Discussiones Mathematicae
General Algebra and Applications 36 (2016) 5–13
doi:10.7151/dmgaa.1246

AN EQUATIONAL AXIOMATIZATION OF POST ALMOST DISTRIBUTIVE LATTICES

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Abstract

In this paper, we prove that the class of P_2-Almost Distributive Lattices and Post Almost Distributive Lattices are equationally definable.

Keywords: Almost Distributive Lattices (ADL), P_2-algebras, P_2-Almost Distributive Lattices (P_2-ADL), Post algebras, Post Almost Distributive Lattices (Post ADL).

2010 Mathematics Subject Classification: 03G20, 06D99.

1. Introduction

A complete list of all closed classes of Boolean functions was first given by Emil Post in 1941, see [4]. This list of closed classes of Boolean functions is nowadays called "Posts". Moreover, Emil Post proved that each of them has a finite basis and he obtained a list of bases for all closed classes [3]. Later on, G. Epstein, T. Traczyk and Ph. Dwinger developed the concept of Post algebra, see [2]. Thus, the Post lattice has become a useful tool in complexity which examines the Boolean circuits and propositional formulas. The Post lattice can be a very helpful tool for complexity studies in Boolean circuits and propositional formulas. The Boolean circuits and Boolean functions attract and deserve a lot of attention in theorectical computer science, and the theory behind them is exhaustively used in circuit design and various other important fields. The mathematical operations that correspond to the operations carried out when soldering the Boolean circuits.

While studying the properties of Post algebra, G. Epstein introduced the concepts of P_2-lattices and $P2$-lattices [7] which are interesting in computer science. These two kinds of lattices can be applied to the theory of machines with m_i-stable devices. The P_2-lattices provide the complete multiple-valued logics. In 1981, Swamy and Rao [16] introduced the concept of an "Almost Distributive Lattice" (or, in brevity simply write ADL) as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. The theory of ADL have many applications in chemistry, applied chemistry, botany, automata theory, cryptography and modeling.

In this paper, we derive the properties of a P_2-almost distributive lattice [7] and also we prove that the class of P_2-almost distributive lattices as well as Post almost distributive lattices are equationally definable [7]. These properties will help further investigations of possible applications of Post almost distributive lattice in logic and computer science.

2. Preliminaries

In this section, we give the necessary definitions and important properties of an ADL, Birkhoff center of an ADL, Heyting ADL and BL-ADL [5]. We begin with the following definition. For more information in theory of lattice,the reader is referred to Birkhoff [1].

Definition 1. An algebra $(A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is said to be an almost distributive Lattice (ADL) (see [16]) if it satisfies the following axioms:

(i) $x \vee 0 = x$

(ii) $0 \wedge x = 0$

(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(vi) $(x \vee y) \wedge y = y$, for all $x, y, z \in A$.

Theorem 2 [15]**.** *Let m be a maximal element in an ADL A and $x \in A$. Then the following statements are equivalent:*

(i) *x is a maximal element of $(A, \leq)$.*

(ii) $x \wedge m = m$.

(iii) $x \wedge a = a$, *for all* $a \in A$.

For other properties of an Post ADL, we refer the readers to [13].

The concept of Birkhoff Center of an almost distributive lattice was introduced by Swamy and Ramesh in [17]. The following definition is taken from [17].

Definition 3 [17]**.** Let A be an ADL with a maximal element m and $B(A) = \{x \in A \mid x \wedge y = 0$ and $x \vee y$ is maximal for some $y \in A\}$. Then $(B(A), \vee, \wedge)$ is a relatively complemented ADL and it is called the Birkhoff center of A. We use the symbol B instead of $B(A)$ when there is no ambiguity.

For any $b \in B,\ b \wedge m$ is a complemented element in the distributive lattice $[0, m]$, whose complement will be denoted by b^m.

In [5], Rao and Berhanu introduced the concept of Heyting almost distributive lattice [5] as a generalization of a Heyting algebra as follows.

Definition 4 [5]**.** An $(A, \vee, \wedge, 0)$ is an ADL with a maximal element m said to be a Heyting ADL (or, simply a H-ADL) (see [5]) if there exists a binary operation $\rightarrow$ on A such that he following conditions hold:

(a) $x \rightarrow x = m$

(b) $(x \rightarrow y) \wedge y = y$

(c) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$

(d) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

(e) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

The concepts of pseudo-supplemented almost distributive lattice (or, simply PSADL) [15], B-almost distributive lattice (or, simply B-ADL) [10] BL-almost distributive lattice (or, simply BL-ADL) [11] see the papers [10] and [11, 12] respectively and derived their properties.

Definition 5 [12]**.** An ADL $(A, \vee, \wedge, 0)$ with a maximal element m and with a Birkhoff center B is called a B-ADL [10] if for any $x, y \in A$, there exists an element $b \in B$ such that

(i) $y \wedge x \wedge b = x \wedge b$.

(ii) If $c \in B$ such that $y \wedge x \wedge c = x \wedge c$, then $b \wedge c = c$.
In this case, $b \wedge m$ is denoted by $x \Rightarrow y$. A B-ADL $(A, \vee, \wedge, 0)$ with a maximal element m and Birkhoff center B is said to be a BL-ADL, if for any $x, y \in A, (x \Rightarrow y) \vee (y \Rightarrow x) = m$. Here $m \Rightarrow x$ is called the pseudo-supplement of x and it is denoted by $x!$.

For other properties of PSADL,B-ADL and BL-ADL, we refer the reader to [15, 10, 11] and [12].

3. P_2-ADLs

As mentioned earlier, While studying the Post algebra,the concept of P_0-lattice, P_1-lattice and P_2-lattice was introduced by G. Epstein as follows.

Definition 6. Let A be a distributive lattice with $0, 1$ and let a sublattice $B \subset A$ be a Boolean algebra of complemented elements of A. If there exists an ascending sequence $0 = e_0 \leq e_1 \leq e_2 \leq \cdots \leq e_{n-1} = 1$, where n is an integer ≥ 2, of elements of A such that every $x \in A$ can be written in the form $x = \bigvee_{i=1}^{n-1} b_i \wedge e_i$, where $b_1, b_2, \ldots, b_{n-1} \in B$, then A will be called a P_0-lattice. A P_0-algebra A is called a P_1-algebra if for every i, $(e_{i+1} \to e_i = e_i)$ exists. A P_1-algebra is called a P_2-algebra if for every $i, e_i!$ exists.

Since the class of P_2-algeba are very much interesting in computers and logic. For this reason, we introduced the concepts of P_1-Almost Distributive Lattice as well as P_2-Almost Distributive Lattice and derived its properties.

Definition 7. Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m and Birkhoff center B. Then A is said to be a P_0-almost Distributive Lattice (or, simply a P_0-ADL) if and only if there exist elements $0 = e_0, e_1, e_2, \ldots, e_{n-1}$ in A such that:

(i) $e_{n-1} \wedge m = m$

(ii) $e_i \wedge e_{i-1} = e_{i-1}$, for $1 \leq i \leq n-1$

(iii) For any $x \in A$, there exist $b_i \in B$ such that $x \wedge m = \bigvee_{i=1}^{n-1}(b_i \wedge e_i \wedge m)$.

A set $\{0 = e_0, e_1, e_2, \ldots, e_{n-1}\}$ of elements in a P_0-ADL A satisfying conditions (i), (ii) and (iii) is called a chain base of A.

A P_0-ADL $(A; e_0, e_1, \ldots, e_{n-1})$ with a maximal element m and Birkhoff center B is said to be a P_1-Almost Distributive Lattice (or, simply a P_1-ADL) if for every i, $e_{i+1} \to e_i = e_i$ exists [9]. A P_1-ADL $(A; e_0, e_1, \ldots, e_{n-1})$ with a maximal element m and Birkhoff center B i is called a P_2-almost distributive lattice (or, simply a P_2-ADL) [7] if for every $i, e_i!$ exists.

Definition 8. Let $(A; e_0, e_1, \ldots, e_{n-1})$ be a P_0-ADL with a maximal element m, Birkhoff center B and $x \in A$ such that

$$x \wedge m = \bigvee_{i=1}^{n-1} (b_i \wedge e_i \wedge m) \text{ where } b_i \in B. \quad \ldots\ldots\ldots \quad (\star)$$

(i) If $b_i \wedge b_{i+1} = b_{i+1}$ for $1 \leq i \leq n-2$, then $(\star)$ is called a monotone representation of x, or simply as mono. rep.

(ii) If $b_i \wedge b_j = 0$ for $i \neq j$, then $(\star)$ is called a disjoint representation of x, or simply as dis. rep.

We observe that every element in a P_0-ADL has both a mono. and dis. representation. For other properties of P_0-ADL, refer [6] and [14].

Through out this section, we write $(A; e_0, e_1, \ldots, e_{n-1})$ to stand for a P_2-ADL with a maximal element m and Birkhoff center B unless mentioned otherwise.

In the following theorem, we give an important property of a P_2-ADL which will be useful to characterize a P_2-ADL.

Theorem 9. *Let $(A; e_0, e_1, \ldots, e_{n-1})$ be a P_2-ADL. Then $D_i(x) = e_i \Rightarrow x$, $C_i(x) = (D_i(x) \wedge {D_{i+1}}^m(x))$ and for $i < n-1$, $C_i(x) = ((x \Rightarrow e_i) \wedge (e_i \Rightarrow x)$ $\wedge\{(x \wedge e_i)!\}^m)$.*

Proof. The proof is straightforward and is hence omitted. ■

We observe that every P_2-ADL [7] is a BL-ADL [10] and by Theorem 3.10. [11], we have $D_i(x \vee y) = D_i(x) \vee D_i(y)$ and $D_i(x \wedge y) = D_i(x) \wedge D_i(y)$.

Now, we prove the following theorem.

Theorem 10. *Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m and the Birkhoff center B. Then A is a P_2-ADL if and only if A satisfies the following:*

A_1 : (a) $x \wedge e_0 = e_0$.

(b) $e_j \wedge e_i = e_i$ *for* $0 \leq i \leq j \leq n-1$.

(c) $e_{n-1} \wedge x = x$.

A_2 : (a) $C_i(x) \wedge C_j(x) = e_0$ *for* $i \neq j$

(b) $(C_0(x) \vee C_1(x) \vee C_2(x) \vee \cdots \vee C_{n-1}(x)) \wedge m = e_{n-1} \wedge m$.

A_3 : (a) $C_i(x \wedge y) = \{C_i(x) \wedge \bigvee_{j=1}^{n-1} C_j(y)\} \vee \{C_i(y) \wedge \bigvee_{j=1}^{n-1} C_j(x)\}$.

(b) $C_{n-1}(x \vee y) = C_{n-1}(x) \vee C_{n-1}(y)$.

A_4 : (a) $C_i(e_j) = e_0$ *for* $j \neq i$ *and* $i < n-1$.

(b) $C_{n-1}(e_0) = e_0$.

A_5 : $x \wedge m = \bigvee_{i=1}^{n-1}(C_i(x) \wedge e_i \wedge m)$.

Proof. Suppose that $(A; e_0, e_1, \ldots, e_{n-1})$ is a P_2-ADL. Then we get A_1 and A_5. Since $C_{n-1}(x) = x! = D_{n-1}(x)$, we get $D_{n-1}(x \wedge e_0) = D_{n-1}(x) \wedge D_{n-1}(e_0)$ and hence $C_{n-1}(e_0) = e_0$. We have $C_i(e_j) = \{(e_j \Rightarrow e_i) \wedge (e_i \Rightarrow e_j) \wedge \{(e_i \wedge e_j)!\}^m\} = 0$. Then $C_i(x) \wedge C_j(x) = \{D_i(x) \wedge \{D_{i+1}(x)\}^m \wedge D_j(x) \wedge \{D_{j+1}(x)\}^m\}$ and hence we get $C_i(x) \wedge C_j(x) = 0$. We can routinely verify that $(C_0(x) \vee C_1(x) \vee \cdots \vee C_{n-1}(x)) \wedge m = e_{n-1} \wedge m$ and $C_i(x \wedge y) = (C_i(x) \wedge \bigvee_{j=1}^{n-1} C_j(y)) \vee (C_i(y) \wedge \bigvee_{j=1}^{n-1} C_j(x))$.

Conversely, suppose that the conditions A_1 to A_5 hold. Then,by using conditions A_1 and A_5, we deduce that $(A; e_0, e_1, \ldots, e_{n-1})$ is a P_0-ADL. To prove A is a P_2-ADL, it suffices to show that for every i, $(e_i \to e_{i-1})$ exist and $e_i!$. We have $C_{n-1}(x) = x! = D_{n-1}(x)$. Now, suppose that $x \wedge e_i \wedge m = e_{i-1} \wedge m$. Then, by condition A_3 and A_4, we obtain that $x \wedge m = e_{i-1} \wedge m$ and hence $e_i \to e_{i-1}$ exist for all i. Therefore A is indeed a P_2-ADL. ■

We now proceed to prove another characterization theorem of a P_2-ADL [7].

Theorem 11. *Let $(A; e_0, e_1, \ldots, e_{n-1})$ be a P_2-ADL. Then we have the following properties.*

R_1 : $(A, \vee, \wedge, \Rightarrow, e_0, e_1, \ldots, e_{n-1})$ *is a BL-ADL, see* [12].

R_2 : $e_j \wedge e_i = e_i$ *for* $i \leq j$.

R_3 : $(e_{i+1} \wedge (e_i \Rightarrow e_j)) \leq e_j \wedge m$ *for* $j < i < n-1$.

R_4 : $x \wedge m = \bigvee_{i=1}^{n-1}(e_i \wedge (e_i \Rightarrow x))$.

Proof. Suppose that $(A; e_0, e_1, \ldots, e_{n-1})$ is a P_2-ADL [7]. Then we get A_1, A_2, A_5 and $A_4(b)$ by the conditions of Theorem 3.5. Since every P_2-ADL is a BL-ADL, $D_i(x \vee y) = D_i(x) \vee D_i(y)$ and $D_i(x \wedge y) = D_i(x) \wedge D_i(y)$ and hence we prove the condition A_3. In a P_2-ADL, for every i, $(e_{i+1} \Rightarrow e_i)$ exist if and only if $(e_i \Rightarrow e_j)$ exist for all i and j. Hence, we obtain $A_4.(a)$. Therefore, $(A; e_0, e_1, \ldots, e_{n-1})$ is a P_2-ADL. ■

4. Post ADL

A Post algebra is a P_2-algebra $(A; e_0, e_1, \ldots, e_{n-1})$ such that $e_{n-2}! = 0$. As mentioned above, the concept of a Post Algebra is very much interesting in computers and logic. For this reason, Rao introduced the concept of Post almost distributive lattice [13] as a generalization of Post algebra in [8]. Now we begin with the following definition.

Definition 12. A post almost distributive lattice (or, simply a Post ADL) is a P_2-almost distributive lattice $(A; e_0, e_1, \ldots, e_{n-1})$ with a maximal element m and Birkhoff center B such that $e_{n-2}! = 0$.

The following theorem was taken from the paper [13] and it will be useful to characterize a Post ADL.

Theorem 13. *Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m, Birkhoff center B and let n be a fixed integer satisfying $n \geq 2$. Then A is a post almost distributive lattice if and only if it satisfying the following conditions:*

A_1 : *For every element $x \in A$, there exist n elements $C_0(x), C_1(x), \ldots, c_{n-1}(x)$ which are pairwise disjoint and $(C_0(x) \vee C_1(x) \vee \cdots \vee c_{n-1}(x)) \wedge m = m$.*

A_2 : *There exist n fixed elements of A, denoted by $0 = e_0, e_1, e_2, \ldots, e_{n-1} \wedge m = m$ with the following properties:*

$2a.$ $e_i \wedge e_{i-1} = e_{i-1}$, *for* $1 \leq i \leq n-1$.

$2b.$ *If* $x \in A$ *and* $x \wedge e_1 = 0$, *then* $x = 0$.

$2c.$ *If* $x \in A$ *and some* i, $(x \vee e_{i-1}) \wedge m = e_i \wedge m$, *then* $x \wedge m = e_i \wedge m$.

A_3 : *For every* $x \in A$, $x \wedge m = \bigvee_{i=0}^{n-1} (C_i(x) \wedge e_i \wedge m)$.

Now we prove that the class of post almost distributive lattices is equationally definable where the only unary operations are the disjoint operators C_i, $i = 1, 2, \ldots, n-1$.

Theorem 14. *Let $(A, \vee, \wedge, 0)$ be an ADL with a maximal element m and Birkhoff center B. Then A is a Post ADL* [13] *if and only if it satisfying the following conditions:*

R_1 : (i) $e_0 \vee x = x$.

(ii) $e_j \wedge e_i = e_i$ *for* $i \leq j$.

(iii) $x \wedge e_{n-1} \wedge m = x \wedge m$.

R_2 : (i) $C_i(x) \wedge C_j(x) = e_0$ *for* $i \neq j$.

(ii) $(C_0(x) \vee C_1(x) \vee C_2(x) \vee \cdot \vee C_{n-1}(x)) \wedge m = e_{n-1} \wedge m$.

R_3 : (i) $C_i(x \vee y) = \{C_i(x) \wedge \bigvee_{j=0}^{n-1} C_j(y)\} \vee \{C_i(y) \wedge \bigvee_{j=0}^{n-1} C_j(x)\}$.

(ii) $C_0(x \wedge y) = C_0(x) \vee C_0(y)$.

R_4 : $C_i(e_j) = e_0$ *for* $i \neq j$

R_5 : $x \wedge m = \bigvee_{i=1}^{n-1} C_i(x) \wedge e_i \wedge m$.

Proof. Suppose that conditions R_1 to R_5 hold. Then, by conditions R_1, R_2 and R_5, we deduce that $(A; e_0, e_1, \ldots, e_{n-1})$ is a P_0-ADL. Since $(C_0(x) \vee C_1(x) \vee C_2(x) \vee \cdots \vee C_{n-1}(x)) \wedge m = m$, we get $C_i(e_i) \wedge m = m$.

Suppose that $x \wedge e_1 = 0$.

$$\begin{aligned} &\text{Then } \{C_0(x \wedge e_1)\} \wedge m = \{C_0(0)\} \wedge m \\ &\Longrightarrow \{C_0(x) \vee C_0(e_1)\} \wedge m = \{C_0(e_0)\} \wedge m \\ &\Longrightarrow \{C_0(x) \vee 0\} \wedge m = m \\ &\Longrightarrow C_0(x) \wedge m = m. \end{aligned}$$

Thus, by condition R_2, we deduce that $C_1(x) = C_2(x) = \cdots = C_{n-1}(x) = 0$. Since $x \wedge m = \bigvee_{i=1}^{n-1} (C_i(x) \wedge e_i \wedge m)$ and $x \wedge e_1 \wedge m = 0$, we get $x = 0$. To prove that A is a Post ADL, it suffices to show that if $(x \vee e_{i-1}) \wedge m = e_i \wedge m$ implies $x \wedge m = e_i \wedge m$.

Suppose that $(x \vee e_{i-1}) \wedge m = e_i \wedge m$.

$$\text{Then } \{C_i(x \vee e_{i-1})\} \wedge m = \{C_i(e_i)\} \wedge m$$
$$\Longrightarrow \left\{\left\{C_i(x) \wedge \bigvee_{j=0}^{n-1} C_j(e_{i-1})\right\} \vee \left\{C_i(e_{i-1}) \wedge \bigvee_{j=0}^{n-1} C_j(x)\right\}\right\} \wedge m = m$$
$$\Longrightarrow \{C_i(x)\} \wedge m = m \text{ for } i > 1$$
$$\text{and } \{C_j(x \vee e_{i-1})\} \wedge m = \{C_j(x)\} \wedge m = \{C_j(e_i)\} \wedge m = 0 \text{ for } j > i.$$

$$\text{So that we have } x \wedge m = \left[\left\{\bigvee_{j=1}^{i-1} C_j(x) \wedge e_j \wedge m\right\} \vee e_i\right] \wedge m = e_i \wedge m.$$

Hence, A is a Post ADL. Conversely suppose that A is a Post ADL. Then we get R_1, R_2 and R_5.

Because every Post ADL is a P_2-ADL and hence A must be a BL-ADL. Thus we obtain the conditions R_3 and R_4. ■

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Received 22 October 2014
Revised 11 March 2016