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M-SOLID GENERALIZED NON-DETERMINISTIC VARIETIES

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Abstract

A generalized non-deterministic hypersubstitution is a mapping which maps operation symbols of type τ to the set of terms of the same type which does not necessarily preserve the arity. We apply the generalized non-deterministic hypersubstitution to an algebra of type τ and obtain a class of derived algebras of type τ. The generalized non-deterministic hypersubstitutions can be also applied to sets of equations of type τ. We obtain two closure operators which turn out to be a conjugate pair of completely additive closure operators. This allows us to apply the theory of conjugate pairs of additive closure operators to characterize M-solid generalized non-deterministic varieties of algebras.

Keywords: generalized non-deterministic hypersubstitution, conjugate pair of additive closure operators, M-solid generalized non-deterministic variety.

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1. Introduction

Let $\tau = (n_i)_{i \in I}$ be a *type* of algebras, indexed by a set I. Let $(f_i)_{i \in I}$ be an indexed set of operation symbols where f_i is n_i-ary and let $X := \{x_1, \ldots, x_n, \ldots\}$ be a countably infinite set of variables, and for each $n \geq 1$ let $X_n := \{x_1, \ldots, x_n\}$. Denoted by $W_\tau(X)$ and $W_\tau(X_n)$ the set of all the terms of type τ and of all n-ary terms of type τ, respectively. These two sets can be use as the universe of the *absolutely free algebras* of type τ

$$\mathcal{F}_\tau(X) := (W_\tau(X); (\bar{f}_i)_{i \in I}) \text{ and } \mathcal{F}_\tau(X_n) := (W_\tau(X_n); (\bar{f}_i)_{i \in I})$$

here the operations $\bar{f}_i$ are defined by setting $\bar{f}_i(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i})$.

In 2000, Leeratanavalee and Denecke [6] gave the concept of generalized superposition operation of terms

$$S^n : (W_\tau(X))^{n+1} \to W_\tau(X)$$

by the following steps: For any term $t \in W_\tau(X)$

(i) If $t = x_j$ for $1 \leq j \leq n$, then $S^n(x_j, t_1, \ldots, t_n) := t_j$.

(ii) If $t = x_j$ for $n < j$, then $S^n(x_j, t_1, \ldots, t_n) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$ and assume that $S^n(s_q, t_1, \ldots, t_{n_i})$ for $1 \leq q \leq n_i$, are already defined, then

$$S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) = f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n)).$$

In 2007, Denecke and Glubudom [2] studied solid non-deterministic varieties of algebras and characterized of M-solid non-deterministic varieties. They gave the concept of superposition operation of the set of terms

$$\hat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times \mathcal{P}(W_\tau(X_m))^n \to \mathcal{P}(W_\tau(X_m))$$

inductively by the following steps: Let $m, n \in \mathrm{N}^+ (:= \mathrm{N} - \{0\})$ and let $B \in \mathcal{P}(W_\tau(X_n))$ and $B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m))$ such that $B, B_1, \ldots, B_n$ are non-empty.

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{S}^n_m(\{x_j\}, B_1, \ldots, B_n) := B_j$.

(ii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and assume that $\hat{S}^n_m(\{t_q\}, B_1, \ldots, B_n)$ for $1 \leq q \leq n_i$, are already defined, then $\hat{S}^n_m(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) \mid r_q \in \hat{S}^n_m(\{t_q\}, B_1, \ldots, B_n)\}$.

(iii) If B is an arbitrary subset of $W_\tau(X_n)$, then

$$\hat{S}^n_m(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}^n_m(\{b\}, B_1, \ldots, B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}^n_m(B, B_1, \ldots, B_n) := \emptyset$.

For the rest of this paper, if X is a set we mean $\mathcal{P}(X)$ the set of all subsets of X. In this paper, we extend the concept of solid non-deterministic varieties which studied in [2] to solid generalized non-deterministic varieties and we also characterize M-solid generalized non-deterministic varieties.

2. Generalized superposition operations of the set of terms

In this section, we give the concept of generalized superposition operations of the 9set of terms and study some properties of such superposition operations.

Definition. Let $B, B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X))$ be such that $B, B_1, \ldots, B_n$ are non-empty. Then the generalized superposition operation

$$\hat{S}^n : (\mathcal{P}(W_\tau(X)))^{n+1} \to \mathcal{P}(W_\tau(X))$$

is inductively defined as follows:

(i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{S}^n(\{x_j\}, B_1, \ldots, B_n) := B_j$.

(ii) If $B = \{x_j\}$ for $n < j$, then $\hat{S}^n(\{x_j\}, B_1, \ldots, B_n) := \{x_j\}$.

(iii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and assume that $\hat{S}^n(\{t_q\}, B_1, \ldots, B_n)$ for $1 \leq q \leq n_i$, are already defined, then $\hat{S}^n(\{f_i(t_1, \ldots, t_{n_i})\}, B_1, \ldots, B_n) :=$ $\{f_i(r_1, \ldots, r_{n_i}) \mid r_q \in \hat{S}^n(\{t_q\}, B_1, \ldots, B_n)\}$.

(iv) If B is an arbitrary subset of $W_\tau(X_n)$, then

$$\hat{S}^n(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}^n(\{b\}, B_1, \ldots, B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}^n(B, B_1, \ldots, B_n) := \emptyset$.

Our next aim is to prove that the generalized superation operation $\hat{S}^n$ satisfy the superassociative law but we need the following lemma.

Lemma 1. *Let S be a subset of $W_\tau(X)$. Then*

$$\hat{S}^n\Big(\bigcup_{s \in S} \hat{S}^n(\{s\}, L_1, \ldots, L_n), T_1, \ldots, T_n\Big) = \bigcup_{s \in S} \hat{S}^n(\hat{S}^n(\{s\}, L_1, \ldots, L_n), T_1, \ldots, T_n).$$

Proof. Let $x \in \hat{S}^n(\bigcup_{s \in S} \hat{S}^n(\{s\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$\Leftrightarrow x \in \hat{S}^n(\hat{S}^n(\{u\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$ for some $u \in S$
$\Leftrightarrow x \in \bigcup_{s \in S} \hat{S}^n(\hat{S}^n(\{s\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$. ■

Lemma 2. *Let S be a subset of $W_\tau(X)$. Then*
$\hat{S}^n(S, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$
$= \hat{S}^n(\hat{S}^n(S, L_1, \ldots, L_n), T_1, \ldots, T_n)$.

Proof. If S is empty, the claim is clearly true.

(1) If S is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of S.

(1.1) If $S = \{x_j\}$ for $1 \leq j \leq n$, then
$\hat{S}^n(\hat{S}^n(S, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\hat{S}^n(\{x_j\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(L_j, T_1, \ldots, T_n)$
$= \hat{S}^n(\{x_j\}, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$
$= \hat{S}^n(S, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$.

(1.2) If $S = \{x_j\}$ for $n < j$, then
$\hat{S}^n(\hat{S}^n(S, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\hat{S}^n(\{x_j\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\{x_j\}, T_1, \ldots, T_n)$
$= \{x_j\}$
$= \hat{S}^n(\{x_j\}, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$
$= \hat{S}^n(S, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$.

(1.3) If $S = \{f_i(s_1, \ldots, s_{n_i})\}$ and we assume that the equations
$\hat{S}^n(\hat{S}^n(\{s_q\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\{s_q\}, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$
where $1 \leq q \leq n_i$, then
$\hat{S}^n(\hat{S}^n(S, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\hat{S}^n(\{f_i(s_1, \ldots, s_{n_i})\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\{f_i(u_1, \ldots, u_{n_i}) \mid u_q \in \hat{S}^n(\{s_q\}, L_1, \ldots, L_n)\}, T_1, \ldots, T_n)$
$= \{f_i(r_1, \ldots, r_{n_i}) \mid r_q \in \hat{S}^n(\{u_q \mid u_q \in \hat{S}^n(\{s_q\}, L_1, \ldots, L_n)\},$
$T_1, \ldots, T_n)\}$
$= \{f_i(r_1, \ldots, r_{n_i}) \mid r_q \in \hat{S}^n(\hat{S}^n(\{s_q\}, L_1, \ldots, L_n), T_1, \ldots, T_n)\}$
$= \{f_i(r_1, \ldots, r_{n_i}) \mid r_q \in \hat{S}^n(\{s_q\}, \hat{S}^n(L_n, T_1, \ldots, T_n), \ldots,$
$\hat{S}^n(L_1, T_1, \ldots, T_n))\}$
$= \hat{S}^n(\{f_i(s_1, \ldots, s_{n_i})\}, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$
$= \hat{S}^n(S, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$

(2) If S is arbitrary subset of $W_\tau(X)$, then
$\hat{S}^n(\hat{S}^n(S, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \hat{S}^n(\bigcup_{s \in S} \hat{S}^n(\{s\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$
$= \bigcup_{s \in S} \hat{S}^n(\hat{S}^n(\{s\}, L_1, \ldots, L_n), T_1, \ldots, T_n)$

$$= \bigcup_{s\in S} \hat{S}^n(\{s\}, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$$
$$= \hat{S}^n(S, \hat{S}^n(L_1, T_1, \ldots, T_n), \ldots, \hat{S}^n(L_n, T_1, \ldots, T_n))$$

∎

The following identity is also satisfied.

Lemma 3. *Let T be a subset of $W_\tau(X)$. Then*

$$\hat{S}^n(T, \{x_1\}, \ldots, \{x_n\}) = T.$$

Proof. If T is empty, the claim is clearly true.

(1) If T is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of T.

(1.1) If $T = \{x_j\}$ for $1 \le j \le n$, then

$$\begin{aligned} \hat{S}^n(T, \{x_1\}, \ldots, \{x_n\}) &= \hat{S}^n(\{x_j\}, \{x_1\}, \ldots, \{x_n\}) \\ &= \{x_j\} \\ &= T. \end{aligned}$$

(1.2) If $T = \{x_j\}$ for $n < j$, then

$$\begin{aligned} \hat{S}^n(T, \{x_1\}, \ldots, \{x_n\}) &= \hat{S}^n(\{x_j\}, \{x_1\}, \ldots, \{x_n\}) \\ &= \{x_j\} \\ &= T. \end{aligned}$$

(1.3) If $T = \{f_i(s_1, \ldots, s_{n_i})\}$ and we assume that the equations

$$\hat{S}^n(\{s_q\}, \{x_1\}, \ldots, \{x_n\}) = \{s_q\}$$

where $1 \le q \le n_i$, then

$$\begin{aligned} \hat{S}^n(T, \{x_1\}, \ldots, \{x_n\}) &= \hat{S}^n(\{f_i(s_1, \ldots, s_{n_i})\}, \{x_1\}, \ldots, \{x_n\}) \\ &= \{f_i(u_1, \ldots, u_{n_i}) \mid u_q \in \hat{S}^n(\{s_q\}, \{x_1\}, \ldots, \\ &\quad \{x_n\})\} \\ &= \{f_i(u_1, \ldots, u_{n_i}) \mid u_q \in \{s_q\}\} \\ &= \{f_i(s_1, \ldots, s_{n_i})\} \\ &= T. \end{aligned}$$

(2) If T is arbitrary subset of $W_\tau(X)$, then

$$\begin{aligned} \hat{S}^n(T, \{x_1\}, \ldots, \{x_n\}) &= \bigcup_{t\in T} \hat{S}^n(\{t\}, \{x_1\}, \ldots, \{x_n\}) \\ &= \{t \mid t \in T\} \\ &= T. \end{aligned}$$

∎

Using the generalized superposition operation $\hat{S}^n$ we can form the new algebra

$$(\mathcal{P}(W_\tau(X)); \hat{S}^n).$$

We call this algebra the *g-power-clone*.

3. Generalized non-deterministic hypersubstitutions

Hypersubstitutions for terms over one-sorted algebras were introduced by Graczyńska and Schweigert ([5]). Our definitions and the properties of generalized superposition operations can be used to define generalized non-deterministic hypersubstitutions and their extensions.

Definition. A generalized non-deterministic hypersubstitution (for short *gnd*-hypersubstitution) of type τ is a mapping from the set $\{f_i \mid i \in I\}$ of n_i-ary operation symbols of type τ to the set $\mathcal{P}(W_\tau(X))$.

We denote by $Hyp_G^{nd}(\tau)$ the set of all *gnd*-hypersubstitutions of type τ. Any *gnd*-hypersubstitution σ induces a mapping $\hat{\sigma}$ defined on the set $\mathcal{P}(W_\tau(X))$, as follows.

Definition. Let σ be a *gnd*-hypersubstitution of type τ. Then σ induces a mapping $\hat{\sigma} : \mathcal{P}(W_\tau(X)) \to \mathcal{P}(W_\tau(X))$, by setting

(i) $\hat{\sigma}[\emptyset] := \emptyset$.

(ii) $\hat{\sigma}[\{x_j\}] := \{x_j\}$ for every $x_j \in X$.

(iii) $\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] = \hat{S}^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])$ if we inductively assume that $\hat{\sigma}[\{t_q\}]$ where $1 \leq q \leq n_i$, are already defined.

(iv) $\hat{\sigma}[B] := \bigcup_{b \in B} \hat{\sigma}[\{b\}]$ for $B \subseteq W_\tau(X)$.

Then we have:

Lemma 4. *Let $\sigma \in Hyp_G^{nd}(\tau)$. Then $\hat{\sigma}$ is an endomorphism on the g-power-clone.*

Proof. We have to shows that

$$\hat{\sigma}[S^n(T, S_1, \ldots, S_n)] = S^n(\hat{\sigma}[T], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]).$$

If T is empty, then the claim is clearly true.

(1) If T is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of T.

(1.1) If $T = \{x_j\}$ for $1 \leq j \leq n$, then

$$\begin{aligned}\hat{\sigma}[S^n(T, S_1, \ldots, S_n)] &= \hat{\sigma}[S^n(\{x_j\}, S_1, \ldots, S_n)] \\ &= \hat{\sigma}[S_j] \\ &= \hat{S}^n(\{x_j\}, \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]) \\ &= \hat{S}^n(\hat{\sigma}[\{x_j\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]) \\ &= \hat{S}^n(\hat{\sigma}[T], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]).\end{aligned}$$

(1.2) If $T = \{x_j\}$ for $n < j$, then

$$\begin{aligned}\hat{\sigma}[\hat{S}^n(T, S_1, \ldots, S_n)] &= \hat{\sigma}[S^n(\{x_j\}, S_1, \ldots, S_n)] \\ &= \hat{\sigma}[\{x_j\}] \\ &= \hat{S}^n(\{x_j\}, \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]) \\ &= \hat{S}^n(\hat{\sigma}[\{x_j\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]) \\ &= \hat{S}^n(\hat{\sigma}[T], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]).\end{aligned}$$

(1.3) If $T = \{f_i(t_1, \ldots, t_{n_i})\}$ and we assume that the equations

$$\hat{\sigma}[\hat{S}^n(\{t_q\}, S_1, \ldots, S_n)] = \hat{S}^n(\hat{\sigma}[\{t_q\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]).$$

where $1 \leq q \leq n_i$, then

$$\begin{aligned}&\hat{\sigma}[\hat{S}^n(T, S_1, \ldots, S_n)] \\ &= \hat{\sigma}[\hat{S}^n(\{f_i(t_1, \ldots, t_{n_i})\}, S_1, \ldots, S_n)] \\ &= \hat{\sigma}[\{f_i(u_1, \ldots, u_{n_i}) \mid u_q \in \hat{S}^n(\{t_q\}, S_1, \ldots, S_n)\}] \\ &= \hat{S}^{n_i}(\sigma(f_i), \hat{\sigma}[\{u_1 \mid u_1 \in \hat{S}^n(\{t_1\}, S_1, \ldots, S_n)\}], \ldots, \\ &\quad \hat{\sigma}[\{u_{n_i} \mid u_{n_i} \in \hat{S}^n(\{t_{n_i}\}, S_1, \ldots, S_n)\}]) \\ &= \hat{S}^{n_i}(\sigma(f_i), \hat{\sigma}[\hat{S}^n(\{t_1\}, S_1, \ldots, S_n)], \ldots, \hat{\sigma}[\hat{S}^n(\{t_{n_i}\}, S_1, \ldots, S_n)]) \\ &= \hat{S}^{n_i}(\sigma(f_i), \hat{S}^n(\hat{\sigma}[\{t_1\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]), \ldots, \hat{S}^n(\hat{\sigma}[\{t_{n_i}\}], \hat{\sigma}[S_1], \ldots, \\ &\quad \hat{\sigma}[S_n)])) \\ &= \hat{S}^n(\hat{S}^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]), \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n)]) \\ &= \hat{S}^n(\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n)]) \\ &= \hat{S}^n(\hat{\sigma}[T], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n)]).\end{aligned}$$

(2) If T is arbitrary subset of $W_\tau(X)$, then

$$\begin{aligned}\hat{\sigma}[S^n(T, S_1, \ldots, S_n)] &= \bigcup_{t \in T} \hat{\sigma}[S^n(\{t\}, S_1, \ldots, S_n)] \\ &= \bigcup_{t \in T} \hat{S}^n(\hat{\sigma}[\{t\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]) \\ &= \hat{S}^n(\bigcup_{t \in T} \hat{\sigma}[\{t\}], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]) \\ &= \hat{S}^n(\hat{\sigma}[T], \hat{\sigma}[S_1], \ldots, \hat{\sigma}[S_n]).\end{aligned}$$

■

Let σ_1, σ_2 be elements in Hyp_G^{nd}. Since the extension of *gnd*-hypersubstitution maps $\mathcal{P}(W_\tau(X))$ to $\mathcal{P}(W_\tau(X))$ we may define a product $\sigma_1 \circ_g \sigma_2$ by

$$\sigma_1 \circ_g \sigma_2 := \hat{\sigma}_1 \circ \sigma_2.$$

Here $\circ$ is the usual composition of mappings. Since $\hat{\sigma}_1 \circ \sigma_2$ maps $\{f_i \mid i \in I\}$ to $\mathcal{P}(W_\tau(X))$, it is a *gnd*-hypersubstitution of type τ. The following lemmas shows that the extension of this product is the product of the extensions of σ_1 and σ_2.

Lemma 5. *Let σ_1, σ_2 be elements in Hyp_G^{nd}. Then we have*

$$(\sigma_1 \circ_g \sigma_2)\hat{\ } = \hat{\sigma}_1 \circ \hat{\sigma}_2.$$

Proof. We will show that $(\sigma_1 \circ_g \sigma_2)\hat{\ }[T] = (\hat{\sigma}_1 \circ \hat{\sigma}_2)[T]$ for all $T \subseteq W_\tau(X)$.

If T is empty, then the claim is clearly true.

(1) If T is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of T.

(1.1) If $T = \{x_j\}$, then

$$\begin{aligned}(\sigma_1 \circ_g \sigma_2)\hat{\ }[T] &= (\sigma_1 \circ_g \sigma_2)\hat{\ }[\{x_j\}]\\ &= \{x_j\}\\ &= \hat{\sigma}_1[\{x_j\}]\\ &= \hat{\sigma}_1[\hat{\sigma}_2[\{x_j\}]]\\ &= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[\{x_j\}]\\ &= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[T].\end{aligned}$$

(1.2) If $T = \{f_i(t_1, \ldots, t_{n_i})\}$ and we assume that the equations

$$(\sigma_1 \circ_g \sigma_2)\hat{\ }[\{t_q\}] = (\hat{\sigma}_1 \circ \hat{\sigma}_2)[\{t_q\}]$$

where $1 \leq q \leq n_i$, then

$$\begin{aligned}(\sigma_1 \circ_g \sigma_2)\hat{\ }[T] &= (\sigma_1 \circ_g \sigma_2)\hat{\ }[f_i(t_1, \ldots, t_{n_i})]\\ &= \hat{S}^{n_i}((\sigma_1 \circ_g \sigma_2)(f_i), (\sigma_1 \circ_g \sigma_2)\hat{\ }[t_1], \ldots, (\sigma_1 \circ_g \sigma_2)\hat{\ }[t_{n_i}])\\ &= \hat{S}^{n_i}((\hat{\sigma}_1 \circ \sigma_2)(f_i), (\hat{\sigma}_1 \circ \hat{\sigma}_2)[\{t_1\}], \ldots, (\hat{\sigma}_1 \circ \hat{\sigma}_2)[\{t_{n_i}\}])\\ &= \hat{S}^{n_i}((\hat{\sigma}_1[\sigma_2(f_i)], \hat{\sigma}_1[\hat{\sigma}_2[\{t_1\}]], \ldots, \hat{\sigma}_1[\hat{\sigma}_2[\{t_{n_i}\}]])\\ &= \hat{\sigma}_1[\hat{S}^{n_i}(\sigma_2(f_i), \hat{\sigma}_2[\{t_1\}], \ldots, \hat{\sigma}_2[\{t_{n_i}\}])]\\ &= \hat{\sigma}_1[\hat{\sigma}_2[\{f_i(t_1, \ldots, t_{n_i})\}]]\\ &= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[\{f_i(t_1, \ldots, t_{n_i})\}]\\ &= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[T].\end{aligned}$$

(2) If T is arbitrary subset of $W_\tau(X)$, then

$$\begin{aligned}(\sigma_1 \circ_g \sigma_2)\hat{}[T] &= \bigcup_{t\in T} (\sigma_1 \circ_g \sigma_2)\hat{}[\{t\}] \\ &= \bigcup_{t\in T} (\hat{\sigma}_1 \circ \hat{\sigma}_2)[\{t\}] \\ &= \bigcup_{t\in T} \hat{\sigma}_1[\hat{\sigma}_2[\{t\}]] \\ &= \hat{\sigma}_1[\bigcup_{t\in T} \hat{\sigma}_2[\{t\}]] \\ &= \hat{\sigma}_1[\hat{\sigma}_2[T]] \\ &= (\hat{\sigma}_1 \circ \hat{\sigma}_2)[T].\end{aligned}$$

■

From Lemma 5, we get that the products $\circ_g$ are associative. In fact, for each σ_1, σ_2 and σ_3 in $Hyp_G^{nd}(\tau)$, we have

$$\begin{aligned}(\sigma_1 \circ_g \sigma_2) \circ_g \sigma_3 &= (\sigma_1 \circ_g \sigma_2)\hat{} \circ \sigma_3 \\ &= (\hat{\sigma}_1 \circ \hat{\sigma}_2) \circ \sigma_3 \\ &= \hat{\sigma}_1 \circ (\hat{\sigma}_2 \circ \sigma_3) \\ &= \hat{\sigma}_1 \circ (\sigma_2 \circ_g \sigma_3)\hat{} \\ &= \sigma_1 \circ_g (\sigma_2 \circ_g \sigma_3).\end{aligned}$$

This shows that the set of all gnd-hypersubstitutions forms a semigroup with respect to the associative binary operation $\circ_g$.

If we consider σ_{gid} is an element in $Hyp_G^{nd}(\tau)$ defined by

$$\sigma_{gid}(f_i) := \{f_i(x_1, \ldots, x_{n_i})\},$$

then it is an identity element with respect to $\circ_g$ (see [2]).

Theorem 6. *The algebra* $(Hyp_G^{nd}(\tau); \circ_g, \sigma_{gid})$ *forms a monoid.*

4. Solid generalized non-deterministic varieties

In this section we want to apply the theory of conjugate pairs of additive closure operators to algebras and identities and want to define generalized non-deterministic hyperidentities and solid generalized non-deterministic varieties of algebras.

Definition. Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$ be an algebra, and $B \in \mathcal{P}(W_\tau(X))$. Then we define the set $B^{\mathcal{A}}$ of terms operations on $\mathcal{A}$ induced by terms as follows:

(1) If $B = \{x_j\}$, then $B^{\mathcal{A}} := \{x_j^{\mathcal{A}}\}$.

(2) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$, then $B^{\mathcal{A}} := \{f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}})\}$ where $f_i^{\mathcal{A}}$ is the fundamental operation of $\mathcal{A}$ corresponding to the operation symbol f_i and where $t_q^{\mathcal{A}}$ are the term operations on $\mathcal{A}$ which are induced in the usual way by the t_q *'s* for $1 \leq q \leq n_i$.

(3) If B is an arbitrary subset of $W_\tau(X)$, then we define $B^{\mathcal{A}} := \bigcup\limits_{b\in B} \{b\}^{\mathcal{A}}$.

If B is empty, then we define $B^{\mathcal{A}} := \emptyset$.

Definition. Let $\mathcal{A}$ be an algebra and let $T, T_j \in \mathcal{P}(W_\tau(X))$ for $1 \leq j \leq n$ such that T, T_j are non-empty. Then the superposition operation

$$\hat{S}^{n,A} : (\mathcal{P}(W_\tau(X))^A)^{n+1} \to \mathcal{P}(W_\tau(X))^A,$$

is inductively defined in the following way:

(1) If $T = \{x_j\}$ for $1 \leq j \leq n$, then

$$\hat{S}^{n,A}(\{x_j\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := T_j^{\mathcal{A}}.$$

(2) If $T = \{x_j\}$ for $n < j$, then

$$\hat{S}^{n,A}(\{x_j\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := \{x_j\}^{\mathcal{A}}.$$

(3) If $T = \{f_i(t_1, \ldots, t_{n_i})\}$ and we assume that $S^{n,A}(\{t_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})$ where $1 \leq q \leq n_i$, are already defined, then
$\hat{S}^{n,A}(\{f_i(t_1, \ldots, t_{n_i})\}_n^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})$
$:= \{f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, \ldots, r_{n_i}^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in \hat{S}^{n,A}(\{t_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}})\}$.

(4) If T is an arbitrary subset of $W_\tau(X)$, then

$$\hat{S}^{n,A}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}) := \bigcup_{t\in T} \hat{S}^{n,A}(\{t\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}).$$

Lemma 7. *Let $\mathcal{A}$ be an agebra of type τ and let T be a subset of $W_\tau(X)$. Then*

$$(\hat{S}^n(T, T_1, \ldots, T_n))^{\mathcal{A}} = \hat{S}^{n,A}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}).$$

Proof. If T is empty, then all is clear.

(1) If T is a one-element set, then we will give a proof by induction on the complexity of term which is the only element of the one-element set T.

(1.1) If $T = \{x_j\}$ where $1 \leq j \leq n$, then

$$\begin{aligned}(\hat{S}^n(T, T_1, \ldots, T_n))^{\mathcal{A}} &= (\hat{S}^n(\{x_j\}, T_1, \ldots, T_n))^{\mathcal{A}} \\ &= T_j^{\mathcal{A}} \\ &= \hat{S}^{n,A}(\{x_j\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \ldots, T_n^{\mathcal{A}}).\end{aligned}$$

(1.2) If $T = \{x_j\}$ where $n < j$, then

$$\begin{aligned}(\hat{S}^n(T, T_1, \dots, T_n))^{\mathcal{A}} &= (\hat{S}^n(\{x_j\}, T_1, \dots, T_n))^{\mathcal{A}} \\ &= \{x_j\}^{\mathcal{A}} \\ &= \hat{S}^{n,A}(\{x_j\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}}).\end{aligned}$$

(1.3) If $T = \{f_i(t_1, \dots, t_{n_i})\}$ and if we assume that the equations

$$(\hat{S}^n(\{s_q\}, T_1, \dots, T_n)^{\mathcal{A}} = \hat{S}^{n,A}(\{s_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}}),$$

where $1 \le q \le n_i$, are satisfied, then

$$\begin{aligned}&(\hat{S}^n(T, T_1, \dots, T_n))^{\mathcal{A}} \\ &= (\hat{S}^n(\{f_i(t_1, \dots, t_{n_i})\}, T_1, \dots, T_n))^{\mathcal{A}} \\ &= (\{f_i(r_1, \dots, r_{n_i}) \mid r_q \in \hat{S}^n(\{t_q\}, T_1, \dots, T_n)\})^{\mathcal{A}} \\ &= \{f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, \dots, r_{n_i}^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in (\hat{S}^n(\{t_q\}, T_1, \dots, T_n))^{\mathcal{A}}\} \\ &= \{f_i^{\mathcal{A}}(r_1^{\mathcal{A}}, \dots, r_{n_i}^{\mathcal{A}}) \mid r_q^{\mathcal{A}} \in \hat{S}^{n,A}(\{t_q\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}})\} \\ &= \hat{S}^{n,A}(\{f_i(t_1, \dots, t_{n_i})\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}}\}) \\ &= \hat{S}^{n,A}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}}).\end{aligned}$$

(2) If T is an arbitrary subset of $W_\tau(X)$, then

$$\begin{aligned}(\hat{S}^n(T, T_1, \dots, T_n))^{\mathcal{A}} &= (\bigcup_{t \in T} \hat{S}^n(\{t\}, T_1, \dots, T_n))^{\mathcal{A}} \\ &= \bigcup_{t \in T} (\hat{S}^n(\{t\}, T_1, \dots, T_n))^{\mathcal{A}} \\ &= \bigcup_{t \in T} \hat{S}^{n,A}(\{t\}^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}}) \\ &= \hat{S}^{n,A}(T^{\mathcal{A}}, T_1^{\mathcal{A}}, \dots, T_n^{\mathcal{A}}).\end{aligned}$$

■

Let B be a subset of $W_\tau(X)$ and let K be a subset of $Alg(\tau)$. Then we set

$$B^K := \bigcup_{\mathcal{A} \in K} B^{\mathcal{A}}.$$

Definition. Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ and $(B_1, B_2) \in \mathcal{P}(W_\tau(X))^2$, written as $B_1 \approx B_2$. Then $\mathcal{A} \models B_1 \approx B_2$ iff $B_1^{\mathcal{A}} = B_2^{\mathcal{A}}$ and $K \models B_1 \approx B_2$ iff $B_1^K = B_2^K$ where $K \subseteq Alg(\tau)$.

Let $\mathcal{K}$ be a subset of $\mathcal{P}(Alg(\tau))$. Then

$$\mathcal{K} \models B_1 \approx B_2 \text{ iff } \forall K \in \mathcal{K}(K \models B_1 \approx B_2).$$

Let PL be a subset of $\mathcal{P}(W_\tau(X))^2 = \mathcal{P}(W_\tau(X) \times W_\tau(X))$. Then $\mathcal{K} \models PL$ iff $K \models B_1 \approx B_2$ for all $K \in \mathcal{K}$ and for all $B_1 \approx B_2 \in PL$.

Let $\mathcal{K}, PL$ be a subset of $\mathcal{P}(W_\tau(X))$, a subset of $\mathcal{P}(W_\tau(X))^2$, respectively. Then we define a mapping $PId : \mathcal{P}(\mathcal{P}(Alg(\tau))) \to \mathcal{P}(\mathcal{P}(W_\tau(X))^2)$ by

$$PId(\mathcal{K}) := \{B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))^2 \mid \forall K \in \mathcal{K}(K \models B_1 \approx B_2)\},$$

and a mapping $PMod : \mathcal{P}(\mathcal{P}(W_\tau(X))^2) \to \mathcal{P}(\mathcal{P}(Alg(\tau)))$ by

$$PMod(PL) := \{K \in \mathcal{P}(Alg(\tau)) \mid \forall B_1 \approx B_2 \in PL(K \models B_1 \approx B_2)\}.$$

In the next lemmas we will show that these two mapping satisfy the Galois-conection properties.

Lemma 8. *Let $\mathcal{P}(Alg(\tau))$ be the class of all subsets of $Alg(\tau)$ and let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \in \mathcal{P}(Alg(\tau))$. Then*

(1) *If $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then $PId\mathcal{K}_2 \subseteq PId\mathcal{K}_1$.*

(2) *$\mathcal{K} \subseteq PModPId\mathcal{K}$.*

Proof. (1) Assume that $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and let $B_1 \approx B_2 \in PId\mathcal{K}_2$. Then for all $K \in \mathcal{K}_2, K \models B_1 \approx B_2$, but we have $\mathcal{K}_1 \subseteq \mathcal{K}_2$, so that $K \models B_1 \approx B_2$ for all $K \in \mathcal{K}_1$. It follows that $B_1 \approx B_2 \in PId\mathcal{K}_1$, and then $PId\mathcal{K}_2 \subseteq PId\mathcal{K}_1$.

(2) Let $K \in \mathcal{K}$. Then $K \models PId\mathcal{K}$, means that $K \in PModPId\mathcal{K}$, and then $\mathcal{K} \subseteq PModPId\mathcal{K}$. ■

In the similarly method, we have

Lemma 9. *Let $\mathcal{P}(W_\tau(X))$ be the set of all subsets of $W_\tau(X)$ and let PL, PL_1, PL_2 be subsets of $\mathcal{P}(W_\tau(X))^2$. Then*

(1) *If $PL_1 \subseteq PL_2$, then $PModPL_2 \subseteq PModPL_1$.*

(2) *$PL \subseteq PIdPModPL$.*

From both lemmas we have that $(PMod, PId)$ is a Galois connection between $\mathcal{P}(Alg(\tau))$ and $\mathcal{P}(W_\tau(X))^2$ with respect to the relation

$$R_{\models} := \{(K, B_1 \approx B_2) \in \mathcal{P}(Alg(\tau)) \times \mathcal{P}(W_\tau(X))^2 \mid K \models B_1 \approx B_2\}.$$

We have two closure operators $PModPId$ and $PIdPMod$ and their sets

$$\{PL \subseteq \mathcal{P}(W_\tau(X))^2 \mid PIdPModPL = PL\}$$

and

$$\{\mathcal{K} \subseteq \mathcal{P}(Alg(\tau)) \mid PModPId\mathcal{K} = \mathcal{K}\}$$

form two complete lattices $\mathcal{E}$ and $\mathcal{L}$, respectively.

If $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ is an algebra of type τ and if $\sigma \in Hyp_G^{nd}(\tau)$ is *gnd*-hypersubstitution, then we define

$$\sigma(\mathcal{A}) := \{A; (l_i^{\sigma(\mathcal{A})})_{i \in I} \mid l_i \in \sigma(f_i)\}.$$

The set $\sigma(\mathcal{A})$ is called the set of derived algebras. Since for every sequence $(l_i)_{i \in I}$ of terms there is a generalized non-deterministic hypersubstitution mapping f_i to l_i, we can write $\sigma(\mathcal{A})$ also in the form

$$\sigma(\mathcal{A}) := \{\rho(\mathcal{A}) \mid \rho \in Hyp_G(\tau) \text{ with } \rho(f_i)^{\mathcal{A}} \in \sigma(f_i)^{\mathcal{A}} \text{ for } i \in I\}.$$

For a class K of algebras of type τ we define

$$\sigma(K) := \bigcup_{\mathcal{A} \in K} \sigma(\mathcal{A}).$$

Definition. Let $B \in \mathcal{P}(W_\tau(X))$, let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ is an algebra of type τ. Let $\sigma \in Hyp_G^{nd}(\tau)$ be *gnd*-hypersubstitution and let $\sigma(\mathcal{A})$ be the set of derived algebras. Then we define the set $B^{\sigma(\mathcal{A})}$ of term operations induced by the set $\sigma(\mathcal{A})$ of derived algebras as follows:

(1) If $B = \{x_j \mid x_j \in X_n\}$, then

$$B^{\sigma(\mathcal{A})} := \{x_j^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A}) \text{ and } \rho \in Hyp_G(\tau)\} = \{e_j^{n,A}\},$$

where $e_j^{n,A} : (a_1, \ldots, a_n) \mapsto a_j$ is an n-ary projection onto the j-th coordinate.

(2) If $B = \{x_j \mid x_j \in X \backslash X_n\}$, then

$$B^{\sigma(\mathcal{A})} := \{x_j^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A}) \text{ and } \rho \in Hyp_G(\tau)\} = \{c_a^n\},$$

where c_a^n is the n-ary constant operation on A with value a and each element from A is uniquely by an element from $X \backslash X_n$.

(3) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$, and if we assume that $\{t_q\}^{\sigma(\mathcal{A})}$ where $1 \le q \le n_i$, are already defined, then

$$B^{\sigma(\mathcal{A})} := \hat{S}^{n,A}(\{f^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})})$$

(4) If B is an arbitrary subset of $W_\tau(X)$, then $B^{\sigma(\mathcal{A})} := \bigcup_{b \in B} \{b\}^{\sigma(\mathcal{A})}$.

If B is empty, then we set $B^{\sigma(\mathcal{A})} := \emptyset$.

Lemma 10. *Let $\mathcal{A}$ be an algebra of type τ, and let $B \in \mathcal{P}(W_\tau(X))$. Let $\sigma \in Hyp_G^{nd}(\tau)$ be a gnd-hypersubstitution of type τ. Then*

$$\hat{\sigma}[B]^{\mathcal{A}} = B^{\sigma(\mathcal{A})}.$$

Proof. If B is empty, then the claim is clearly true.

(1) If B is a one-element set, then we will give a proof by induction on the complexity of the term which forms the only element of the one-element of B.

(1.1) If $B = \{x_j \mid x_j \in X_n\}$, then

$$\begin{aligned}\hat{\sigma}[B]^{\mathcal{A}} &= \hat{\sigma}[\{x_j \mid x_j \in X_n \mid x_j \in X_n\}]^{\mathcal{A}} \\ &= \{x_j \mid x_j \in X_n\}^{\mathcal{A}} \\ &= \{x_j^{\mathcal{A}} \mid x_j \in X_n\} \\ &= \{e_j^{n,A}\} \\ &= \{x_j^{\sigma(\mathcal{A})} \mid x_j \in X_n\} \\ &= \{x_j \mid x_j \in X_n\}^{\sigma(\mathcal{A})} \\ &= B^{\sigma(\mathcal{A})}.\end{aligned}$$

(1.2) If $B = \{x_j \mid x_j \in X \backslash X_n\}$, then

$$\begin{aligned}\hat{\sigma}[B]^{\mathcal{A}} &= \hat{\sigma}[\{x_j \mid x_j \in X \backslash X_n\}]^{\mathcal{A}} \\ &= \{x_j \mid x_j \in X \backslash X_n\}^{\mathcal{A}} \\ &= \{x_j^{\mathcal{A}} \mid x_j \in X \backslash X_n\} \\ &= \{c_a^n \mid a \in A\} \\ &= \{x_j^{\sigma(\mathcal{A})} \mid x_j \in X \backslash X_n\} \\ &= \{x_j \mid x_j \in X \backslash X_n\}^{\sigma(\mathcal{A})} \\ &= B^{\sigma(\mathcal{A})}.\end{aligned}$$

(1.3) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$, and we assume that the equations

$$\hat{\sigma}[\{t_q\}]^{\mathcal{A}} = \{t_q\}^{\sigma(\mathcal{A})}$$

where $1 \leq q \leq n_i$, then

$$\begin{aligned}\hat{\sigma}[B]^{\mathcal{A}} &= \hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}]^{\mathcal{A}} \\ &= (\hat{S}^n(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]))^{\mathcal{A}} \\ &= \hat{S}^{n,A}(\sigma(f_i)^{\mathcal{A}}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}}) \\ &= \hat{S}^{n,A}(\{l_i \mid l_i \in \sigma(f_i)\}^{\mathcal{A}}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}})\end{aligned}$$

$$
\begin{aligned}
&= \hat{S}^{n,A}(\{l_i^{\mathcal{A}} \mid l_i \in \sigma(f_i)\}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}}) \\
&= \bigcup_{l_i \in \sigma(f_i)} \hat{S}^{n,A}(\{l_i^{\mathcal{A}}\}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \ldots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}}) \\
&= \bigcup_{l_i \in \sigma(f_i)} \hat{S}^{n,A}(\{\rho(f_i)^{\mathcal{A}} \mid \rho(f_i) = l_i \text{ for some } \rho \in Hyp_G(\tau) \\
&\quad \text{and } \rho(\mathcal{A}) \in \sigma(\mathcal{A})\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})}) \\
&= \hat{S}^{n,A}(\{f_i^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\}, \{t_1\}^{\sigma(\mathcal{A})}, \ldots, \{t_{n_i}\}^{\sigma(\mathcal{A})}) \\
&= \{f_i(t_1, \ldots, t_{n_i})\}^{\sigma(\mathcal{A})} \\
&= B^{\sigma(\mathcal{A})}.
\end{aligned}
$$

(2) If B is an arbitrary subset of $W_\tau(X)$, then

$$
\begin{aligned}
\hat{\sigma}[B]^{\mathcal{A}} &= (\bigcup_{b \in B} \hat{\sigma}[\{b\}])^{\mathcal{A}} \\
&= \bigcup_{b \in B} \{b\}^{\sigma(\mathcal{A})} \\
&= (\bigcup_{b \in B} \{b\})^{\sigma(\mathcal{A})} \\
&= B^{\sigma(\mathcal{A})}.
\end{aligned}
$$

■

Lemma 11. *Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ and $\sigma_1, \sigma_2 \in Hyp_G^{nd}(\tau)$. Then we have $\sigma_1(\sigma_2(\mathcal{A})) = (\sigma_1 \circ_g \sigma_2)(\mathcal{A})$.*

Proof. By Lemma 10, we have

$$
\begin{aligned}
\sigma_1(\sigma_2(\mathcal{A})) &= \{\beta(\rho(\mathcal{A})) \mid \rho(f_i)^{\beta(\mathcal{A})} \in \sigma_1(f_i)^{\sigma_2(\mathcal{A})}, \beta(\mathcal{A}) \in \sigma_2(\mathcal{A}), i \in I\} \\
&= \{(\beta \circ_G \rho)(\mathcal{A}) \mid (\beta \circ_G \rho)(f_i)^{\mathcal{A}} \in \hat{\sigma}_2[\sigma_1(f_i)]^{\mathcal{A}}, \beta(\mathcal{A}) \in \sigma_2(\mathcal{A}), i \in I\} \\
&= \{(\beta \circ_G \rho)(\mathcal{A}) \mid (\beta \circ_G \rho)(f_i)^{\mathcal{A}} \in (\sigma_2 \circ_g \sigma_1)(f_i)^{\mathcal{A}}, \beta(\mathcal{A}) \in \sigma_2(\mathcal{A}), \\
&\quad i \in I\} \\
&= (\sigma_2 \circ_g \sigma_1)(\mathcal{A}).
\end{aligned}
$$

■

Lemma 12. *Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra and $\sigma_{id} \in Hyp_G^{nd}(\tau)$. Then $\sigma_{id}(\mathcal{A}) = \{\mathcal{A}\}$.*

Proof. By Lemma 11, we have

$$
\begin{aligned}
\sigma_{gid}(\mathcal{A}) &= \{\rho(\mathcal{A}) \mid \rho(f_i)^{\mathcal{A}} \in \sigma_{gid}(f_i)^{\mathcal{A}}, i \in I\} \\
&= \{\rho(\mathcal{A}) \mid \rho(f_i)^{\mathcal{A}} \in \{f_i(x_1, \ldots, x_{n_i})\}^{\mathcal{A}}, i \in I\} \\
&= \{\rho(\mathcal{A}) \mid \rho(f_i)^{\mathcal{A}} \in \{f_i^{\mathcal{A}}(x_1^{\mathcal{A}}, \ldots, x_{n_i}^{\mathcal{A}})\}, i \in I\} \\
&= \{\rho(\mathcal{A}) \mid \rho(f_i)^{\mathcal{A}} = f_i^{\mathcal{A}} = \sigma_{gid}(f_i)^{\mathcal{A}} \in \{f_i^{\mathcal{A}}\}, i \in I\} \\
&= \{\sigma_{gid}(\mathcal{A}) \mid \rho(f_i)^{\mathcal{A}} = f_i^{\mathcal{A}} = \sigma_{gid}(f_i)^{\mathcal{A}} \in \{f_i^{\mathcal{A}}\}, i \in I\} \\
&= \{\mathcal{A} \mid \rho(f_i)^{\mathcal{A}} = f_i^{\mathcal{A}} = \sigma_{gid}(f_i)^{\mathcal{A}} \in \{f_i^{\mathcal{A}}\}, i \in I\}.
\end{aligned}
$$

■

Let $(M; \circ_g, \sigma_{gid})$ be a submonoid of $(Hyp_G^{nd}(\tau); \circ_g, \sigma_{gid})$ and let $B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))$. For every $\mathcal{A} \models B_1 \approx B_2$ such that $\mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2]$ for all $\sigma \in M$, we written as, $\mathcal{A} \models_{M-gnd-hyp} B_1 \approx B_1$ and $K \models_{M-gnd-hyp} B_1 \approx B_2$ for all $\mathcal{A} \in K$.

Now we define two mappings which give a second Galois connection. Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\tau))$ and $PL \subseteq \mathcal{P}(W_\tau(X))^2$. Then we define a mapping

$$H_M PId : \mathcal{P}(\mathcal{P}(Alg(\tau))) \to \mathcal{P}(\mathcal{P}(W_\tau(X))^2),$$

by

$$H_M PId\mathcal{K} := \{B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))^2 \mid \forall K \in \mathcal{K}(K \models_{M-gnd-hyp} B_1 \approx B_2)\}$$

and define a mapping

$$H_M PMod : \mathcal{P}(\mathcal{P}(W_\tau(X))^2) \to \mathcal{P}(\mathcal{P}(Alg(\tau))),$$

by

$$H_M PModPL := \{K \in \mathcal{P}(Alg(\tau)) \mid \forall B_1 \approx B_2 \in PL(K \models_{M-gnd-hyp} B_1 \approx B_2)\}.$$

It is easy to see that $(H_M PMod, H_M PId)$ is a Galois connection between $\mathcal{P}(Alg(\tau))$ and $\mathcal{P}(W_\tau(X))^2$ with respect to the second relation

$$R_{\models_{M-gnd-hyp}}$$
$$:= \{(K, B_1 \approx B_2) \in \mathcal{P}(Alg(\tau)) \times \mathcal{P}(W_\tau(X))^2 \mid K \models_{M-gnd-hyp} B_1 \approx B_2\}.$$

We have two closure operators $H_M PModH_M PId$ and $H_M PIdH_M PMod$ and their sets

$$\{PL \subseteq \mathcal{P}(W_\tau(X))^2 \mid H_M PIdH_M PModPL = PL\}$$

and

$$\{\mathcal{K} \subseteq \mathcal{P}(Alg(\tau)) \mid H_M PModH_M PId\mathcal{K} = \mathcal{K}\}$$

form two complete sublattices $\mathcal{SE}$, $\mathcal{SL}$ of $\mathcal{E}$, $\mathcal{L}$, respectively.

Theorem 13. *Let $\mathcal{A}$ be an algebra of type τ, and $\sigma \in Hyp_G^{nd}(\tau)$. Let $B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))^2$. Then*

$$\sigma(\mathcal{A}) \models B_1 \approx B_2 \Leftrightarrow \mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2].$$

Proof. By Lemma 10, we have

$$\begin{aligned} \sigma(\mathcal{A}) \models B_1 \approx B_2 &\Leftrightarrow B_1^{\sigma(\mathcal{A})} = B_2^{\sigma(\mathcal{A})} \\ &\Leftrightarrow \hat{\sigma}[B_1]^{\mathcal{A}} = \hat{\sigma}[B_2]^{\mathcal{A}} \\ &\Leftrightarrow \mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2]. \end{aligned}$$

■

Definition. Let $\mathcal{K} \subseteq \mathcal{P}(Alg(\tau))$ and $PL \subseteq \mathcal{P}(W_\tau(X))^2$. Then we set

$$\chi_M^E[B_1 \approx B_2] := \{\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \mid \sigma \in M\}$$

and

$$\chi_M^A[K] := \{\sigma(K) \mid \sigma \in M\}.$$

We define two operators in the following way:

$$\chi_M^E : \mathcal{P}(\mathcal{P}(W_\tau(X))^2) \to \mathcal{P}(\mathcal{P}(W_\tau(X))^2)$$

by $\chi_M^E[PL] := \{\chi_M^E[B_1 \approx B_2] \mid B_1 \approx B_2 \in PL\}$ and

$$\chi_M^A : \mathcal{P}(\mathcal{P}(Alg(\tau))) \to \mathcal{P}(\mathcal{P}(Alg(\tau)))$$

by $\chi_M^A[\mathcal{K}] := \{\chi_M^A[K] \mid K \in \mathcal{K}\}$.

In the next lemmas we will show that the both operators are closure operators.

Lemma 14. *Let PL, PL_1, PL_2 be subsets of $\mathcal{P}(W_\tau(X))^2$. Then*

(i) $PL \subseteq \chi_M^E[PL]$.

(ii) $PL_1 \subseteq PL_2 \Rightarrow \chi_M^E[PL_1] \subseteq \chi_M^E[PL_2]$.

(iii) $\chi_M^E[PL] = \chi_M^E[\chi_M^E[PL]]$.

Proof. (i) Let $B_1 \approx B_2 \in PL$. Then, since $B_1 = \hat{\sigma}_{gid}[B_1]$ and $B_2 = \hat{\sigma}_{gid}[B_2]$, we have $\hat{\sigma}[B_1] = B_1 \approx B_2 = \hat{\sigma}[B_2] \in \chi_M^E[PL]$ and then $PL \subseteq \chi_M^E[PL]$.

(ii) Assume that $PL_1 \subseteq PL_2$ and let $\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \in \chi_M^E[PL_1]$. Then $B_1 \approx B_2 \in PL_2$ but $PL_1 \subseteq PL_2$, so that $B_1 \approx B_2 \in PL_2$ and $\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \in \chi_M^E[PL_2]$. We have $\chi_M^E[PL_1] \subseteq \chi_M^E[PL_2]$.

(iii) By (i) we have $\chi_M^E[PL] \subseteq \chi_M^E[[\chi_M^E[PL]]$. Let $\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \in \chi_M^E[\chi_M^E[PL]]$. Then $B_1 \approx B_2 \in \chi_M^E[PL]$, and there exists $\rho \in M$ and $C_1 \approx C_2 \in PL$ such that $B_1 = \hat{\rho}[C_1]$ and $B_2 = \hat{\rho}[C_2]$, and then

$$\begin{aligned}\hat{\sigma}[B_1] &= \hat{\sigma}[\hat{\rho}[C_1]] \\ &= (\hat{\sigma} \circ \hat{\rho})[C_1] \\ &= (\sigma \circ_g \rho)\hat{}[C_1] \\ &= \hat{\lambda}[C_1] \text{ where } \lambda = \sigma \circ_g \rho \in M \text{ and}\end{aligned}$$

$$\begin{aligned}\hat{\sigma}[B_2] &= \hat{\sigma}[\hat{\rho}[C_2]] \\ &= (\hat{\sigma} \circ \hat{\rho})[C_2] \\ &= (\sigma \circ_g \rho)\hat{}[C_2] \\ &= \hat{\lambda}[C_2] \text{ where } \lambda = \sigma \circ_g \rho \in M.\end{aligned}$$

Then we set $\hat{\lambda}[C_1] = \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] = \hat{\lambda}[C_2] \in \chi_M^E[PL]$, and obtain $\chi_M^E[\chi_M^E[PL]] \subseteq \chi_M^E[PL]$. ∎

Similarly method we have

Lemma 15. *Let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be subset of $\mathcal{P}(Alg(\tau))$. Then*

(i) $\mathcal{K} \subseteq \chi_M^A[\mathcal{K}]$,

(ii) $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \chi_M^A[\mathcal{K}_1] \subseteq \chi_M^A[\mathcal{K}_2]$,

(iii) $\chi_M^A[\mathcal{K}] = \chi_M^A[\chi_M^A[\mathcal{K}]]$.

The next theorem needs the concept of a conjugate pair of additive closure operators.

Theorem 16. *The pair (χ_M^A, χ_M^E) is a conjugate pair of completely additive closure operators with respect to the relation $R_{\models}$.*

Proof. By Theorem 13, Lemma 14 and Lemma 15. ∎

Definition. Let $(M; \circ_g, \sigma_{gid})$ be a submonoid of $(Hyp_G^{nd}(\tau); \circ_g, \sigma_{gid})$. A variety V of type τ is said to be an M-solid generalized non-deterministic variety, for short an M-*gnd*-solid variety, if $\{\{\mathcal{A}\} \mid \mathcal{A} \in V\} \models \{\hat{\sigma}[\{s\}] \approx \hat{\sigma}[\{t\}] \mid s \approx t \in IdV, \sigma \in M\}$ and $s \approx t$ is also said to be an M-solid generalized non-deterministic hyperidentity in V, for short an M-*gnd* hyperidentity in V. In this case that $M = Hyp_G^{nd}(\tau)$ we will speak of a solid generalized non-deterministic variety, for short of a *gnd*-solid variety and generalized non-deterministic hyperidentity, for short of a *gnd* hyperidentity, respectively.

Now we may apply the theory of conjugate pairs of additive closure operators (see [4]) and obtain the following propositions:

Lemma 17. *Let $V \subseteq Alg(\tau)$ be a class of algebras and $\Sigma \subseteq W_\tau(X)^2$. Let $V^* = \{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$ and $\Sigma^* = \{\hat{\sigma}[\{s\}] \approx \hat{\sigma}[\{t\}] \mid s \approx t \in \Sigma, \sigma \in M\}$. Then the following properties hold:*

(i) $H_M PIdV^* = PId\chi_M^A[V^*]$.

(ii) $H_M PIdV^* \subseteq PIdV^*$.

(iii) $\chi_M^E[H_M PIdV^*] = H_M PIdV^*$.

(iv) $\chi_M^A[PModH_M PIdV^*] = PModH_M PIdV^*$.

(v) $H_M PIdH_M PMod\Sigma^* = PIdPMod\chi_M^E[\Sigma^*]$.

Using these propositions one obtains the following characterization of M-solid generalized non-deterministic varieties.

Theorem 18. *Let V be a subset of $Alg(\tau)$. Let $V^* = \{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$. Then the following properties are equivalent:*

(i) $H_M PModH_M PIdV^* = V^*$.

(ii) $\chi_M^A[V^*] = V^*$ (*i.e.* V^* *is an* M*-gnd-solid variety*).

(iii) $PIdV^* = H_M PIdV^*$ (*i.e. every identity in* V^* *is satisfied as an* M*-gnd hyperidentity*).

(v) $\chi_M^E[PIdV^*] = PIdV^*$.

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