# $M$-SOLID GENERALIZED NON-DETERMINISTIC VARIETIES 

Somsak Lekkoksung<br>Department of Mathematics, Faculty of Engineering<br>Rajamangala University of Technology Isan Khon Kaen Campus. 40000, Thailand<br>e-mail: lekkoksung_somsak@hotmail.com


#### Abstract

A generalized non-deterministic hypersubstitution is a mapping which maps operation symbols of type $\tau$ to the set of terms of the same type which does not necessarily preserve the arity. We apply the generalized nondeterministic hypersubstitution to an algebra of type $\tau$ and obtain a class of derived algebras of type $\tau$. The generalized non-deterministic hypersubstitutions can be also applied to sets of equations of type $\tau$. We obtain two closure operators which turn out to be a conjugate pair of completely additive closure operators. This allows us to apply the theory of conjugate pairs of additive closure operators to characterize $M$-solid generalized non-deterministic varieties of algebras.


Keywords: generalized non-deterministic hypersubstitution, conjugate pair of additive closure operators, $M$-solid generalized non-deterministic variety.
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## 1. Introduction

Let $\tau=\left(n_{i}\right)_{i \in I}$ be a type of algebras, indexed by a set $I$. Let $\left(f_{i}\right)_{i \in I}$ be an indexed set of operation symbols where $f_{i}$ is $n_{i}$-ary and let $X:=\left\{x_{1}, \ldots, x_{n}, \ldots\right\}$ be a countably infinite set of variables, and for each $n \geq 1$ let $X_{n}:=\left\{x_{1}, \ldots, x_{n}\right\}$. Denoted by $W_{\tau}(X)$ and $W_{\tau}\left(X_{n}\right)$ the set of all the terms of type $\tau$ and of all $n$-ary terms of type $\tau$, respectively. These two sets can be use as the universe of the absolutely free algebras of type $\tau$

$$
\mathcal{F}_{\tau}(X):=\left(W_{\tau}(X) ;\left(\bar{f}_{i}\right)_{i \in I}\right) \text { and } \mathcal{F}_{\tau}\left(X_{n}\right):=\left(W_{\tau}\left(X_{n}\right) ;\left(\bar{f}_{i}\right)_{i \in I}\right)
$$

here the operations $\bar{f}_{i}$ are defined by setting $\bar{f}_{i}\left(t_{1}, \ldots, t_{n_{i}}\right):=f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)$.
In 2000, Leeratanavalee and Denecke [6] gave the concept of generalized superposition operation of terms

$$
S^{n}:\left(W_{\tau}(X)\right)^{n+1} \rightarrow W_{\tau}(X)
$$

by the following steps: For any term $t \in W_{\tau}(X)$
(i) If $t=x_{j}$ for $1 \leq j \leq n$, then $S^{n}\left(x_{j}, t_{1}, \ldots, t_{n}\right):=t_{j}$.
(ii) If $t=x_{j}$ for $n<j$, then $S^{n}\left(x_{j}, t_{1}, \ldots, t_{n}\right):=x_{j}$.
(iii) If $t=f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)$ and assume that $S^{n}\left(s_{q}, t_{1}, \ldots, t_{n_{i}}\right)$ for $1 \leq q \leq n_{i}$, are already defined, then

$$
S^{n}\left(f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right), t_{1}, \ldots, t_{n}\right)=f_{i}\left(S^{n}\left(s_{1}, t_{1}, \ldots, t_{n}\right), \ldots, S^{n}\left(s_{n_{i}}, t_{1}, \ldots, t_{n}\right)\right)
$$

In 2007, Denecke and Glubudom [2] studied solid non-deterministic varieties of algebras and characterized of $M$-solid non-deterministic varieties. They gave the concept of superposition operation of the set of terms

$$
\hat{S}_{m}^{n}: \mathcal{P}\left(W_{\tau}\left(X_{n}\right)\right) \times \mathcal{P}\left(W_{\tau}\left(X_{m}\right)\right)^{n} \rightarrow \mathcal{P}\left(W_{\tau}\left(X_{m}\right)\right)
$$

inductively by the following steps: Let $m, n \in \mathrm{~N}^{+}(:=\mathrm{N}-\{0\})$ and let $B \in$ $\mathcal{P}\left(W_{\tau}\left(X_{n}\right)\right)$ and $B_{1}, \ldots, B_{n} \in \mathcal{P}\left(W_{\tau}\left(X_{m}\right)\right)$ such that $B, B_{1}, \ldots, B_{n}$ are nonempty.
(i) If $B=\left\{x_{j}\right\}$ for $1 \leq j \leq n$, then $\hat{S}_{m}^{n}\left(\left\{x_{j}\right\}, B_{1}, \ldots, B_{n}\right):=B_{j}$.
(ii) If $B=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$ and assume that $\hat{S}_{m}^{n}\left(\left\{t_{q}\right\}, B_{1}, \ldots, B_{n}\right)$ for $1 \leq q \leq$ $n_{i}$, are already defined, then $\hat{S}_{m}^{n}\left(\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}, B_{1}, \ldots, B_{n}\right):=$

$$
\left\{f_{i}\left(r_{1}, \ldots, r_{n_{i}}\right) \mid r_{q} \in \hat{S}_{m}^{n}\left(\left\{t_{q}\right\}, B_{1}, \ldots, B_{n}\right)\right\}
$$

(iii) If $B$ is an arbitrary subset of $W_{\tau}\left(X_{n}\right)$, then

$$
\hat{S}_{m}^{n}\left(B, B_{1}, \ldots, B_{n}\right):=\bigcup_{b \in B} \hat{S}_{m}^{n}\left(\{b\}, B_{1}, \ldots, B_{n}\right) .
$$

If one of the sets $B, B_{1}, \ldots, B_{n}$ is empty, we define $\hat{S}_{m}^{n}\left(B, B_{1}, \ldots, B_{n}\right):=\emptyset$.
For the rest of this paper, if $X$ is a set we mean $\mathcal{P}(X)$ the set of all subsets of $X$. In this paper, we extend the concept of solid non-deterministic varieties which studied in [2] to solid generalized non-deterministic varieties and we also characterize $M$-solid generalized non-deterministic varieties.

## 2. Generalized superposition operations of the set of terms

In this section, we give the concept of generalized superposition operations of the 9 set of terms and study some properties of such superposition operations.

Definition. Let $B, B_{1}, \ldots, B_{n} \in \mathcal{P}\left(W_{\tau}(X)\right)$ be such that $B, B_{1}, \ldots, B_{n}$ are nonempty. Then the generalized superposition operation

$$
\hat{S}^{n}:\left(\mathcal{P}\left(W_{\tau}(X)\right)\right)^{n+1} \rightarrow \mathcal{P}\left(W_{\tau}(X)\right)
$$

is inductively defined as follows:
(i) If $B=\left\{x_{j}\right\}$ for $1 \leq j \leq n$, then $\hat{S}^{n}\left(\left\{x_{j}\right\}, B_{1}, \ldots, B_{n}\right):=B_{j}$.
(ii) If $B=\left\{x_{j}\right\}$ for $n<j$, then $\hat{S}^{n}\left(\left\{x_{j}\right\}, B_{1}, \ldots, B_{n}\right):=\left\{x_{j}\right\}$.
(iii) If $B=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$ and assume that $\hat{S}^{n}\left(\left\{t_{q}\right\}, B_{1}, \ldots, B_{n}\right)$ for $1 \leq q \leq$ $n_{i}$, are already defined, then $\hat{S}^{n}\left(\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}, B_{1}, \ldots, B_{n}\right):=$

$$
\left\{f_{i}\left(r_{1}, \ldots, r_{n_{i}}\right) \mid r_{q} \in \hat{S}^{n}\left(\left\{t_{q}\right\}, B_{1}, \ldots, B_{n}\right)\right\}
$$

(iv) If $B$ is an arbitrary subset of $W_{\tau}\left(X_{n}\right)$, then

$$
\hat{S}^{n}\left(B, B_{1}, \ldots, B_{n}\right):=\bigcup_{b \in B} \hat{S}^{n}\left(\{b\}, B_{1}, \ldots, B_{n}\right) .
$$

If one of the sets $B, B_{1}, \ldots, B_{n}$ is empty, we define $\hat{S}^{n}\left(B, B_{1}, \ldots, B_{n}\right):=\emptyset$.
Our next aim is to prove that the generalized superation operation $\hat{S}^{n}$ satisfy the superassociative law but we need the following lemma.

Lemma 1. Let $S$ be a subset of $W_{\tau}(X)$. Then
$\hat{S}^{n}\left(\bigcup_{s \in S} \hat{S}^{n}\left(\{s\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)=\bigcup_{s \in S} \hat{S}^{n}\left(\hat{S}^{n}\left(\{s\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$.
Proof. Let $x \in \hat{S}^{n}\left(\bigcup_{s \in S} \hat{S}^{n}\left(\{s\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$\Leftrightarrow x \in \hat{S}^{n}\left(\hat{S}^{n}\left(\{u\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$ for some $u \in S$
$\Leftrightarrow x \in \bigcup_{s \in S} \hat{S}^{n}\left(\hat{S}^{n}\left(\{s\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$.
Lemma 2. Let $S$ be a subset of $W_{\tau}(X)$. Then

$$
\hat{S}^{n}\left(S, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)
$$

$$
=\hat{S}^{n}\left(\hat{S}^{n}\left(S, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)
$$

Proof. If $S$ is empty, the claim is clearly true.
(1) If $S$ is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of $S$.
(1.1) If $S=\left\{x_{j}\right\}$ for $1 \leq j \leq n$, then
$\hat{S}^{n}\left(\hat{S}^{n}\left(S, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(\hat{S}^{n}\left(\left\{x_{j}\right\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(L_{j}, T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(\left\{x_{j}\right\}, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)$
$=\hat{S}^{n}\left(S, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)$.
(1.2) If $S=\left\{x_{j}\right\}$ for $n<j$, then
$\hat{S}^{n}\left(\hat{S}^{n}\left(S, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(\hat{S}^{n}\left(\left\{x_{j}\right\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(\left\{x_{j}\right\}, T_{1}, \ldots, T_{n}\right)$
$=\left\{x_{j}\right\}$
$=\hat{S}^{n}\left(\left\{x_{j}\right\}, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)$
$=\hat{S}^{n}\left(S, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)$.
(1.3) If $S=\left\{f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)\right\}$ and we assume that the equations
$\hat{S}^{n}\left(\hat{S}^{n}\left(\left\{s_{q}\right\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(\left\{s_{q}\right\}, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)$
where $1 \leq q \leq n_{i}$, then

$$
\hat{S}^{n}\left(\hat{S}^{n}\left(S, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)
$$

$$
=\hat{S}^{n}\left(\hat{S}^{n}\left(\left\{f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)\right\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)
$$

$$
=\hat{S}^{n}\left(\left\{f_{i}\left(u_{1}, \ldots, u_{n_{i}}\right) \mid u_{q} \in \hat{S}^{n}\left(\left\{s_{q}\right\}, L_{1}, \ldots, L_{n}\right)\right\}, T_{1}, \ldots, T_{n}\right)
$$

$$
=\left\{f_{i}\left(r_{1}, \ldots, r_{n_{i}}\right) \mid r_{q} \in \hat{S}^{n}\left(\left\{u_{q} \mid u_{q} \in \hat{S}^{n}\left(\left\{s_{q}\right\}, L_{1}, \ldots, L_{n}\right)\right\}\right.\right.
$$

$$
\left.\left.T_{1}, \ldots, T_{n}\right)\right\}
$$

$$
=\left\{f_{i}\left(r_{1}, \ldots, r_{n_{i}}\right) \mid r_{q} \in \hat{S}^{n}\left(\hat{S}^{n}\left(\left\{s_{q}\right\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)\right\}
$$

$$
=\left\{f_{i}\left(r_{1}, \ldots, r_{n_{i}}\right) \mid r_{q} \in \hat{S}^{n}\left(\left\{s_{q}\right\}, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right), \ldots\right.\right.
$$

$$
\left.\left.\hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right)\right)\right\}
$$

$$
=\hat{S}^{n}\left(\left\{f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)\right\}, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)
$$

$$
=\hat{S}^{n}\left(S, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)
$$

(2) If $S$ is arbitrary subset of $W_{\tau}(X)$, then
$\hat{S}^{n}\left(\hat{S}^{n}\left(S, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\hat{S}^{n}\left(\bigcup_{s \in S} \hat{S}^{n}\left(\{s\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$
$=\bigcup_{s \in S} \hat{S}^{n}\left(\hat{S}^{n}\left(\{s\}, L_{1}, \ldots, L_{n}\right), T_{1}, \ldots, T_{n}\right)$

$$
\begin{aligned}
& =\bigcup_{s \in S} \hat{S}^{n}\left(\{s\}, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right) \\
& =\hat{S}^{n}\left(S, \hat{S}^{n}\left(L_{1}, T_{1}, \ldots, T_{n}\right), \ldots, \hat{S}^{n}\left(L_{n}, T_{1}, \ldots, T_{n}\right)\right)
\end{aligned}
$$

The following identity is also satisfied.
Lemma 3. Let $T$ be a subset of $W_{\tau}(X)$. Then

$$
\hat{S}^{n}\left(T,\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right)=T
$$

Proof. If $T$ is empty, the claim is clearly true.
(1) If $T$ is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of $T$.
(1.1) If $T=\left\{x_{j}\right\}$ for $1 \leq j \leq n$, then

$$
\begin{aligned}
\hat{S}^{n}\left(T,\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) & =\hat{S}^{n}\left(\left\{x_{j}\right\},\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) \\
& =\left\{x_{j}\right\} \\
& =T .
\end{aligned}
$$

(1.2) If $T=\left\{x_{j}\right\}$ for $n<j$, then

$$
\begin{aligned}
\hat{S}^{n}\left(T,\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) & =\hat{S}^{n}\left(\left\{x_{j}\right\},\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) \\
& =\left\{x_{j}\right\} \\
& =T .
\end{aligned}
$$

(1.3) If $T=\left\{f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)\right\}$ and we assume that the equations

$$
\hat{S}^{n}\left(\left\{s_{q}\right\},\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right)=\left\{s_{q}\right\}
$$

where $1 \leq q \leq n_{i}$, then

$$
\begin{aligned}
\hat{S}^{n}\left(T,\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right)= & \hat{S}^{n}\left(\left\{f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)\right\},\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) \\
= & \left\{f_{i}\left(u_{1}, \ldots, u_{n_{i}}\right) \mid u_{q} \in \hat{S}^{n}\left(\left\{s_{q}\right\},\left\{x_{1}\right\}, \ldots,\right.\right. \\
& \left.\left.\left\{x_{n}\right\}\right)\right\} \\
= & \left\{f_{i}\left(u_{1}, \ldots, u_{n_{i}}\right) \mid u_{q} \in\left\{s_{q}\right\}\right\} \\
& =\left\{f_{i}\left(s_{1}, \ldots, s_{n_{i}}\right)\right\} \\
= & T .
\end{aligned}
$$

(2) If $T$ is arbitrary subset of $W_{\tau}(X)$, then

$$
\begin{aligned}
\hat{S}^{n}\left(T,\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) & =\bigcup_{t \in T} \hat{S}^{n}\left(\{t\},\left\{x_{1}\right\}, \ldots,\left\{x_{n}\right\}\right) \\
& =\{t \mid t \in T\} \\
& =T .
\end{aligned}
$$

Using the generalized superposition operation $\hat{S}^{n}$ we can form the new algebra

$$
\left(\mathcal{P}\left(W_{\tau}(X)\right) ; \hat{S}^{n}\right)
$$

We call this algebra the $g$-power-clone.

## 3. GENERALIZED NON-DETERMINISTIC HYPERSUBSTITUTIONS

Hypersubstitutions for terms over one-sorted algebras were introduced by Graczyńska and Schweigert ([5]). Our definitions and the properties of generalized superposition operations can be used to define generalized non-deterministic hypersubstitutions and their extensions.

Definition. A generalized non-deterministic hypersubstitution (for short gndhypersubstitution) of type $\tau$ is a mapping from the set $\left\{f_{i} \mid i \in I\right\}$ of $n_{i}$-ary operation symbols of type $\tau$ to the set $\mathcal{P}\left(W_{\tau}(X)\right)$.

We denote by $H y p_{G}^{n d}(\tau)$ the set of all $g n d$-hypersubstitutions of type $\tau$. Any $g n d$-hypersubstitution $\sigma$ induces a mapping $\hat{\sigma}$ defined on the set $\mathcal{P}\left(W_{\tau}(X)\right)$, as follows.

Definition. Let $\sigma$ be a $g n d$-hypersubstitution of type $\tau$. Then $\sigma$ induces a mapping $\hat{\sigma}: \mathcal{P}\left(W_{\tau}(X)\right) \rightarrow \mathcal{P}\left(W_{\tau}(X)\right)$, by setting
(i) $\hat{\sigma}[\emptyset]:=\emptyset$.
(ii) $\hat{\sigma}\left[\left\{x_{j}\right\}\right]:=\left\{x_{j}\right\}$ for every $x_{j} \in X$.
(iii) $\hat{\sigma}\left[\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}\right]=\hat{S}^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[\left\{t_{1}\right\}\right], \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]\right)$ if we inductively assume that $\hat{\sigma}\left[\left\{t_{q}\right\}\right]$ where $1 \leq q \leq n_{i}$, are already defined.
(iv) $\hat{\sigma}[B]:=\bigcup_{b \in B} \hat{\sigma}[\{b\}]$ for $B \subseteq W_{\tau}(X)$.

Then we have:
Lemma 4. Let $\sigma \in H y p_{G}^{n d}(\tau)$. Then $\hat{\sigma}$ is an endomorphism on the $g$-power-clone.
Proof. We have to shows that

$$
\hat{\sigma}\left[S^{n}\left(T, S_{1}, \ldots, S_{n}\right)\right]=S^{n}\left(\hat{\sigma}[T], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right)
$$

If $T$ is empty, then the claim is clearly true.
(1) If $T$ is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of $T$.
(1.1) If $T=\left\{x_{j}\right\}$ for $1 \leq j \leq n$, then

$$
\begin{aligned}
\hat{\sigma}\left[S^{n}\left(T, S_{1}, \ldots, S_{n}\right)\right] & =\hat{\sigma}\left[S^{n}\left(\left\{x_{j}\right\}, S_{1}, \ldots, S_{n}\right)\right] \\
& =\hat{\sigma}\left[S_{j}\right] \\
& =\hat{S}^{n}\left(\left\{x_{j}\right\}, \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) \\
& =\hat{S}^{n}\left(\hat{\sigma}\left[\left\{x_{j}\right\}\right], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) \\
& =\hat{S}^{n}\left(\hat{\sigma}[T], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) .
\end{aligned}
$$

(1.2) If $T=\left\{x_{j}\right\}$ for $n<j$, then

$$
\begin{aligned}
\hat{\sigma}\left[\hat{S}^{n}\left(T, S_{1}, \ldots, S_{n}\right)\right] & =\hat{\sigma}\left[S^{n}\left(\left\{x_{j}\right\}, S_{1}, \ldots, S_{n}\right)\right] \\
& =\hat{\sigma}\left[\left\{x_{j}\right\}\right] \\
& =\hat{S}^{n}\left(\left\{x_{j}\right\}, \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) \\
& =\hat{S}^{n}\left(\hat{\sigma}\left[\left\{x_{j}\right\}\right], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) \\
& =\hat{S}^{n}\left(\hat{\sigma}[T], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) .
\end{aligned}
$$

(1.3) If $T=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$ and we assume that the equations

$$
\hat{\sigma}\left[\hat{S}^{n}\left(\left\{t_{q}\right\}, S_{1}, \ldots, S_{n}\right)\right]=\hat{S}^{n}\left(\hat{\sigma}\left[\left\{t_{q}\right\}\right], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right)
$$

where $1 \leq q \leq n_{i}$, then
$\hat{\sigma}\left[\hat{S}^{n}\left(T, S_{1}, \ldots, S_{n}\right)\right]$
$=\hat{\sigma}\left[\hat{S}^{n}\left(\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}, S_{1}, \ldots, S_{n}\right)\right]$
$=\hat{\sigma}\left[\left\{f_{i}\left(u_{1}, \ldots, u_{n_{i}}\right) \mid u_{q} \in \hat{S}^{n}\left(\left\{t_{q}\right\}, S_{1}, \ldots, S_{n}\right)\right\}\right]$
$=\hat{S}^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[\left\{u_{1} \mid u_{1} \in \hat{S}^{n}\left(\left\{t_{1}\right\}, S_{1}, \ldots, S_{n}\right)\right\}\right], \ldots\right.$,
$\left.\hat{\sigma}\left[\left\{u_{n_{i}} \mid u_{n_{i}} \in \hat{S}^{n}\left(\left\{t_{n_{i}}\right\}, S_{1}, \ldots, S_{n}\right)\right\}\right]\right)$
$=\hat{S}^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[\hat{S}^{n}\left(\left\{t_{1}\right\}, S_{1}, \ldots, S_{n}\right)\right], \ldots, \hat{\sigma}\left[\hat{S}^{n}\left(\left\{t_{n_{i}}\right\}, S_{1}, \ldots, S_{n}\right)\right]\right)$
$=\hat{S}^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{S}^{n}\left(\hat{\sigma}\left[\left\{t_{1}\right\}\right], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right), \ldots, \hat{S}^{n}\left(\hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right], \hat{\sigma}\left[S_{1}\right], \ldots\right.\right.$,
$\left.\left.\left.\hat{\sigma}\left[S_{n}\right)\right]\right)\right)$
$\left.=\hat{S}^{n}\left(\hat{S}^{n_{i}}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[\left\{t_{1}\right\}\right], \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]\right), \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right)\right]\right)$
$\left.=\hat{S}^{n}\left(\hat{\sigma}\left[\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}\right], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right)\right]\right)$
$\left.=\hat{S}^{n}\left(\hat{\sigma}[T], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right)\right]\right)$.
(2) If $T$ is arbitrary subset of $W_{\tau}(X)$, then

$$
\begin{aligned}
\hat{\sigma}\left[S^{n}\left(T, S_{1}, \ldots, S_{n}\right)\right] & =\bigcup_{t \in T} \hat{\sigma}\left[S^{n}\left(\{t\}, S_{1}, \ldots, S_{n}\right)\right] \\
& =\bigcup_{t \in T} \hat{S}^{n}\left(\hat{\sigma}[\{t\}], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) \\
& =\hat{S}^{n}\left(\bigcup_{t \in T} \hat{\sigma}[\{t\}], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right) \\
& =\hat{S}^{n}\left(\hat{\sigma}[T], \hat{\sigma}\left[S_{1}\right], \ldots, \hat{\sigma}\left[S_{n}\right]\right)
\end{aligned}
$$

Let $\sigma_{1}, \sigma_{2}$ be elements in $H y p_{G}^{n d}$. Since the extension of $g n d$-hypersubstitution maps $\mathcal{P}\left(W_{\tau}(X)\right)$ to $\mathcal{P}\left(W_{\tau}(X)\right)$ we may define a product $\sigma_{1} \circ_{g} \sigma_{2}$ by

$$
\sigma_{1} \circ_{g} \sigma_{2}:=\hat{\sigma}_{1} \circ \sigma_{2} .
$$

Here $\circ$ is the usual composition of mappings. Since $\hat{\sigma}_{1} \circ \sigma_{2}$ maps $\left\{f_{i} \mid i \in I\right\}$ to $\mathcal{P}\left(W_{\tau}(X)\right)$, it is a $g n d$-hypersubstitution of type $\tau$. The following lemmas shows that the extension of this product is the product of the extensions of $\sigma_{1}$ and $\sigma_{2}$.

Lemma 5. Let $\sigma_{1}, \sigma_{2}$ be elements in $H y p_{G}^{n d}$. Then we have

$$
\left(\sigma_{1} \circ_{g} \sigma_{2}\right)^{\kappa}=\hat{\sigma}_{1} \circ \hat{\sigma}_{2} .
$$

Proof. We will show that $\left(\sigma_{1} \circ_{g} \sigma_{2}\right)[T]=\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)[T]$ for all $T \subseteq W_{\tau}(X)$.
If $T$ is empty, then the claim is clearly true.
(1) If $T$ is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of $T$.
(1.1) If $T=\left\{x_{j}\right\}$, then

$$
\begin{aligned}
\left(\sigma_{1} \circ_{g} \sigma_{2}\right)[T] & =\left(\sigma_{1} \circ_{g} \sigma_{2}\right)\left[\left\{x_{j}\right\}\right] \\
& =\left\{x_{j}\right\} \\
& =\hat{\sigma}_{1}\left[\left\{x_{j}\right\}\right] \\
& =\hat{\sigma}_{1}\left[\hat{\sigma}_{2}\left[\left\{x_{j}\right\}\right]\right] \\
& =\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)\left[\left\{x_{j}\right\}\right] \\
& =\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)[T] .
\end{aligned}
$$

(1.2) If $T=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$ and we assume that the equations

$$
\left(\sigma_{1} \circ_{g} \sigma_{2}\right)\left[\left\{t_{q}\right\}\right]=\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)\left[\left\{t_{q}\right\}\right]
$$

where $1 \leq q \leq n_{i}$, then

$$
\begin{aligned}
\left(\sigma_{1} \circ_{g} \sigma_{2}\right)[T] & =\left(\sigma_{1} \circ_{g} \sigma_{2}\right)\left[f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right] \\
& =\hat{S}^{n_{i}}\left(\left(\sigma_{1} \circ_{g} \sigma_{2}\right)\left(f_{i}\right),\left(\sigma_{1} \circ_{g} \sigma_{2}\right)\left[t_{1}\right], \ldots,\left(\sigma_{1} \circ \circ_{g} \sigma_{2}\right)\left[t_{n_{i}}\right]\right) \\
& =\hat{S}^{n_{i}}\left(\left(\hat{\sigma}_{1} \circ \sigma_{2}\right)\left(f_{i}\right),\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)\left[\left\{t_{1}\right\}\right], \ldots,\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)\left[\left[t_{n_{i}}\right\}\right]\right) \\
& =\hat{S}^{n_{i}}\left(\left(\hat{\sigma}_{1}\left[\sigma_{2}\left(f_{i}\right)\right], \hat{\sigma}_{1}\left[\hat{\sigma}_{2}\left[\left\{t_{1}\right\}\right]\right], \ldots, \hat{\sigma}_{1}\left[\hat{\sigma}_{2}\left[\left\{t_{n_{i}}\right\}\right]\right]\right)\right. \\
& \left.=\hat{\sigma}_{1}\left[\hat{S}_{i}^{n_{i}}\left(\sigma_{2}\left(f_{i}\right), \hat{\sigma}_{2}\left[\left\{t_{1}\right\}\right]\right], \ldots, \hat{\sigma}_{2}\left[\left\{t_{n_{i}}\right\}\right]\right)\right] \\
& \left.=\hat{\sigma}_{1} \hat{\sigma}_{2}\left[\left\{f_{2}\left(t_{1}, \ldots, t_{1}\right)\right\}\right]\right] \\
& =\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)\left[\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}\right] \\
& =\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)[T] .
\end{aligned}
$$

(2) If $T$ is arbitrary subset of $W_{\tau}(X)$, then

$$
\begin{aligned}
\left(\sigma_{1} \circ_{g} \sigma_{2}\right)[T] & =\bigcup_{t \in T}\left(\sigma_{1} \circ_{g} \sigma_{2}\right)\{\{t\}] \\
& =\bigcup_{t \in T}\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)[\{t\}] \\
& =\bigcup_{t \in T} \hat{\sigma}_{1}\left[\hat{\sigma}_{2}[\{t\}]\right] \\
& =\hat{\sigma}_{1}\left[\bigcup \hat{\sigma}_{2}[\{t\}]\right] \\
& =\hat{\sigma}_{1}\left[\hat{\sigma}_{2}[T]\right] \\
& =\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right)[T] .
\end{aligned}
$$

From Lemma 5, we get that the products $\circ_{g}$ are associative. In fact, for each $\sigma_{1}, \sigma_{2}$ and $\sigma_{3}$ in $H y p_{G}^{n d}(\tau)$, we have

$$
\begin{aligned}
\left(\sigma_{1} \circ_{g} \sigma_{2}\right) \circ_{g} \sigma_{3} & =\left(\sigma_{1} \circ_{g} \sigma_{2}\right) \circ \sigma_{3} \\
& =\left(\hat{\sigma}_{1} \circ \hat{\sigma}_{2}\right) \circ \sigma_{3} \\
& =\hat{\sigma}_{1} \circ\left(\hat{\sigma}_{2} \circ \sigma_{3}\right) \\
& =\hat{\sigma}_{1} \circ\left(\sigma_{2} \circ_{g} \sigma_{3}\right)^{r} \\
& =\sigma_{1} \circ_{g}\left(\sigma_{2} \circ_{g} \sigma_{3}\right) .
\end{aligned}
$$

This shows that the set of all gnd-hypersubstitutions forms a semigroup with respect to the associative binary operation $\circ_{g}$.

If we consider $\sigma_{g i d}$ is an element in $\operatorname{Hyp}_{G}^{n d}(\tau)$ defined by

$$
\sigma_{g i d}\left(f_{i}\right):=\left\{f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right)\right\}
$$

then it is an identity element with respect to $\circ_{g}$ (see [2]).
Theorem 6. The algebra $\left(H_{y p}^{n d}(\tau) ; \circ_{g}, \sigma_{g i d}\right)$ forms a monoid.

## 4. Solid generalized non-Deterministic varieties

In this section we want to apply the theory of conjugate pairs of additive closure operators to algebras and identities and want to define generalized nondeterministic hyperidentities and solid generalized non-deterministic varieties of algebras.

Definition. Let $\mathcal{A}=\left(A ;\left(f_{i}^{\mathcal{A}}\right)_{i \in I}\right)$ be an algebra, and $B \in \mathcal{P}\left(W_{\tau}(X)\right)$. Then we define the set $B^{\mathcal{A}}$ of terms operations on $\mathcal{A}$ induced by terms as follows:
(1) If $B=\left\{x_{j}\right\}$, then $B^{\mathcal{A}}:=\left\{x_{j}^{\mathcal{A}}\right\}$.
(2) If $B=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$, then $B^{\mathcal{A}}:=\left\{f_{i}^{\mathcal{A}}\left(t_{1}^{\mathcal{A}}, \ldots, t_{n_{i}}^{\mathcal{A}}\right)\right\}$ where $f_{i}^{\mathcal{A}}$ is the fundamental operation of $\mathcal{A}$ corresponding to the operation symbol $f_{i}$ and where $t_{q}^{\mathcal{A}}$ are the term operations on $\mathcal{A}$ which are induced in the usual way by the $t_{q}$ 's for $1 \leq q \leq n_{i}$.
(3) If $B$ is an arbitrary subset of $W_{\tau}(X)$, then we define $B^{\mathcal{A}}:=\bigcup_{b \in B}\{b\}^{\mathcal{A}}$.

If $B$ is empty, then we define $B^{\mathcal{A}}:=\emptyset$.
Definition. Let $\mathcal{A}$ be an algebra and let $T, T_{j} \in \mathcal{P}\left(W_{\tau}(X)\right)$ for $1 \leq j \leq n$ such that $T, T_{j}$ are non-empty. Then the superposition operation

$$
\hat{S}^{n, A}:\left(\mathcal{P}\left(W_{\tau}(X)\right)^{A}\right)^{n+1} \rightarrow \mathcal{P}\left(W_{\tau}(X)\right)^{A},
$$

is inductively defined in the following way:
(1) If $T=\left\{x_{j}\right\}$ for $1 \leq j \leq n$, then

$$
\hat{S}^{n, A}\left(\left\{x_{j}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right):=T_{j}^{\mathcal{A}}
$$

(2) If $T=\left\{x_{j}\right\}$ for $n<j$, then

$$
\hat{S}^{n, A}\left(\left\{x_{j}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right):=\left\{x_{j}\right\}^{\mathcal{A}} .
$$

(3) If $T=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$ and we assume that $S^{n, A}\left(\left\{t_{q}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)$ where $1 \leq q \leq n_{i}$, are already defined, then

$$
\begin{aligned}
& \hat{S}^{n, A}\left(\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}_{n}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right) \\
& :=\left\{f_{i}^{\mathcal{A}}\left(r_{1}^{\mathcal{A}}, \ldots, r_{n_{i}}^{\mathcal{A}}\right) \mid r_{q}^{\mathcal{A}} \in \hat{S}^{n, A}\left(\left\{t_{q}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)\right\} .
\end{aligned}
$$

(4) If $T$ is an arbitrary subset of $W_{\tau}(X)$, then

$$
\hat{S}^{n, A}\left(T^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right):=\bigcup_{t \in T} \hat{S}^{n, A}\left(\{t\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)
$$

Lemma 7. Let $\mathcal{A}$ be an agebra of type $\tau$ and let $T$ be a subset of $W_{\tau}(X)$. Then

$$
\left(\hat{S}^{n}\left(T, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}}=\hat{S}^{n, A}\left(T^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right) .
$$

Proof. If $T$ is empty, then all is clear.
(1) If $T$ is a one-element set, then we will give a proof by induction on the complexity of term which is the only element of the one-element set $T$.
(1.1) If $T=\left\{x_{j}\right\}$ where $1 \leq j \leq n$, then

$$
\begin{aligned}
\left(\hat{S}^{n}\left(T, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} & =\left(\hat{S}^{n}\left(\left\{x_{j}\right\}, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} \\
& =T_{j}^{\mathcal{A}} \\
& =\widehat{S}^{n, A}\left(\left\{x_{j}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right) .
\end{aligned}
$$

(1.2) If $T=\left\{x_{j}\right\}$ where $n<j$, then

$$
\begin{aligned}
\left(\hat{S}^{n}\left(T, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} & =\left(\hat{S}^{n}\left(\left\{x_{j}\right\}, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} \\
& =\left\{x_{j}\right\}^{\mathcal{A}} \\
& =\hat{S}^{n, A}\left(\left\{x_{j}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)
\end{aligned}
$$

(1.3) If $T=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$ and if we assume that the equations

$$
\left(\hat{S}^{n}\left(\left\{s_{q}\right\}, T_{1}, \ldots, T_{n}\right)^{\mathcal{A}}=\hat{S}^{n, A}\left(\left\{s_{q}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)\right.
$$

where $1 \leq q \leq n_{i}$, are satisfied, then

$$
\left(\hat{S}^{n}\left(T, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}}
$$

$$
=\left(\hat{S}^{n}\left(\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}}
$$

$$
=\left(\left\{f_{i}\left(r_{1}, \ldots, r_{n_{i}}\right) \mid r_{q} \in \hat{S}^{n}\left(\left\{t_{q}\right\}, T_{1}, \ldots, T_{n}\right)\right\}\right)^{\mathcal{A}}
$$

$$
=\left\{f_{i}^{\mathcal{A}}\left(r_{1}^{\mathcal{A}}, \ldots, r_{n_{i}}^{\mathcal{A}}\right) \mid r_{q}^{\mathcal{A}} \in\left(\hat{S}^{n}\left(\left\{t_{q}\right\}, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}}\right\}
$$

$$
=\left\{f_{i}^{\mathcal{A}}\left(r_{1}^{\mathcal{A}}, \ldots, r_{n_{i}}^{\mathcal{A}}\right) \mid r_{q}^{\mathcal{A}} \in \hat{S}^{n, A}\left(\left\{t_{q}\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)\right\}
$$

$$
\left.=\hat{S}^{n, A}\left(\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right\}\right)
$$

$$
=\hat{S}^{n, A}\left(T^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)
$$

(2) If $T$ is an arbitrary subset of $W_{\tau}(X)$, then

$$
\begin{aligned}
\left(\hat{S}^{n}\left(T, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} & =\left(\bigcup_{t \in T} \hat{S}^{n}\left(\{t\}, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} \\
& =\bigcup_{t \in T}\left(\hat{S}^{n}\left(\{t\}, T_{1}, \ldots, T_{n}\right)\right)^{\mathcal{A}} \\
& =\bigcup_{t \in T} \hat{S}^{n, A}\left(\{t\}^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right) \\
& =\widehat{S}^{n, A}\left(T^{\mathcal{A}}, T_{1}^{\mathcal{A}}, \ldots, T_{n}^{\mathcal{A}}\right)
\end{aligned}
$$

Let $B$ be a subset of $W_{\tau}(X)$ and let $K$ be a subset of $\operatorname{Alg}(\tau)$. Then we set

$$
B^{K}:=\bigcup_{\mathcal{A} \in K} B^{\mathcal{A}}
$$

Definition. Let $\mathcal{A}=\left(A ;\left(f_{i}^{\mathcal{A}}\right)_{i \in I}\right)$ be an algebra of type $\tau$ and $\left(B_{1}, B_{2}\right) \in$ $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$, written as $B_{1} \approx B_{2}$. Then $\mathcal{A} \vDash B_{1} \approx B_{2}$ iff $B_{1}^{\mathcal{A}}=B_{2}^{\mathcal{A}}$ and $K \models B_{1} \approx B_{2}$ iff $B_{1}^{K}=B_{2}^{K}$ where $K \subseteq \operatorname{Alg}(\tau)$.

Let $\mathcal{K}$ be a subset of $\mathcal{P}(A l g(\tau))$. Then

$$
\mathcal{K} \mid=B_{1} \approx B_{2} \text { iff } \forall K \in \mathcal{K}\left(K \models B_{1} \approx B_{2}\right)
$$

Let $P L$ be a subset of $\mathcal{P}\left(W_{\tau}(X)\right)^{2}=\mathcal{P}\left(W_{\tau}(X) \times W_{\tau}(X)\right)$. Then $\mathcal{K} \models P L$ iff $K \models B_{1} \approx B_{2}$ for all $K \in \mathcal{K}$ and for all $B_{1} \approx B_{2} \in P L$.

Let $\mathcal{K}, P L$ be a subset of $\mathcal{P}\left(W_{\tau}(X)\right)$, a subset of $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$, respectively. Then we define a mapping PId: $\mathcal{P}(\mathcal{P}(\operatorname{Alg}(\tau))) \rightarrow \mathcal{P}\left(\mathcal{P}\left(W_{\tau}(X)\right)^{2}\right)$ by

$$
\operatorname{PId}(\mathcal{K}):=\left\{B_{1} \approx B_{2} \in \mathcal{P}\left(W_{\tau}(X)\right)^{2} \mid \forall K \in \mathcal{K}\left(K \models B_{1} \approx B_{2}\right)\right\},
$$

and a mapping PMod: $\mathcal{P}\left(\mathcal{P}\left(W_{\tau}(X)\right)^{2}\right) \rightarrow \mathcal{P}(\mathcal{P}(\operatorname{Alg}(\tau)))$ by

$$
\operatorname{PMod}(P L):=\left\{K \in \mathcal{P}(\operatorname{Alg}(\tau)) \mid \forall B_{1} \approx B_{2} \in P L\left(K \models B_{1} \approx B_{2}\right)\right\} .
$$

In the next lemmas we will show that these two mapping satisfy the Galoisconection properties.

Lemma 8. Let $\mathcal{P}(\operatorname{Alg}(\tau))$ be the class of all subsets of $\operatorname{Alg}(\tau)$ and let $\mathcal{K}, \mathcal{K}_{1}, \mathcal{K}_{2} \in$ $\mathcal{P}(\operatorname{Alg}(\tau))$. Then
(1) If $\mathcal{K}_{1} \subseteq \mathcal{K}_{2}$, then $\operatorname{PId} \mathcal{K}_{2} \subseteq$ PIdK $_{1}$.
(2) $\mathcal{K} \subseteq$ PModPIdK.

Proof. (1) Assume that $\mathcal{K}_{1} \subseteq \mathcal{K}_{2}$ and let $B_{1} \approx B_{2} \in \operatorname{PId}_{2}$. Then for all $K \in \mathcal{K}_{2}, K \models B_{1} \approx B_{2}$, but we have $\mathcal{K}_{1} \subseteq \mathcal{K}_{2}$, so that $K \models B_{1} \approx B_{2}$ for all $K \in \mathcal{K}_{1}$. It follows that $B_{1} \approx B_{2} \in P I d \mathcal{K}_{1}$, and then $\operatorname{PId}_{2} \subseteq P I d \mathcal{K}_{1}$.
(2) Let $K \in \mathcal{K}$. Then $K \models P I d \mathcal{K}$, means that $K \in P M o d P I d \mathcal{K}$, and then $\mathcal{K} \subseteq P M o d P I d \mathcal{K}$.

In the similarly method, we have
Lemma 9. Let $\mathcal{P}\left(W_{\tau}(X)\right)$ be the set of all subsets of $W_{\tau}(X)$ and let $P L, P L_{1}, P L_{2}$ be subsets of $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$. Then
(1) If $P L_{1} \subseteq P L_{2}$, then $P M o d P L_{2} \subseteq P M o d P L_{1}$.
(2) $P L \subseteq P I d P M o d P L$.

From both lemmas we have that ( $P M o d, P I d$ ) is a Galois connection between $\mathcal{P}(\operatorname{Alg}(\tau))$ and $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$ with respect to the relation

$$
R_{\models}:=\left\{\left(K, B_{1} \approx B_{2}\right) \in \mathcal{P}(\operatorname{Alg}(\tau)) \times \mathcal{P}\left(W_{\tau}(X)\right)^{2} \mid K \models B_{1} \approx B_{2}\right\} .
$$

We have two closure operators PModPId and PIdPMod and their sets

$$
\left\{P L \subseteq \mathcal{P}\left(W_{\tau}(X)\right)^{2} \mid P I d P M o d P L=P L\right\}
$$

and

$$
\{\mathcal{K} \subseteq \mathcal{P}(\operatorname{Alg}(\tau)) \mid P M o d P I d \mathcal{K}=\mathcal{K}\}
$$

form two complete lattices $\mathcal{E}$ and $\mathcal{L}$, respectively.
If $\mathcal{A}=\left(A ;\left(f_{i}^{\mathcal{A}}\right)_{i \in I}\right)$ is an algebra of type $\tau$ and if $\sigma \in \operatorname{Hyp}_{G}^{n d}(\tau)$ is $g n d$ hypersubstitution, then we define

$$
\sigma(\mathcal{A}):=\left\{A ;\left(l_{i}^{\sigma(\mathcal{A})}\right)_{i \in I} \mid l_{i} \in \sigma\left(f_{i}\right)\right\} .
$$

The set $\sigma(\mathcal{A})$ is called the set of derived algebras. Since for every sequence $\left(l_{i}\right)_{i \in I}$ of terms there is a generalized non-deterministic hypersubstitution mapping $f_{i}$ to $l_{i}$, we can write $\sigma(\mathcal{A})$ also in the form

$$
\sigma(\mathcal{A}):=\left\{\rho(\mathcal{A}) \mid \rho \in \operatorname{Hyp}_{G}(\tau) \text { with } \rho\left(f_{i}\right)^{\mathcal{A}} \in \sigma\left(f_{i}\right)^{\mathcal{A}} \text { for } i \in I\right\} .
$$

For a class $K$ of algebras of type $\tau$ we define

$$
\sigma(K):=\bigcup_{\mathcal{A} \in K} \sigma(\mathcal{A}) .
$$

Definition. Let $B \in \mathcal{P}\left(W_{\tau}(X)\right)$, let $\mathcal{A}=\left(A ;\left(f_{i}^{\mathcal{A}}\right)_{i \in I}\right)$ is an algebra of type $\tau$. Let $\sigma \in H y p_{G}^{n d}(\tau)$ be $g n d$-hypersubstitution and let $\sigma(\mathcal{A})$ be the set of derived algebras. Then we define the set $B^{\sigma(\mathcal{A})}$ of term operations induced by the set $\sigma(\mathcal{A})$ of derived algebras as follows:
(1) If $B=\left\{x_{j} \mid x_{j} \in X_{n}\right\}$, then

$$
B^{\sigma(\mathcal{A})}:=\left\{x_{j}^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A}) \text { and } \rho \in \operatorname{Hyp}_{G}(\tau)\right\}=\left\{e_{j}^{n, A}\right\}
$$

where $e_{j}^{n, A}:\left(a_{1}, \ldots, a_{n}\right) \mapsto a_{j}$ is an $n$-ary projection onto the $j$-th coordinate.
(2) If $B=\left\{x_{j} \mid x_{j} \in X \backslash X_{n}\right\}$, then

$$
B^{\sigma(\mathcal{A})}:=\left\{x_{j}^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A}) \text { and } \rho \in \operatorname{Hyp}_{G}(\tau)\right\}=\left\{c_{a}^{n}\right\},
$$

where $c_{a}^{n}$ is the $n$-ary constant operation on $A$ with value $a$ and each element from $A$ is uniquely by an element from $X \backslash X_{n}$.
(3) If $B=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$, and if we assume that $\left\{t_{q}\right\}^{\sigma(\mathcal{A})}$ where $1 \leq q \leq n_{i}$, are already defined, then

$$
B^{\sigma(\mathcal{A})}:=\hat{S}^{n, A}\left(\left\{f^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\right\},\left\{t_{1}\right\}^{\sigma(\mathcal{A})}, \ldots,\left\{t_{n_{i}}\right\}^{\sigma(\mathcal{A})}\right)
$$

(4) If $B$ is an arbitrary subset of $W_{\tau}(X)$, then $B^{\sigma(\mathcal{A})}:=\bigcup_{b \in B}\{b\}^{\sigma(\mathcal{A})}$.

If $B$ is empty, then we set $B^{\sigma(\mathcal{A})}:=\emptyset$.

Lemma 10. Let $\mathcal{A}$ be an algebra of type $\tau$, and let $B \in \mathcal{P}\left(W_{\tau}(X)\right)$. Let $\sigma \in$ $H y p_{G}^{n d}(\tau)$ be a gnd-hypersubstitution of type $\tau$. Then

$$
\hat{\sigma}[B]^{\mathcal{A}}=B^{\sigma(\mathcal{A})}
$$

Proof. If $B$ is empty, then the claim is clearly true.
(1) If $B$ is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of $B$.
(1.1) If $B=\left\{x_{j} \mid x_{j} \in X_{n}\right\}$, then

$$
\begin{aligned}
\hat{\sigma}[B]^{\mathcal{A}} & =\hat{\sigma}\left[\left\{x_{j}\left|x_{j} \in X_{n}\right| x_{j} \in X_{n}\right\}\right]^{\mathcal{A}} \\
& =\left\{x_{j} \mid x_{j} \in X_{n}\right\}^{\mathcal{A}} \\
& =\left\{x_{j}^{\mathcal{A}} \mid x_{j} \in X_{n}\right\} \\
& =\left\{e_{j}^{n, A}\right\} \\
& =\left\{x_{j}^{\sigma(\mathcal{A})} \mid x_{j} \in X_{n}\right\} \\
& =\left\{x_{j} \mid x_{j} \in X_{n}\right\}^{\sigma(\mathcal{A})} \\
& =B^{\sigma(\mathcal{A})} .
\end{aligned}
$$

(1.2) If $B=\left\{x_{j} \mid x_{j} \in X \backslash X_{n}\right\}$, then

$$
\begin{aligned}
\hat{\sigma}[B]^{\mathcal{A}} & =\hat{\sigma}\left[\left\{x_{j} \mid x_{j} \in X \backslash X_{n}\right\}\right]^{\mathcal{A}} \\
& =\left\{x_{j} \mid x_{j} \in X \backslash X_{n}\right\}^{\mathcal{A}} \\
& =\left\{x_{j}^{\mathcal{A}} \mid x_{j} \in X \backslash X_{n}\right\} \\
& =\left\{c_{a}^{n} \mid a \in A\right\} \\
& =\left\{x_{j}^{\sigma(\mathcal{A})} \mid x_{j} \in X \backslash X_{n}\right\} \\
& =\left\{x_{j} \mid x_{j} \in X \backslash X_{n}\right\}^{\sigma(\mathcal{A})} \\
& =B^{\sigma(\mathcal{A})} .
\end{aligned}
$$

(1.3) If $B=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}$, and we assume that the equations

$$
\hat{\sigma}\left[\left\{t_{q}\right\}\right]^{\mathcal{A}}=\left\{t_{q}\right\}^{\sigma(\mathcal{A})}
$$

where $1 \leq q \leq n_{i}$, then

$$
\begin{aligned}
\hat{\sigma}[B]^{\mathcal{A}} & =\hat{\sigma}\left[\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}\right]^{\mathcal{A}} \\
& =\left(\hat{S}^{n}\left(\sigma\left(f_{i}\right), \hat{\sigma}\left[\left\{t_{1}\right\}\right], \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]\right)\right)^{\mathcal{A}} \\
& =\hat{S}^{n, A}\left(\sigma\left(f_{i}\right)^{\mathcal{A}}, \hat{\sigma}\left[\left\{t_{1}\right\}\right]^{\mathcal{A}}, \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]^{\mathcal{A}}\right) \\
& =\hat{S}^{n, A}\left(\left\{l_{i} \mid l_{i} \in \sigma\left(f_{i}\right)\right\}^{\mathcal{A}}, \hat{\sigma}\left[\left\{t_{1}\right\}\right]^{\mathcal{A}}, \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]^{\mathcal{A}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \hat{S}^{n, A}\left(\left\{l_{i}^{\mathcal{A}} \mid l_{i} \in \sigma\left(f_{i}\right)\right\}, \hat{\sigma}\left[\left\{t_{1}\right\}\right]^{\mathcal{A}}, \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]^{\mathcal{A}}\right) \\
&= \bigcup_{l_{i} \in \sigma\left(f_{i}\right)} \hat{S}^{n, A}\left(\left\{l_{i}^{\mathcal{A}}\right\}, \hat{\sigma}\left[\left\{t_{1}\right\}\right]^{\mathcal{A}}, \ldots, \hat{\sigma}\left[\left\{t_{n_{i}}\right\}\right]^{\mathcal{A}}\right) \\
&= \bigcup_{l_{i} \in \sigma\left(f_{i}\right)} \hat{S}^{n, A}\left(\left\{\rho\left(f_{i}\right)^{\mathcal{A}} \mid \rho\left(f_{i}\right)=l_{i} \text { for some } \rho \in H y p_{G}(\tau)\right.\right. \\
&\left.\quad \operatorname{and} \rho(\mathcal{A}) \in \sigma(\mathcal{A})\},\left\{t_{1}\right\}^{\sigma(\mathcal{A})}, \ldots,\left\{t_{n_{i}}\right\}^{\sigma(\mathcal{A})}\right) \\
&= \hat{S}^{n, A}\left(\left\{f_{i}^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\right\},\left\{t_{1}\right\}^{\sigma(\mathcal{A})}, \ldots,\left\{t_{n_{i}}\right\}^{\sigma(\mathcal{A})}\right) \\
&=\left\{f_{i}\left(t_{1}, \ldots, t_{n_{i}}\right)\right\}^{\sigma(\mathcal{A})} \\
&=B^{\sigma(\mathcal{A})}
\end{aligned}
$$

(2) If $B$ is an arbitrary subset of $W_{\tau}(X)$, then

$$
\begin{aligned}
\hat{\sigma}[B]^{\mathcal{A}} & =\left(\bigcup_{b \in B} \hat{\sigma}[\{b\}]\right)^{\mathcal{A}} \\
& =\bigcup_{b \in B}\{b\}^{\sigma(\mathcal{A})} \\
& =\left(\bigcup_{b \in B}\{b\}\right)^{\sigma(\mathcal{A})} \\
& =B^{\sigma(\mathcal{A})} .
\end{aligned}
$$

Lemma 11. Let $\mathcal{A}=\left(A ;\left(f_{i}^{\mathcal{A}}\right)_{i \in I}\right)$ be an algebra of type $\tau$ and $\sigma_{1}, \sigma_{2} \in \operatorname{Hyp}_{G}^{n d}(\tau)$. Then we have $\sigma_{1}\left(\sigma_{2}(\mathcal{A})\right)=\left(\sigma_{1} \circ_{g} \sigma_{2}\right)(\mathcal{A})$.

Proof. By Lemma 10, we have

$$
\begin{aligned}
\sigma_{1}\left(\sigma_{2}(\mathcal{A})\right)= & \left\{\beta(\rho(\mathcal{A})) \mid \rho\left(f_{i}\right)^{\beta(\mathcal{A})} \in \sigma_{1}\left(f_{i}\right)^{\sigma_{2}(\mathcal{A})}, \beta(\mathcal{A}) \in \sigma_{2}(\mathcal{A}), i \in I\right\} \\
= & \left\{\left(\beta \circ_{G} \rho\right)(\mathcal{A}) \mid\left(\beta \circ_{G} \rho\right)\left(f_{i}\right)^{\mathcal{A}} \in \hat{\sigma}_{2}\left[\sigma_{1}\left(f_{i}\right)\right]^{\mathcal{A}}, \beta(\mathcal{A}) \in \sigma_{2}(\mathcal{A}), i \in I\right\} \\
= & \left\{\left(\beta \circ_{G} \rho\right)(\mathcal{A}) \mid\left(\beta \circ_{G} \rho\right)\left(f_{i}\right)^{\mathcal{A}} \in\left(\sigma_{2} \circ_{g} \sigma_{1}\right)\left(f_{i}\right)^{\mathcal{A}}, \beta(\mathcal{A}) \in \sigma_{2}(\mathcal{A})\right. \\
& i \in I\} \\
= & \left(\sigma_{2} \circ_{g} \sigma_{1}\right)(\mathcal{A})
\end{aligned}
$$

Lemma 12. Let $\mathcal{A}=\left(A ;\left(f_{i}^{\mathcal{A}}\right)_{i \in I}\right)$ be an algebra and $\sigma_{i d} \in H y p_{G}^{n d}(\tau)$. Then $\sigma_{\text {id }}(\mathcal{A})=\{\mathcal{A}\}$.

Proof. By Lemma 11, we have

$$
\begin{aligned}
\sigma_{g i d}(\mathcal{A}) & =\left\{\rho(\mathcal{A}) \mid \rho\left(f_{i}\right)^{\mathcal{A}} \in \sigma_{\text {gid }}\left(f_{i}\right)^{\mathcal{A}}, i \in I\right\} \\
& =\left\{\rho(\mathcal{A}) \mid \rho\left(f_{i}\right)^{\mathcal{A}} \in\left\{f_{i}\left(x_{1}, \ldots, x_{n_{i}}\right) \mathcal{A}^{\mathcal{A}}, i \in I\right\}\right. \\
& =\left\{\rho(\mathcal{A}) \mid \rho\left(f_{i}\right)^{\mathcal{A}} \in\left\{f_{i}^{\mathcal{A}}\left(x_{1}^{\mathcal{A}}, \ldots, x_{n_{i}}^{\mathcal{A}}\right)\right\}, i \in I\right\} \\
& =\left\{\rho(\mathcal{A}) \mid \rho\left(f_{i}\right)^{\mathcal{A}}=f_{i}^{\mathcal{A}}=\sigma_{\text {gid }}\left(f_{i}\right)^{\mathcal{A}} \in\left\{f_{i}^{\mathcal{A}}\right\}, i \in I\right\} \\
& =\left\{\sigma_{g i d}(\mathcal{A}) \mid \rho\left(f_{i}\right)^{\mathcal{A}}=f_{i}^{\mathcal{A}}=\sigma_{g i d}\left(f_{i}\right)^{\mathcal{A}} \in\left\{f_{i}^{\mathcal{A}}\right\}, i \in I\right\} \\
& =\left\{\mathcal{A} \mid \rho\left(f_{i}\right)^{\mathcal{A}}=f_{i}^{\mathcal{A}}=\sigma_{\text {gid }}\left(f_{i}\right)^{\mathcal{A}} \in\left\{f_{i}^{\mathcal{A}}\right\}, i \in I\right\} .
\end{aligned}
$$

Let $\left(M ; \circ_{g}, \sigma_{g i d}\right)$ be a submonoid of $\left(H y p_{G}^{n d}(\tau) ; \circ_{g}, \sigma_{g i d}\right)$ and let $B_{1} \approx B_{2} \in$ $\mathcal{P}\left(W_{\tau}(X)\right)$. For every $\mathcal{A} \vDash B_{1} \approx B_{2}$ such that $\mathcal{A} \vDash \hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right]$ for all $\sigma \in M$, we written as, $\mathcal{A} \models_{M-g n d-h y p} B_{1} \approx B_{1}$ and $K \models_{M-g n d-h y p} B_{1} \approx B_{2}$ for all $\mathcal{A} \in K$.

Now we define two mappings which give a second Galois connection. Let $\mathcal{K} \subseteq$ $\mathcal{P}(A l g(\tau))$ and $P L \subseteq \mathcal{P}\left(W_{\tau}(X)\right)^{2}$. Then we define a mapping

$$
H_{M} P I d: \mathcal{P}(\mathcal{P}(\operatorname{Alg}(\tau))) \rightarrow \mathcal{P}\left(\mathcal{P}\left(W_{\tau}(X)\right)^{2}\right),
$$

by

$$
H_{M} P I d \mathcal{K}:=\left\{B_{1} \approx B_{2} \in \mathcal{P}\left(W_{\tau}(X)\right)^{2} \mid \forall K \in \mathcal{K}\left(K \models_{M-\text { gnd-hyp }} B_{1} \approx B_{2}\right)\right\}
$$

and define a mapping

$$
H_{M} P M o d: \mathcal{P}\left(\mathcal{P}\left(W_{\tau}(X)\right)^{2}\right) \rightarrow \mathcal{P}(\mathcal{P}(\operatorname{Alg}(\tau))),
$$

by

$$
H_{M} P M o d P L:=\left\{K \in \mathcal{P}(\operatorname{Alg}(\tau)) \mid \forall B_{1} \approx B_{2} \in P L\left(\left.K\right|_{M-g n d-h y p} B_{1} \approx B_{2}\right)\right\} .
$$

It is easy to see that ( $\left.H_{M} P M o d, H_{M} P I d\right)$ is a Galois connection between $\mathcal{P}(\operatorname{Alg}(\tau))$ and $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$ with respect to the second relation

$$
\begin{aligned}
& R_{\models_{M-g n d-h y p}} \\
& :=\left\{\left(K, B_{1} \approx B_{2}\right) \in \mathcal{P}(\operatorname{Alg}(\tau)) \times \mathcal{P}\left(W_{\tau}(X)\right)^{2} \mid K \models_{M-g n d-h y p} B_{1} \approx B_{2}\right\} .
\end{aligned}
$$

We have two closure operators $H_{M} P M o d H_{M} P I d$ and $H_{M} P I d H_{M} P M o d$ and their sets

$$
\left\{P L \subseteq \mathcal{P}\left(W_{\tau}(X)\right)^{2} \mid H_{M} P I d H_{M} P M o d P L=P L\right\}
$$

and

$$
\left\{\mathcal{K} \subseteq \mathcal{P}(\operatorname{Alg}(\tau)) \mid H_{M} P M o d H_{M} P I d \mathcal{K}=\mathcal{K}\right\}
$$

form two complete sublattices $\mathcal{S E}, \mathcal{S} \mathcal{L}$ of $\mathcal{E}, \mathcal{L}$, respectively.
Theorem 13. Let $\mathcal{A}$ be an algebra of type $\tau$, and $\sigma \in \operatorname{Hyp}_{G}^{n d}(\tau)$. Let $B_{1} \approx B_{2} \in$ $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$. Then

$$
\sigma(\mathcal{A}) \models B_{1} \approx B_{2} \Leftrightarrow \mathcal{A} \models \hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right] .
$$

Proof. By Lemma 10, we have

$$
\begin{aligned}
\sigma(\mathcal{A}) \models B_{1} \approx B_{2} & \Leftrightarrow B_{1}^{\sigma(\mathcal{A})}=B_{2}^{\sigma(\mathcal{A})} \\
& \Leftrightarrow \hat{\sigma}\left[B_{1}\right]^{\mathcal{A}}=\hat{\sigma}\left[B_{2}\right]^{\mathcal{A}} \\
& \Leftrightarrow \mathcal{A} \models \hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right] .
\end{aligned}
$$

Definition. Let $\mathcal{K} \subseteq \mathcal{P}(\operatorname{Alg}(\tau))$ and $P L \subseteq \mathcal{P}\left(W_{\tau}(X)\right)^{2}$. Then we set

$$
\chi_{M}^{E}\left[B_{1} \approx B_{2}\right]:=\left\{\hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right] \mid \sigma \in M\right\}
$$

and

$$
\chi_{M}^{A}[K]:=\{\sigma(K) \mid \sigma \in M\} .
$$

We define two operators in the following way:

$$
\chi_{M}^{E}: \mathcal{P}\left(\mathcal{P}\left(W_{\tau}(X)\right)^{2}\right) \rightarrow \mathcal{P}\left(\mathcal{P}\left(W_{\tau}(X)\right)^{2}\right)
$$

by $\chi_{M}^{E}[P L]:=\left\{\chi_{M}^{E}\left[B_{1} \approx B_{2}\right] \mid B_{1} \approx B_{2} \in P L\right\}$ and

$$
\chi_{M}^{A}: \mathcal{P}(\mathcal{P}(\operatorname{Alg}(\tau))) \rightarrow \mathcal{P}(\mathcal{P}(\operatorname{Alg}(\tau)))
$$

by $\chi_{M}^{A}[\mathcal{K}]:=\left\{\chi_{M}^{A}[K] \mid K \in \mathcal{K}\right\}$.
In the next lemmas we will show that the both operators are closure operators.
Lemma 14. Let $P L, P L_{1}, P L_{2}$ be subsets of $\mathcal{P}\left(W_{\tau}(X)\right)^{2}$. Then
(i) $P L \subseteq \chi_{M}^{E}[P L]$.
(ii) $P L_{1} \subseteq P L_{2} \Rightarrow \chi_{M}^{E}\left[P L_{1}\right] \subseteq \chi_{M}^{E}\left[P L_{2}\right]$.
(iii) $\chi_{M}^{E}[P L]=\chi_{M}^{E}\left[\chi_{M}^{E}[P L]\right]$.

Proof. (i) Let $B_{1} \approx B_{2} \in P L$. Then, since $B_{1}=\hat{\sigma}_{\text {gid }}\left[B_{1}\right]$ and $B_{2}=\hat{\sigma}_{\text {gid }}\left[B_{2}\right]$, we have $\hat{\sigma}\left[B_{1}\right]=B_{1} \approx B_{2}=\hat{\sigma}\left[B_{2}\right] \in \chi_{M}^{E}[P L]$ and then $P L \subseteq \chi_{M}^{E}[P L]$.
(ii) Assume that $P L_{1} \subseteq P L_{2}$ and let $\hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right] \in \chi_{M}^{E}\left[P L_{1}\right]$. Then $B_{1} \approx B_{2} \in P L_{2}$ but $P L_{1} \subseteq P L_{2}$, so that $B_{1} \approx B_{2} \in P L_{2}$ and $\hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right] \in$ $\chi_{M}^{E}\left[P L_{2}\right]$. We have $\chi_{M}^{E}\left[P L_{1}\right] \subseteq \chi_{M}^{E}\left[P L_{2}\right]$.
(iii) By (i) we have $\chi_{M}^{E}[P L] \subseteq \chi_{M}^{E}\left[\left[\chi_{M}^{E}[P L]\right]\right.$. Let $\hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right] \in \chi_{M}^{E}\left[\chi_{M}^{E}[P L]\right]$. Then $B_{1} \approx B_{2} \in \chi_{M}^{E}[P L]$, and there exists $\rho \in M$ and $C_{1} \approx C_{2} \in P L$ such that $B_{1}=\hat{\rho}\left[C_{1}\right]$ and $B_{2}=\hat{\rho}\left[C_{2}\right]$, and then

$$
\begin{aligned}
\hat{\sigma}\left[B_{1}\right] & =\hat{\sigma}\left[\hat{\rho}\left[C_{1}\right]\right] \\
& =(\hat{\sigma} \circ \hat{\rho})\left[C_{1}\right] \\
& =(\sigma \circ g \rho)\left[C_{1}\right] \\
& =\hat{\lambda}\left[C_{1}\right] \text { where } \lambda=\sigma \circ_{g} \rho \in M \text { and } \\
\hat{\sigma}\left[B_{2}\right] & =\hat{\sigma}\left[\hat{\rho}\left[C_{2}\right]\right] \\
& =(\hat{\sigma} \circ \hat{\rho})\left[C_{2}\right] \\
& =(\sigma \circ g \rho)\left[C_{2}\right] \\
& =\hat{\lambda}\left[C_{2}\right] \text { where } \lambda=\sigma \circ_{g} \rho \in M .
\end{aligned}
$$

Then we set $\hat{\lambda}\left[C_{1}\right]=\hat{\sigma}\left[B_{1}\right] \approx \hat{\sigma}\left[B_{2}\right]=\hat{\lambda}\left[C_{2}\right] \in \chi_{M}^{E}[P L]$, and obtain $\chi_{M}^{E}\left[\chi_{M}^{E}[P L]\right] \subseteq$ $\chi_{M}^{E}[P L]$.

Similarly method we have
Lemma 15. Let $\mathcal{K}, \mathcal{K}_{1}, \mathcal{K}_{2}$ be subset of $\mathcal{P}(\operatorname{Alg}(\tau))$. Then
(i) $\mathcal{K} \subseteq \chi_{M}^{A}[\mathcal{K}]$,
(ii) $\mathcal{K}_{1} \subseteq \mathcal{K}_{2} \Rightarrow \chi_{M}^{A}\left[\mathcal{K}_{1}\right] \subseteq \chi_{M}^{A}\left[\mathcal{K}_{2}\right]$,
(iii) $\chi_{M}^{A}[\mathcal{K}]=\chi_{M}^{A}\left[\chi_{M}^{A}[\mathcal{K}]\right]$.

The next theorem needs the concept of a conjugate pair of additive closure operators.

Theorem 16. The pair $\left(\chi_{M}^{A}, \chi_{M}^{E}\right)$ is a conjugate pair of completely additive closure operators with respect to the relation $R_{\models}$.

Proof. By Theorem 13, Lemma 14 and Lemma 15.
Definition. Let $\left(M ; \circ_{g}, \sigma_{g i d}\right)$ be a submonoid of $\left(H y p_{G}^{n d}(\tau) ; \circ_{g}, \sigma_{g i d}\right)$. A variety $V$ of type $\tau$ is said to be an $M$-solid generalized non-deterministic variety, for short an $M$-gnd-solid variety, if $\{\{\mathcal{A}\} \mid \mathcal{A} \in V\} \models\{\hat{\sigma}[\{s\}] \approx \hat{\sigma}[\{t\}] \mid s \approx t \in$ $I d V, \sigma \in M\}$ and $s \approx t$ is also said to be an $M$-solid generalized non-deterministic hyperidentity in $V$, for short an $M$-gnd hyperidentity in $V$. In this case that $M=H y p_{G}^{n d}(\tau)$ we will speak of a solid generalized non-deterministic variety, for short of a $g n d$-solid variety and generalized non-deterministic hyperidentity, for short of a gnd hyperidentity, respectively.

Now we may apply the theory of conjugate pairs of additive closure operators (see [4]) and obtain the following propositions:

Lemma 17. Let $V \subseteq \operatorname{Alg}(\tau)$ be a class of algebras and $\Sigma \subseteq W_{\tau}(X)^{2}$. Let $V^{*}=\{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$ and $\Sigma^{*}=\{\hat{\sigma}[\{s\}] \approx \hat{\sigma}[\{t\}] \mid s \approx t \in \Sigma, \sigma \in M\}$. Then the following properties hold:
(i) $H_{M} P I d V^{*}=P I d \chi_{M}^{A}\left[V^{*}\right]$.
(ii) $H_{M} P I d V^{*} \subseteq P I d V^{*}$.
(iii) $\chi_{M}^{E}\left[H_{M} P I d V^{*}\right]=H_{M} P I d V^{*}$.
(iv) $\chi_{M}^{A}\left[P M o d H_{M} P I d V^{*}\right]=P M o d H_{M} P I d V^{*}$.
(v) $H_{M} P I d H_{M} P M o d \Sigma^{*}=P I d P M o d \chi_{M}^{E}\left[\Sigma^{*}\right]$.

Using these propositions one obtains the following characterization of $M$-solid generalized non-deterministic varieties.

Theorem 18. Let $V$ be a subset of $\operatorname{Alg}(\tau)$. Let $V^{*}=\{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$. Then the following properties are equivalent:
(i) $H_{M} P M o d H_{M} P I d V^{*}=V^{*}$.
(ii) $\chi_{M}^{A}\left[V^{*}\right]=V^{*}$ (i.e. $V^{*}$ is an $M$-gnd-solid variety).
(iii) $P I d V^{*}=H_{M} P I d V^{*}$ (i.e. every identity in $V^{*}$ is satisfied as an $M$-gnd hyperidentity).
(v) $\chi_{M}^{E}\left[P I d V^{*}\right]=P I d V^{*}$.

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