1.5 and then Lemma 1.3, we have $f(x,y)[s,t] = 0,\ x,y,s,t \in S$. Now fix $x,y$ then by Lemma 1.6, there exist ideal $I \subset Z(S)$ such that $f = f(x,y) \in I \subset Z(S)$. This implies that $bf, fb \in Z(S), \forall b \in S$, thus we have

$$xfy = xyf = fxy = yfx \ \text{ and} \tag{21}$$

$$xf^2y = f^2xy = yf^2x = f^2yx. \tag{22}$$

Replace $y$ with $f^2y$ in (8), we get

$$2T(xf^2y + f^2yx) + 2T(x)f^2\acute{y} + 2T(f^2y)\acute{x} = 0.$$

Using (22), we get

$$2T(yf^2x + f^2xy) + 2T(x)f^2\acute{y} + 2T(f^2y)\acute{x} = 0. \tag{23}$$

By Lemma 1.1, (8), (7) and (23), we have

$$2T(y)f^2x + 2T(f^2x)y + 2T(x)f^2\acute{y} + 2T(f^2y)\acute{x} = 0, \quad \text{or}$$
$$2T(y)f^2x + T(f^2x + f^2x)y + 2T(x)f^2\acute{y} + T(f^2y + f^2y)\acute{x} = 0, \quad \text{or}$$
$$2T(y)f^2x + T(f^2x + xf^2)y + 2T(x)f^2\acute{y} + T(f^2y + yf^2)\acute{x} = 0, \quad \text{or}$$
$$2T(y)f^2x + T(f^2)xy + T(x)f^2y + 2T(x)f^2\acute{y} + T(f^2)y\acute{x} + T(y)f^2\acute{x} = 0, \quad \text{or}$$
$$2T(y)f^2x + T(y)f^2\acute{x} + T(f^2)xy + T(x)f^2y + 2T(x)f^2\acute{y} + T(f^2)y\acute{x} = 0, \quad \text{or}$$
$$T(y)f^2x + T(f)fxy + T(x)f^2\acute{y} + T(f)fy\acute{x} = 0, \quad \text{or}$$
$$T(y)f^2x + T(x)f^2\acute{y} + T(f)fy(\acute{x} + x) = 0. \tag{24}$$

Now replace $x$ with $xy$ and $y$ with $f^2$ in (8) and then using (21) and (22), we get

$$2T(fxfy + fyfx) + 2T(xy)\acute{f^2} + 2T(f^2)\acute{x}y = 0.$$

By Lemma 1.1, (8) and (7), we have

$$2T(fx)fy + 2T(fy)fx + 2T(xy)\acute{f^2} + 2T(f^2)\acute{x}y = 0, \quad \text{or}$$

$$T(fx + fx)fy + T(fy + fy)fx + 2T(xy)\acute{f^2} + 2T(f^2)\acute{x}y = 0$$

$$T(fx+xf)fy+T(fy+yf)fx+2T(xy)\acute{f^2}+2T(f^2)\acute{x}y=0$$

$$T(f)xfy+T(x)ffy+T(f)yfx+T(y)ffx+2T(xy)\acute{f^2}+2T(f^2)\acute{x}y=0$$

$$T(f)xfy+T(x)f^2y+T(f)fxy+T(y)f^2x+2T(xy)\acute{f^2}+2T(f)f\acute{x}y=0$$

$$T(f)fxy+2T(f)f\acute{x}y+T(f)fxy+T(x)f^2y+T(y)f^2x+2T(xy)\acute{f^2}=0$$

$$T(f)fy(\acute{x}+x)+T(x)f^2y+T(y)f^2x+2T(xy)\acute{f^2}=0.$$

Using Lemma 1.1 in (24) and using the result in last equation, we get

$$2T(x)f^2y+2T(xy)\acute{f^2}=0, \text{ or}$$

$$T(x)f^2y+T(xy)\acute{f^2}=0, \text{ or} \tag{25}$$

$(T(x)\acute{y}+T(xy))f^2=0$ or $f^3=0$ which implies

$$f^2Sf^2=f^4=(0)\Rightarrow f^2=0.$$

Thus $fSf=f^2S=(0)\Rightarrow f=0$. Therefore $T(xy)+T(x)\acute{y}=0$ and then Lemma 1.1 implies that $T$ is a left centralizer.

**Theorem 2.2.** *Let $S$ be a 2-torsion free semiprime inverse semiring and let $T: S\to S$ be an additive mapping such that*

$$T(xyx)+xT(y)\acute{x}=0,\ \forall x,y\in S. \tag{26}$$

*Then $T$ is a centralizer.*

***Proof.*** First we show that

$$[[T(x),x],x]=0.$$

Linearization of (26) gives

$$T(xyz+zyx)+xT(y)\acute{z}+zT(y)\acute{x}=0,\ \forall x,y,z\in S. \tag{27}$$

Replace $y$ with $x$ and $z$ with $y$ in last equation, we get

$$T(x^2y+yx^2)+xT(x)\acute{y}+yT(x)\acute{x}=0. \tag{28}$$

Replace $z$ with $x^3$ in (27), we get

$$T(xyx^3+x^3yx)+xT(y)\acute{x^3}+x^3T(y)\acute{x}=0. \tag{29}$$

Replace $y$ with $xyx$ in (28), we get

$$T(x^3yx + xyx^3) + xyxT(x)\acute{x} + xT(x)xy\acute{x} = 0. \tag{30}$$

Replace $y$ with $x^2y + yx^2$ in (26), we have

$$T(x^3yx + xyx^3) + xT(x^2y + yx^2)\acute{x} = 0. \tag{31}$$

Using Lemma 1.1 in (30) and using the result in (31), we get

$$\begin{aligned} xyxT(x)x + xT(x)xyx + xT(x^2y + yx^2)\acute{x} = 0, \text{ or} \\ x[T(x), x]yx + x\acute{y}[T(x), x]x = 0. \end{aligned} \tag{32}$$

Using Lemma 1.7 in (32), we have

$$(x[T(x), x] + [T(x), x]\acute{x})yx = 0, \text{ or}$$

$$[[T(x), x], x]yx = 0. \tag{33}$$

Replace $y$ with $y[T(x), x]$ in (33), we have

$$[[T(x), x], x]y[T(x), x]x = 0. \tag{34}$$

Post multiplication (33) with $[T(x), x]$ gives

$$[[T(x), x], x]yx[T(x), x] = 0. \tag{35}$$

Adding pseudo inverse of (35) and (34), we have $[[T(x), x], x]y[[T(x), x], x] = 0$ and then semiprimeness of $S$ implies that

$$[[T(x), x], x] = 0, \ \forall x \in S \text{ or} \tag{36}$$

$$[T(x), x]x + \acute{x}[T(x), x] = 0 \text{ or}$$

$$[T(x), x]x + (x + \acute{x})[T(x), x] = x[T(x), x], \text{ or}$$

$$[T(x), x]x + [T(x), x](x + \acute{x}) = x[T(x), x], \text{ or}$$

$$[T(x), x]x = x[T(x), x], \ \forall x \in S. \tag{37}$$

Linearization of (36) gives

$$\begin{aligned}[[T(x),x],y]+[[T(x),y],x]+[[T(y),y],x]+[[T(y),x],y]\\ +[[T(x),y],y]+[[T(y),x],x]=0.\end{aligned} \tag{38}$$

Replace $x$ with $\acute{x}$ in (38) and using again (38) and the fact that $(T(x))\acute{} = T(\acute{x})$ we have

$$\begin{aligned}2[[T(x),x],y]+2[[T(x),y],x]+[[T(y),y],x+\acute{x}]+[[T(y),x],y+\acute{y}]\\ +[[T(x),y],y+\acute{y}]+2[[T(y),x],x]=0.\end{aligned} \tag{39}$$

Adding (38) in (39) and then using (38) again, we get

$$2[[T(x),x],y]+2[[T(x),y],x]+2[[T(y),x],x]=0,\ \forall x,y\in S.$$

$$[[T(x),x],y]+[[T(x),y],x]+[[T(y),x],x]=0,\ \forall x,y\in S. \tag{40}$$

Replace $y$ with $xyx$ in (40), we have

$$[[T(x),x],xyx]+[[T(x),xyx],x]+[[T(xyx),x],x]=0, \text{ or}$$

Using Lemma 1.1 in (26) and using it in last equation, we get

$$[[T(x),x],xyx]+[[T(x),xyx],x]+[[xT(y)x,x],x]=0.$$

Using Lemma 1.2, we have

$$\begin{aligned}[[T(x),x],x]yx+x[[T(x),x],yx]+[[T(x),xy]x,x]+[xy[T(x),x],x]\\ +[[xT(y),x]x,x]=0.\end{aligned}$$

Using (36) and Lemma 1.2, we get

$$x[[T(x),x],y]x+[[xT(y),x]x,x]+[[T(x),xy],x]x+x[y[T(x),x],x]=0.$$

Again using Lemma 1.2, and (36) we have

$$\begin{aligned}x[[T(x),x],y]x+x[[T(y),x],x]x+[T(x),x][y,x]x\\ +x[[T(x),y],x]x+x[y,x][T(x),x]=0.\end{aligned}$$

Using (40) in last equation, we get

$$[T(x),x][y,x]x+x[y,x][T(x),x]=0$$

$$[T(x),x](yx+\acute{x}y)x+x(yx+\acute{x}y)[T(x),x]=0$$

$$[T(x),x]yx^2+[T(x),x]\acute{x}yx+xyx[T(x),x]+\acute{x^2}y[T(x),x]=0.$$

Using (37), we get

$$[T(x),x]yx^2+x^2\acute{y}[T(x),x]+\acute{x}[T(x),x]yx+xy[T(x),x]x=0.$$

Using (32), we have

$$[T(x),x]yx^2+x^2\acute{y}[T(x),x]=0. \tag{41}$$

Pre multiply (41) by $x$ gives

$$x[T(x),x]yx^2+x^3\acute{y}[T(x),x]=0. \tag{42}$$

Using Lemma 1.1 in (32) and using it in (42), we get

$$xy[T(x),x]x^2+x^3\acute{y}[T(x),x]=0. \tag{43}$$

Pre multiply last equation by $T(x)$, we get

$$T(x)xy[T(x),x]x^2+T(x)x^3\acute{y}[T(x),x]=0. \tag{44}$$

Replace $y$ with $T(x)y$ in (43), we get

$$xT(x)y[T(x),x]x^2+x^3T(x)\acute{y}[T(x),x]=0. \tag{45}$$

Adding pseudo inverse of (45) and (44), we get

$$[T(x),x]y[T(x),x]x^2+[T(x),x^3]\acute{y}[T(x),x]=0. \tag{46}$$

By applying Lemma 1.7 in (46), we get

$$([T(x),x]\acute{x^2}+[T(x),x^3])y[T(x),x]=0$$

$$([T(x),x]\acute{x^2}+[T(x),x]x^2+x[T(x),x^2])y[T(x),x]=0$$

$$([T(x),x]\acute{x^2}+[T(x),x]x^2+x[T(x),x]x+x^2[T(x),x])y[T(x),x]=0.$$

Using (37) and the fact that $S$ is inverse semiring, we have

$$x[T(x), x]xy[T(x), x] = 0.$$

And then semiprimeness of $S$ implies that

$$x[T(x), x]x = 0, \forall x \in S. \tag{47}$$

Replace $y$ with $yx$ in (32) and using (47) we have

$$x[T(x), x]yx^2 = 0. \tag{48}$$

Replace $y$ with $yT(x)$ in (48), we get

$$x[T(x), x]yT(x)x^2 = 0. \tag{49}$$

Post multiplying (48) by $T(x)$, we get

$$x[T(x), x]yx^2T(x) = 0. \tag{50}$$

Adding pseudo inverse of (50) in (49), we get

$$x[T(x), x]y[T(x), x^2] = 0$$

$$x[T(x), x]y([T(x), x]x + x[T(x), x]) = 0.$$

Using (37) and the fact that $S$ is 2-torsion free, we have

$$x[T(x), x] = 0 = [T(x), x]x, \ x \in S. \tag{51}$$

As (40) obtained from (36), we can get following from (51)

$$[T(x), x]y + [T(x), y]x + [T(y), x]x = 0. \tag{52}$$

Post multiplying (52) by $[T(x), x]$ and using (51), we get $[T(x), x]y[T(x), x] = 0, \ \forall y \in S$ which implies that

$$[T(x), x] = 0. \tag{53}$$

Replace $y$ with $xy + yx$ in (26), we have

$$T(x^2yx + xyx^2) + xT(xy + yx)\acute{x} = 0. \tag{54}$$

Replace $z$ with $x^2$ in (27), we get

$$T(xyx^2 + x^2yx) + xT(y)\acute{x^2} + x^2T(y)\acute{x} = 0. \tag{55}$$

Using Lemma 1.1 in (54) and using the result in (55) we get

$$x(T(xy+yx)+\acute{x}T(y)+T(y)\acute{x})x=0.$$

Now if we define biadditive function $g: S\times S\to S$ by $g(x,y)=T(xy+yx)+T(y)\acute{x}+\acute{x}T(y)$ then last equation can be written as

$$xg(x,y)x=0. \tag{56}$$

As (40) obtained from (36), we can obtain following from (56)

$$xg(x,y)z+xg(z,y)x+zg(x,y)x=0,\ \ \forall x,y,z\in S. \tag{57}$$

Post multiplication (57) by $g(x,y)x$ and using (56) we get

$$xg(x,y)zg(x,y)x=0. \tag{58}$$

Linearization of (53) gives

$$[T(x),y]+[T(y),x]=0. \tag{59}$$

Replace $y$ with $xy+yx$ in above equation and using (53) we get

$$[T(xy+yx),x]+x[T(x),y]+[T(x),y]x=0.$$

Using Lemma 1.1 in (59) and using the result in last equation, we get

$$\acute{x}[T(y),x]+[T(y),x]\acute{x}+[T(xy+yx),x]=0.$$

Using Lemma 1.2 in last equation, we get

$$[\acute{x}T(y),x]+[T(y)\acute{x},x]+[T(xy+yx),x]=0,\ \ \text{or}$$

$$[\acute{x}T(y)+T(y)\acute{x}+T(xy+yx),x]=0,\ \ \text{or}$$

$$[g(x,y),x]=0. \tag{60}$$

which gives

$$g(x,y)x=xg(x,y),\ x,y\in S. \tag{61}$$

By (58) and (61), $g(x,y)xzg(x,y)x=0$ this and (61) implies

$$xg(x,y)=0=g(x,y)x. \tag{62}$$

Linearization of (62) gives $g(x,y)z+g(z,y)x=0$.

Post multiplying last equation by $g(x, y)$ and using (62), we get $g(x, y)zg(x, y) = 0$ and this implies $g(x, y) = 0, x, y \in S$. Put $x = y$, we get

$$2T(x^2) + \acute{x}T(x) + T(x)\acute{x} = 0. \tag{63}$$

From (53) we can get $T(x)x = xT(x)$, using this and the fact that $S$ is 2-torsion free, in (63), we get

$$T(x^2) + \acute{x}T(x) = 0 \text{ and } T(x^2) + T(x)\acute{x} = 0.$$

And therefore by Theorem 2.1, it follows that $T$ is right and left centralizer. This completes the proof.

**Acknowledgement**

We are thankful to referee for the valuable suggestions.

## References

[1] H.J. Bandlet and M. Petrich, *Subdirect products of rings and distrbutive lattics*, Proc. Edin Math. Soc. **25** (1982) 135–171. doi:10.1017/s0013091500016643

[2] M. Bresar and Borut Zalar, *On the structure of Jordan *-derivations*, Colloq. Math. **63** (2) (1992) 163–171. http://www.zentralblatt-math.org/zmath/en/advanced/?q=an:0786.46045

[3] J.S. Golan, *The theory of semirings with applications in mathematics and theoretical computer science* (John Wiley and Sons. Inc., New York, 1992). doi:10.1007/978-94-015-9333-5_13

[4] M.A Javed, M. Aslam and M. Hussain, *On condition* $(A_2)$ *of Bandlet and Petrich for inverse Semirings*, Int. Mathematical Forum **7** (59) (2012) 2903–2914. http://www.m-hikari.com/imf/imf-2012/57-60-2012/aslamIMF57-60-2012.pdf

[5] U. Habisch and H.J. Weinert, *Semirings-Algebraic theory and applications in computer science* (World Scientific, 1993). doi:10.1142/3903

[6] P.H. Karvellas, *Inversive semirings*, J. Austral. Math. Soc. **18** (1974) 277–288. doi:10.1017/s1446788700022850

[7] V.J Khanna, Lattices and Boolean Algebras (Vikas Publishing House Pvt. Ltd., 2004).

[8] M.K. Sen, S.K. Maity and K.P Shum, *Clifford semirings and generalized clifford semirings*, Taiwanese J. Math. **9** (2005) 433–444. http://journal.tms.org.tw/index.php/TJM/article/view/1014

[9] M.K. Sen and S.K. Maity, *Regular additively inverse semirings*, Acta Math. Univ. Comenianae **1** (2006) 137–146. http://eudml.org/doc/129834

[10] J. Vukman, *Centralizers of semiprime rings*, Comment. Math. Univ. Carolinae **42** (2001) 101–108. http://hdl.handle.net/10338.dmlcz/118920

[11] J. Vukman, *Jordan left derivation on semiprime rings*, Math. Jour. Okayama Univ. **39** (1997) 1–6. http://www.math.okayama-u.ac.jp/mjou/mjou1-46/mjou_pdf/mjou_39/mjou_39_001.pdf

[12] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolinae **32** (1991), 609–614. http://hdl.handle.net/10338.dmlcz/118440

Received 3 November 2015
Revised 8 March 2016