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# ON THE AUTOTOPISM GROUP OF THE CORDERO-FIGUEROA SEMIFIELD OF ORDER $3^6$

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**Abstract**

In [5] M. Biliotti, V. Jha and N. Johnson were able to completely determine the autotopism group of a generalized twisted field as a subgroup of $\Gamma L(K) \times \Gamma L(K)$, where $K = GF(p^n)$ and $\Gamma L(K)$ is the group of nonsingular semilinear transformations over $K$. In this article, we consider the Cordero-Figueroa semifield of order $3^6$, which is not a generalized twisted field, and we prove that its autotopism group is isomorphic to a subgroup of $\Gamma L(K) \times \Gamma L(K)$, where $K = GF(3^6)$.

**Keywords:** finite presemifield, finite semifield, autotopism, autotopism group.

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## 1. Introduction

The study of finite semifields was initiated about a century ago by L.E. Dickson [9]. Although nowadays it is common to use the term finite semifield (introduced by Knuth [12] in 1965), in the earlier literature was known as *nonassociative division algebra* or *distributive quasifield*. Shortly after the classification of finite fields, the study of semifields took a purely algebraic point of view. By now, the theory of semifields has become of considerable interest in many different areas of mathematics. Besides the numerous links with finite geometry (e.g. translation planes, generalized quadrangles), semifields arise in the context of difference sets, coding theory and group theory.

A *finite semifield* is an algebraic structure $(P, +, *)$ that consists of a set $P \neq \emptyset$, with at least two elements 0 and 1, and two binary operations, addition $(+)$ and multiplication $(*)$, satisfying the following axioms:

1. $(P, +)$ is a group with additive identity 0.
2. $x * y = 0$ implies $x = 0$ or $y = 0$.
3. $x * (y + z) = x * y + x * z$ and $(x + y) * z = x * z + y * z$, for all $x, y, z \in P$.
4. There exists $1 \in P$ such that $x * 1 = 1 * x = x$, for all $x \in P$.

A finite semifield without multiplicative identity is called *finite presemifield*.

Throughout this article, the term semifield (or presemifield) will always be used to refer a finite semifield (or a finite presemifield).

Two semifields (or presemifields) $(P, +, *)$ and $(P', +, \circ)$ are *isotopic* if there exist a triple $(F, G, H)$ of bijective functions from $P$ to $P'$ which are additives and satisfy $G(x * y) = F(x) \circ H(y)$, for all $x, y \in P$. The triple $(F, G, H)$ is called *isotopism* from $P$ to $P'$.

Any presemifield $(P, +, *)$ is isotopic to a semifield by defining a new operation $\circ$ as follows: $(x * e) \circ (e * y) = x * y$, where $e \in P$ is a fixed element, $e \neq 0$, and $x, y \in P$. Thus, $(P, +, \circ)$ is a semifield isotopic to $(P, +, *)$ with multiplicative identity $e * e$.

The notion of isotopy was introduced by A.A. Albert [2] for purely algebraic reasons; however, it has a geometric meaning based on projective geometry: two semifields are isotopic if and only if its corresponding projective translation planes are isomorphic (for more details see [3]).

There are plenty of results concerning isotopism of semifields (or presemifields) (see, for example, [8] and [13]), however, in this article, we are interested in autotopisms of semifields (or presemifields). Any isotopism from a semifield (or presemifield) $P$ to itself is known as *autotopism* of $P$.

The autotopisms of a semifield (or presemifield) $P$ form a group under component-wise composition and it is known in the literature as the *full autotopism group* of $P$. We denote this group by $\mathcal{A}(P)$.

It is not difficult to prove that the set

$$HG = \{(f, g, h) \in \mathcal{A}(P) : g = i\},$$

where $i$ is the identity function from $P$ to $P$, is a normal subgroup of $\mathcal{A}(P)$.

On the other hand, if $P$ is a semifield, the set

$$N_m = \{x \in P : (a * x) * b = a * (x * b) \ for\ all\ a, b \in P\}$$

is known as the *middle nucleus* of $P$. In [11] (Theorem 8.2) is proved that the multiplicative group of $N_m$ and $HG$ are isomorphic. For this reason, from now on, we will denote the subgroup $HG$ by $N_m^*$.

Let $K = GF(p^n)$ and let $\alpha$ and $\beta$ be automorphisms of $K$ with $\alpha \neq 1$ and $\beta \neq 1$. Let $c \in K$ be such that $c \neq x^{\alpha-1}y^{\beta-1}$ for all $x, y \in K^*$. Then the product over $K$

$$x * y = xy - cx^{\alpha}y^{\beta}$$

defines a presemifield which is called *generalized twisted field*.

The study of these presemifields began with A.A. Albert [1], disciple of Dickson, who wrote several seminal papers on twisted and generalized twisted fields in the late 50's and early 60's.

In the late 90's, M. Biliotti, V. Jha and N. Johnson determined the full autotopism group of a generalized twisted field as a subgroup of $\Gamma L(K) \times \Gamma L(K)$ (see Theorem 5.2 in [5]).

## 2. A presemifield defined by a 3-term product

Let $\alpha \neq 1$ and $\beta \neq 1$, $\alpha \neq \beta$, be automorphisms of $K = GF(p^n)$, where $p \geqslant 3$ and $n \geqslant 4$. For constants $A, B \in K^*$, we define over $K$ the product

$$x * y = xy + Ax^{\alpha}y^{\beta} + Bx^{\beta}y^{\alpha}.$$

If there exist $\alpha$, $\beta$, $A$, and $B$ such that $x * y = 0$ implies $x = 0$ or $y = 0$, the triple $(K, +, *)$ is a presemifield. In this case, we denote this presemifield by $P(K, \alpha, \beta, A, B)$.

The first example of a presemifield with a 3-term product as defined above was given by Figueroa in [10]. There in, a presemifield is defined by the product

$$x * y = xy + \gamma x^3 y^{27} + \gamma^{13} x^{27} y^3, \tag{1}$$

where $x, y \in GF(3^6)$ and $\gamma \in GF(3^6)$ is a primitive element such that $\gamma^6 = 1 + \gamma$.

The idea of generating presemifields with this type of product arose in [7], where the authors proved that a semifield of order $p^n \neq 2^6$, for a prime number $p$ and an integer $n \geq 3$, which admits an autotopism of order a $p$-primitive prime divisor $h$ of $p^n - 1$ (that is, $h \mid (p^n - 1)$ but $h \nmid (p^i - 1)$ for each integer $i$ with $1 \leqslant i \leqslant n-1$) is a presemifield with a product of the form

$$x * y = xy + \sum_{i=1}^{n-1} a_i x^{p^i} y^{p^{e_i}},$$

where $x, y \in GF(p^n)$, $a_i \in GF(p^n)$ are constants, and $0 \leqslant e_i \leqslant n-1$.

The presemifield defined by the product (1) is a semifield that admits an autotopism of order a 3-primitive prime divisor of $3^6 - 1$ and it is known in the literature of finite translation planes as *Cordero-Figueroa semifield of order* $3^6$ (see Theorem 37.2 in [6]).

From now on, a presemifield of the type $P(K, \alpha, \beta, A, B)$ will be defined as a *Figueroa's presemifield of order* $p^n$.

Since $\alpha$ and $\beta$ in a Figueroa's presemifield of order $p^n$ are automorphisms such that $\alpha \neq 1$, $\beta \neq 1$, and $\alpha \neq \beta$, in what follows (without loss the generality) we assume that $\alpha = p^a$ and $\beta = p^b$, with $0 < a < b < n$.

In the next section we prove an important lemma about the middle nucleus of a Figueroa's presemifield of order $p^n$ that will culminate in our main result in section 4.

## 3. On the middle nucleus of a Figueroa's presemifield of order $p^n$

The next lemma gives a description of the elements of $N_m^*$ for a Figueroa's presemifield of order $p^n$ and, furthermore, gives a way to compute the order of $N_m^*$. As we assumed at the end of Section 2, $\alpha = p^a$ and $\beta = p^b$, with $0 < a < b < n$.

**Lemma 1.** Let $(f, g, h)$ be an autotopism of a Figueroa's presemifield $P(K, \alpha, \beta, A, B)$ of order $p^n$, and suppose that $\alpha^3 \neq 1$ if $\alpha\beta = 1$. Then, $(f, g, h) \in N_m^*$ if and only if

$$f(x) = cx, \; g(y) = y, \; h(n) = c^{-1}n,$$

where $c \in K$ and $c^\alpha = c^\beta$. Furthermore, the order of $N_m^*$ is $p^{gcd(n, b-a)} - 1$.

***Proof.*** Suppose $(f, g, h) \in N_m^*$. Then, $x * n = f(x) * h(n)$ implies that

$$xn + Ax^\alpha n^\beta + Bx^\beta n^\alpha = f(x)m + Af^\alpha(x)m^\beta + Bf^\beta(x)m^\alpha, \tag{2}$$

where $m = h(n)$. Since $f$ is additive, $f(x) = \sum_{i=0}^{n-1} f_i x^{p^i}$. Thus, we have that

$$Af^{\alpha}(x)m^{\beta} = (Am^{\beta})\sum_{i=0}^{n-1} f_i^{\alpha} x^{p^{i+a}} = \sum_{t=0}^{n-1} Am^{\beta} f_{t-a}^{\alpha} x^{p^t},$$

$$Bf^{\beta}(x)m^{\alpha} = (Bm^{\alpha})\sum_{i=0}^{n-1} f_i^{\beta} x^{p^{i+b}} = \sum_{t=0}^{n-1} Bm^{\alpha} f_{t-b}^{\beta} x^{p^t}.$$

Therefore, the right-hand side of (2) becomes

$$\sum_{t=0}^{n-1}(mf_t + Am^{\beta} f_{t-a}^{\alpha} + Bm^{\alpha} f_{t-b}^{\beta})x^{p^t}.$$

If $t \neq 0, a,\ b \pmod{n}$, then the coefficient of $x^{p^t}$ in the left-hand side of (2) is 0, which implies, for these values of $t$, that $mf_t + Am^{\beta} f_{t-a}^{\alpha} + Bm^{\alpha} f_{t-b}^{\beta} = 0$. Then we have that $f_t = 0$, $f_{t-a} = 0$, and $f_{t-b} = 0$.

If $a + b \neq n$, then $t = a + b \neq 0 \pmod{n}$. Therefore, $f_{t-a} = f_b = 0$ and $f_{t-b} = f_a = 0$.

If $a + b = n$, then $2b = n$ if and only if $2a = n$. Since $a < b$, then $2a \neq 0 \pmod{n}$ and $2b \neq 0 \pmod{n}$. In the case that $2a = b \pmod{n}$ or $2b = a \pmod{n}$, we obtain that $3a = n$, which is impossible by the assumption that $\alpha^3 \neq 1$ if $\alpha\beta = 1$. Thus, for $t = 2a \pmod{n}$, we have $f_{t-a} = f_a = 0$, and for $t = 2b \pmod{n}$, we get $f_b = 0$. Hence $f(x) = f_0 x$. Consequently, from equation (2), $n = f_0 m = f_0 h(n)$ and $f_0^{\alpha} = f_0^{\beta}$. The converse is obvious.

On the other hand, in order to compute the order of $N_m^*$, note that the order of $N_m^*$ is equal to the number of elements $c \in K$, $c \neq 0$, such that $c^{\alpha} = c^{\beta}$. So, $c^{p^a(p^{b-a}-1)} = 1$ and the order of $N_m^*$ is $p^{gcd(n,\,b-a)} - 1$. ■

## 4. The autotopism group of the Cordero-Figueroa semifield of order $3^6$

In the next theorem, we use a method that is purely algebraic and determine that the full autotopism group of the Cordero-Figueroa semifield of order $3^6$ is isomorphic to a particular subgroup of $\Gamma L(K) \times \Gamma L(K)$, where $K = GF(3^6)$, and, furthermore, we compute its order.

**Theorem 1.** Let $P = P(K, \alpha, \beta, A, B)$ be the Cordero-Figueroa semifield of order $3^6$. The full autotopism group $\mathcal{A}(P)$ is isomorphic to the subgroup of $\Gamma L(K) \times \Gamma L(K)$:

$$\langle(\gamma^{13}x, \gamma^{26}y)\rangle \rtimes \langle(\gamma x^3, \gamma y^3)\rangle,$$

where $\langle(\gamma^{13}x, \gamma^{26}y)\rangle$ is normal in the group $\langle(\gamma^{13}x, \gamma^{26}y)\rangle\langle(\gamma x^3, \gamma y^3)\rangle$. Furthermore, the order of $\mathcal{A}(P)$ is 672.

***Proof.*** Let $(f,g,h)\in\mathcal{A}(P)$. Since $f$, $g$, and $h$ are additives:

$$f(x)=\sum_{k=0}^{5} f_k x^{3^k},\quad g(y)=\sum_{k=0}^{5} g_k y^{3^k},\quad h(n)=\sum_{k=0}^{5} h_k n^{3^k}.$$

Let us denote $m=h(n)$. Then $g(x*n)=f(x)*m$ yields the following equations

$$f_0m+Bf_3^{3^3}m^{3^1}+Af_5^{3^1}m^{3^3}=g_0n+g_5A^{3^5}n^{3^2}+g_3B^{3^3}n^{3^4},\tag{3}$$
$$f_1m+Bf_4^{3^3}m^{3^1}+Af_0^{3^1}m^{3^3}=g_1n^{3^1}+g_0An^{3^3}+g_4B^{3^4}n^{3^5},\tag{4}$$
$$f_2m+Bf_5^{3^3}m^{3^1}+Af_1^{3^1}m^{3^3}=g_5B^{3^5}n+g_2n^{3^2}+g_1A^{3^1}n^{3^4},\tag{5}$$
$$f_3m+Bf_0^{3^3}m^{3^1}+Af_2^{3^1}m^{3^3}=g_0Bn^{3^1}+g_3n^{3^3}+g_2A^{3^2}n^{3^5},\tag{6}$$
$$f_4m+Bf_1^{3^3}m^{3^1}+Af_3^{3^1}m^{3^3}=g_3A^{3^3}n+g_1B^{3^1}n^{3^2}+g_4n^{3^4},\tag{7}$$
$$f_5m+Bf_2^{3^3}m^{3^1}+Af_4^{3^1}m^{3^3}=g_4A^{3^4}n^{3^1}+g_2B^{3^2}n^{3^3}+g_5n^{3^5}.\tag{8}$$

Since $N_m^*$ is a normal subgroup of $\mathcal{A}(P)$, we have that for any $(f,g,h)\in\mathcal{A}(P)$ and any $(f_0,i,h_0)\in N_m^*$, there exist $(\tilde{f}_0,i,\tilde{h}_0)\in N_m^*$ such that

$$(f,g,h)^{-1}*(f_0,i,h_0)*(f,g,h)=(\tilde{f}_0,i,\tilde{h}_0).$$

Then, for all $x\in K$, we obtain

$$f_0(f(x))=f(\tilde{f}_0(x)).\tag{9}$$

Since $(f_0,i,h_0),(\tilde{f}_0,i,\tilde{h}_0)\in N_m^*$, from lemma 1 follows that $f_0(x)=c_0x$ and $\tilde{f}_0(x)=\tilde{c}_0x$, where $c_0,\tilde{c}_0\in K$. Also, by lemma 1, as the order of $N_m^*$ is 8, $\tilde{c}_0\in GF^*(3^2)$. Then, from equation (9) we get

$$c_0f_0=f_0\tilde{c}_0,\tag{10}$$
$$c_0f_1=f_1\tilde{c}_0^3,\tag{11}$$
$$c_0f_2=f_2\tilde{c}_0,\tag{12}$$
$$c_0f_3=f_3\tilde{c}_0^3,\tag{13}$$
$$c_0f_4=f_4\tilde{c}_0,\tag{14}$$
$$c_0f_5=f_5\tilde{c}_0^3.\tag{15}$$

Notice that $f(x)\neq 0$. If $f_0\neq 0$, then the equation (10) implies that $c_0=\tilde{c}_0$. Therefore, the equations (11), (13), and (15) imply that $f_1=f_3=f_5=0$. Similarly, if $f_2\neq 0$ or $f_4\neq 0$, then $f_1=f_3=f_5=0$. In the same fashion, if $f_1\neq 0$, $f_3\neq 0$, or $f_5\neq 0$, the equations (10), (12), and (14) imply $f_0=f_2=f_4=0$. Hence, at least one, and at most three, of the coefficients $f_0$, $f_2$, and $f_4$ (or $f_1$, $f_3$, and $f_5$) of $f$ are nonzero.

Consider the case that $f$ has three nonzero coefficients, let say $f_0 \neq 0$, $f_2 \neq 0$, and $f_4 \neq 0$. Then, the equations (3), (5), and (7) imply

$$\frac{g_0}{f_0} = \frac{g_5 B^{3^5}}{f_2} = \frac{g_3 A^{3^3}}{f_4}, \tag{16}$$

$$\frac{g_5 A^{3^5}}{f_0} = \frac{g_2}{f_2} = \frac{g_1 B^{3^1}}{f_4}, \tag{17}$$

$$\frac{g_3 B^{3^3}}{f_0} = \frac{g_1 A^{3^1}}{f_2} = \frac{g_4}{f_4}. \tag{18}$$

From equations (16) and (17) we get

$$g_1 = \frac{g_0 f_2 f_4 A^{3^5}}{f_0^2 B^{3^5} B^{3^1}},$$

and from equations (16) and (18) we obtain

$$g_1 = \frac{g_0 f_2 f_4 B^{3^3}}{f_0^2 A^{3^3} A^{3^1}}.$$

Thus, equating the right hand sides of the last two equations result that

$$g_0(B^{273} - A^{273}) = 0.$$

But, since the order of $\gamma$ is 728, $B^{273} - A^{273} = \gamma^{637} - \gamma^{273} \neq 0$. Hence $g_0 = 0$. Then, equations (16), (17), and (18) imply that $g_1 = g_2 = g_3 = g_4 = g_5 = 0$, and thus $g(y) = 0$ for all $y \in K$, which is a contradiction.

Similarly for the case that the three coefficients $f_1$, $f_3$, and $f_5$ are all nonzero.

Now consider the case that $f$ has two nonzero coefficients, let say $f_0 \neq 0$ and $f_2 \neq 0$. Then, the left hand side of equation (7) is zero, implying that $g_4 = g_3 = g_1 = 0$. Hence, from (4), we get

$$A f_0^{3^1} m^{3^3} = g_0 A n^{3^3}.$$

Since

$$m = h(n) = \sum_{k=0}^{5} h_k n^{3^k},$$

we obtain that all the coefficients of $h$ are zero except $h_0$. Thus $h(n) = h_0 n$.

Using that $m = h(n) = h_0 n$ in (6), we get $h_0 = 0$. Therefore, $h(n) = 0$ for all $n \in K$; a contradiction.

Similarly in the other cases where two of the three coefficients $f_0$, $f_2$, and $f_4$ (or $f_1$, $f_3$, and $f_5$) are nonzero.

Hence $f(x)$ has only one nonzero coefficient, and thus $f$ is of the form

$$f(x) = f_k x^{3^k},$$

where $k$ is an integer with $0 \leq k \leq 5$.

Let $k$, $0 \leq k \leq 5$, such that $f_i = 0$ for all $i \neq k$, then from the equations (3) through (8), we get

$$g(y) = g_k y^{3^k}, \quad h(n) = g_k f_k^{-1} n^{3^k}$$

where $f_k$ and $g_k$ satisfy

$$\begin{aligned} g_k f_k^{3^3} A^{3^k} &= A f_k^{3^1} g_k^{3^3}, \\ g_k f_k^{3^1} B^{3^k} &= B f_k^{3^3} g_k^{3^1}. \end{aligned}$$

Using these equations, it is possible to find $f_k$ and $g_k$, $0 \leq k \leq 5$, as powers of $\gamma$. By direct computation it can be shown that

$$\begin{aligned} f_0 &= \gamma^{13j}, & g_0 &= \gamma^{26j}; \\ f_1 &= \gamma^{13j+1}, & g_1 &= \gamma^{26j+1}; \\ f_2 &= \gamma^{13j+4}, & g_2 &= \gamma^{26j+4}; \\ f_3 &= \gamma^{13j}, & g_3 &= \gamma^{26j+13}; \\ f_4 &= \gamma^{13j+1}, & g_4 &= \gamma^{26j+40}; \\ f_5 &= \gamma^{13j+4}, & g_5 &= \gamma^{26j+121}, \end{aligned}$$

where $j$ is an integer with $0 \leqslant j < 56$.

To find a subgroup of $\Gamma L(K) \times \Gamma L(K)$ isomorphic to $\mathcal{A}(P)$, first let us define the subgroup $H$ of $\Gamma L(K) \times \Gamma L(K)$ as

$$H = \{(f_k x^{3^k},\ g_k y^{3^k}) :\ k \text{ is an integer with } 0 \leq k \leq 5\}.$$

Note that, for each $k$, the function $\Psi : (f_k x,\ g_k y,\ g_k f_k^{-1} n) \mapsto (f_k x,\ g_k y)$ is an isomorphism from $\mathcal{A}(P)$ to $H$.

It is not difficult to verify that

$$H = \langle(\gamma^{13}x, \gamma^{26}y)\rangle\langle(\gamma x^3, \gamma y^3)\rangle.$$

It can be shown that the subgroup $\langle(\gamma^{13}x, \gamma^{26}y)\rangle$ is normal in $H$. Furthermore, the intersection of the subgroups $\langle(\gamma^{13}x, \gamma^{26}y)\rangle$ and $\langle(\gamma x^3, \gamma y^3)\rangle$ is the identity $(x, y)$, and $\langle(\gamma x^3, \gamma y^3)\rangle$ is not normal in $H$. Therefore, by definition of semidirect product of groups, $H$ is the semidirect product of the subgroups $\langle(\gamma^{13}x, \gamma^{26}y)\rangle$ and

$\langle(\gamma x^3, \gamma y^3)\rangle$; i.e., $H = \langle(\gamma^{13}x, \gamma^{26}y)\rangle \rtimes \langle(\gamma x^3, \gamma y^3)\rangle$. Hence $\mathcal{A}(P)$ is isomorphic to $\langle(\gamma^{13}x, \gamma^{26}y)\rangle \rtimes \langle(\gamma x^3, \gamma y^3)\rangle$.

Finally, in order to compute the order of $\mathcal{A}(P)$, notice that the order of $(\gamma^{13}x, \gamma^{26}y)$ is 56 and the order of $(\gamma x^3, \gamma y^3)$ is 12. As the elements of $H$ are of the form

$$(\gamma^{13}x, \gamma^{26}y)^i(\gamma x^3, \gamma y^3)^j,$$

where $0 \leq i < 56$ and $0 \leq j < 12$, the order of $H$ is $56 \times 12 = 672$. It is now clear that the order of $\mathcal{A}(P)$ is 672, and the proof is complete. ■

**Some Remarks.** We conclude this article with some comments:

(a) Cordero-Figueroa's semifield of order $3^6$ was found with the aid of an antique computer program (Basic) in 1994. Because of being the first one of its kind, we consider to study the autotopism group of this particular semifield. Nowadays computers and programs are much more powerful than those of twenty years ago, therefore, now it should be possible to find all the 3-term product semifields of order $3^6$ (up to isotopism). Up to now, it is unknown for us if this was done.

(b) Theorem 1 establish an important result related to the autotopism group of the Cordero-Figueroa semifield of order $3^6$, and its proof provides some evidence that it can be generalized for a Figueroa's presemifield of order $p^n$. To be more specific, we conjecture that the autotopism group of a Figueroa's presemifield of order $p^n$ is isomorphic to a subgroup of $\Gamma L(K) \times \Gamma L(K)$ and its elements are of the form $(ux^\phi, uvy^\phi, vn^\phi)$, where $u, v \in K^*$ and $\phi$ is an automorphism of $K$.

(c) Some parts of the discussion concerning the middle nucleus can be simplified by using recent arguments, as for example, the fact that the middle nucleus of the Cordero-Figueroa semifield of order $3^6$ has order 9 follows directly from Proposition 3 in [4].

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