# SUPERIOR SUBALGEBRAS AND IDEALS OF BCK/BCI-ALGEBRAS 

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#### Abstract

The notions of superior subalgebras and (commutative) superior ideals are introduced, and their relations and related properties are investigated. Conditions for a superior ideal to be commutative are provided.


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## 1. Introduction

Algebras have played an important role in pure and applied mathematics and have its comprehensive applications in many aspects including dynamical systems and genetic code of biology (see [1, 2, 6], and [11]). Starting from the four DNA bases order in the Boolean lattice, Sáanchez et al. [10] proposed a novel Lie Algebra of the genetic code which shows strong connections among algebraic

[^0]relationship, codon assignments and physicochemical properties of amino acids. A BCK/BCI-algebra (see $[3,4,9]$ ) is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers. Jun and Song [5] introduced the notion of BCK-valued functions and investigated several properties. They established block-codes by using the notion of BCK-valued functions, and shown that every finite BCK-algebra determines a block-code.

In this paper, we first introduce the notion of superior mapping by using partially ordered sets. Using the superior mapping, we introduce the concept of superiorsubalgebras and (commutative) superiorideals in BCK/BCI-algebras, and investigate related properties. We discuss relations among a superiorsubalgebra, a superiorideal and a commutative superiorideal.

## 2. Preliminaries

We display basic definitions and properties of BCK/BCI-algebras that will be used in this paper. For more details of BCK/BCI-algebras, we refer the reader to $[3,7,8]$ and $[9]$.

An algebra $\mathcal{L}:=(L ; *, 0)$ of type $(2,0)$ is called a BCI-algebra if it satisfies the following conditions:
(I) $(\forall x, y, z \in L)(((x * y) *(x * z)) *(z * y)=0)$,
(II) $(\forall x, y \in L)((x *(x * y)) * y=0)$,
(III) $(\forall x \in L)(x * x=0)$,
(IV) $(\forall x, y \in L)(x * y=0, y * x=0 \Rightarrow x=y)$.

If a BCI-algebra $\mathcal{L}$ satisfies the following identity:
(V) $(\forall x \in L)(0 * x=0)$,
then $\mathcal{L}$ is called a BCK-algebra.
Any BCK/BCI-algebra $\mathcal{L}$ satisfies the following conditions:

$$
\begin{align*}
& (\forall x \in L)(x * 0=x),  \tag{2.1}\\
& (\forall x, y, z \in L)(x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x),  \tag{2.2}\\
& (\forall x, y, z \in L)((x * y) * z=(x * z) * y),  \tag{2.3}\\
& (\forall x, y, z \in L)((x * z) *(y * z) \leq x * y) \tag{2.4}
\end{align*}
$$

where $x \leq y$ if and only if $x * y=0$.
A BCK-algebra $\mathcal{L}$ is said to be commutative if $x \wedge y=y \wedge x$ for all $x, y \in L$ where $x \wedge y=y *(y * x)$.

A nonempty subset $S$ of a BCK/BCI-algebra $\mathcal{L}$ is called a subalgebra of $\mathcal{L}$ if $x * y \in S$ for all $x, y \in S$. A subset $A$ of a BCK/BCI-algebra $\mathcal{L}$ is called an ideal of $\mathcal{L}$ if it satisfies:

$$
\begin{align*}
& 0 \in A  \tag{2.5}\\
& (\forall x, y \in L)(x * y \in A, y \in A \Rightarrow x \in A) \tag{2.6}
\end{align*}
$$

A subset $A$ of a BCK-algebra $\mathcal{L}$ is called a commutative ideal of $\mathcal{L}$ if it satisfies (2.5) and

$$
\begin{equation*}
(\forall x, y, z \in L)((x * y) * z \in A, z \in A \Rightarrow x *(y *(y * x)) \in A) \tag{2.7}
\end{equation*}
$$

## 3. SUPERIOR MAPPINGS

Let $L$ be a set of parameters and let $U$ be a partially ordered set with the partial ordering $\preceq$ and the first element $e$. For a mapping $\tilde{f}: L \rightarrow \mathcal{P}(U)$, we consider the mapping

$$
\|\tilde{f}\|: L \rightarrow U, x \mapsto \begin{cases}\sup \tilde{f}(x) & \text { if } \exists \sup \tilde{f}(x)  \tag{3.1}\\ e & \text { otherwise }\end{cases}
$$

which is called the superiormapping of $L$ with respect to $(\tilde{f}, L)$. In this case, we say that $(\tilde{f}, L)$ is a pair on $(U, \preceq)$.

Example 3.1. Let $U=\{1,2,3,4,6,8,9,12,18,24\}$ be ordered by the relation " $x$ divides $y$ ". The Hasse diagram of $U$ appears in Figure 1.


Figure 1
For a set $L=\{a, b, c, d\}$ of parameters, let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is
given as follows:

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{2,4,6\} & \text { if } x=a, \\ \{8,12,18\} & \text { if } x=b, \\ \{1,3,6,9\} & \text { if } x=c, \\ \{4,6,8,12\} & \text { if } x=d .\end{cases}
$$

Then the superiormapping of $L$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(a)=12,\|\tilde{f}\|(c)=18$ and $\|\tilde{f}\|(d)=24$, but $\|\tilde{f}\|(b)=1$ because there does not exist the supremum of $\tilde{f}(b)$.

Example 3.2. For any positive integer $m$, we will let $\mathbf{D}_{m}$ denote the set of divisors of $m$ ordered by divisibility. The Hasse diagram of

$$
\mathbf{D}_{36}=\{1,2,3,4,6,9,12,18,36\}
$$

appears in Figure 2.


Figure 2
For a set $L=\left\{a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}\right\}$ of parameters, let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ with $U=\mathbf{D}_{36}$ in which $\tilde{f}$ is defined as follows:

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{1,2,3\} & \text { if } x=a_{1}, \\ \{2,3,6\} & \text { if } x=a_{2}, \\ \{2,3,4,6\} & \text { if } x=a_{3}, \\ \{12,36\} & \text { if } x=a_{4}, \\ \{4,6,9\} & \text { if } x=a_{5}, \\ \{3,4,6,9\} & \text { if } x=a_{6} .\end{cases}
$$

Then the superiormapping of $L$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|\left(a_{1}\right)=\|\tilde{f}\|\left(a_{2}\right)=6,\|\tilde{f}\|\left(a_{3}\right)=12$, and $\|\tilde{f}\|\left(a_{4}\right)=\|\tilde{f}\|\left(a_{5}\right)=\|\tilde{f}\|\left(a_{6}\right)=36$.

## 4. Superior subalgebras and ideals

Definition 4.1. Let $\mathcal{L}:=(L, *, 0)$ be a BCK/BCI-algebra and let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$. By a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$, we mean the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ which satisfies:

$$
\begin{equation*}
(\forall x, y \in L)(\|\tilde{f}\|(x * y) \preceq \sup \{\|\tilde{f}\|(x),\|\tilde{f}\|(y)\}) \tag{4.1}
\end{equation*}
$$

whenever there exists $\sup \{\|\tilde{f}\|(x),\|\tilde{f}\|(y)\}$ for any $x, y \in L$.
Example 4.2. Let $L=\{0, a, b, c\}$ be a set with a binary operation '*' shown in Table 1.

Table 1. Cayley table for the binary operation ' $*$ '.

| $*$ | 0 | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | $a$ |
| $b$ | $b$ | $b$ | 0 | $b$ |
| $c$ | $c$ | $c$ | $c$ | 0 |

Then $\mathcal{L}:=(L, *, 0)$ is a BCK-algebra (see $[9])$. Consider the poset $(U, \preceq)$ which is given in Example 3.1.
(1) Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given by

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{1,2\} & \text { if } x=0 \\ \{4,6,8\} & \text { if } x=a, \\ \{2,3,4,6\} & \text { if } x=b, \\ \{1,2,3,6\} & \text { if } x=c .\end{cases}
$$

Then the superiormapping of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(0)=2,\|\tilde{f}\|(a)=24,\|\tilde{f}\|(b)=12$ and $\|\tilde{f}\|(c)=6$, and it is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.
(2) Let $(\tilde{g}, L)$ be a pair on $(U, \preceq)$ in which $\tilde{g}$ is provided as follows:

$$
\tilde{g}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{2,4,6\} & \text { if } x \in\{0, a\}, \\ \{1,3,6,9\} & \text { if } x=b, \\ \{4,6,8,12\} & \text { if } x=c .\end{cases}
$$

Then the superiormapping of $\mathcal{L}$ with respect to $(\tilde{g}, L)$ is described as follows: $\|\tilde{g}\|(0)=\|\tilde{g}\|(a)=12,\|\tilde{g}\|(b)=18$ and $\|\tilde{g}\|(c)=24$, and it is not a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$ since $\|\tilde{g}\|(b * b)=\|\tilde{g}\|(0)=12$ and $\sup \{\|\tilde{g}\|(b),\|\tilde{g}\|(b)\}=18$ are noncomparable.
(3) Let $(\tilde{h}, L)$ be a pair on $(U, \preceq)$ in which $\tilde{h}$ is given as follows:

$$
\tilde{h}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{2,4,6\} & \text { if } x=0, \\ \{8,12,18\} & \text { if } x=a, \\ \{1,3,6,9\} & \text { if } x=b, \\ \{2,3,9\} & \text { if } x=c .\end{cases}
$$

Then the superiormapping of $\mathcal{L}$ with respect to $(\tilde{h}, L)$ is described as follows: $\|\tilde{g}\|(0)=12,\|\tilde{g}\|(a)=1$, and $\|\tilde{g}\|(b)=\|\tilde{g}\|(c)=18$. Since

$$
\|\tilde{g}\|(a * a)=\|\tilde{g}\|(0)=12 \npreceq 1=\sup \{\|\tilde{g}\|(a),\|\tilde{g}\|(a)\},
$$

$\|\tilde{f}\|$ is not a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.
Example 4.3. Let $L=\{0,1,2, a, b\}$ be a set with a binary operation '*' shown in Table 2.

Table 2. Cayley table for the binary operation ' $*$ '.

| $*$ | 0 | 1 | 2 | $a$ | $b$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | $a$ | $a$ |
| 1 | 1 | 0 | 1 | $b$ | $a$ |
| 2 | 2 | 2 | 0 | $a$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | 0 | 0 |
| $b$ | $b$ | $a$ | $b$ | 1 | 0 |

Then $\mathcal{L}:=(L, *, 0)$ is a BCI-algebra (see [9]). Consider the poset $(U, \preceq)$ which is given in Example 3.2. Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is defined by

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{1\} & \text { if } x=0 \\ \{4,6,9,12\} & \text { if } x \in\{1, b\} \\ \{2,3\} & \text { if } x=2, \\ \{3,6,9\} & \text { if } x=a\end{cases}
$$

Then the superiormapping of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(0)=1,\|\tilde{f}\|(a)=18,\|\tilde{f}\|(b)=\|\tilde{f}\|(1)=36$ and $\|\tilde{f}\|(2)=6$, and it is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.
Definition 4.4. Let $\mathcal{L}:=(L, *, 0)$ be a BCK/BCI-algebra and let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$. By a superiorideal on $(\mathcal{L}, \tilde{f})$, we mean the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ which satisfies the following conditions:

$$
\begin{equation*}
(\forall x \in L)(\|\tilde{f}\|(0) \preceq\|\tilde{f}\|(x)), \tag{4.2}
\end{equation*}
$$

$$
\begin{equation*}
(\forall x, y \in L)(\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(x * y),\|\tilde{f}\|(y)\}), \tag{4.3}
\end{equation*}
$$

whenever there exists $\sup \{\|\tilde{f}\|(x),\|\tilde{f}\|(y)\}$ for any $x, y \in L$.

## Example 4.5.

(1) In Example 4.2(1), the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.
(2) In Example 4.2(2), the superiormapping $\|\tilde{g}\|$ of $\mathcal{L}$ with respect to $(\tilde{g}, L)$ is not a superiorideal on $(\mathcal{L}, \tilde{g})$.
(3) In Example 4.2(3), the superiormapping $\|\tilde{g}\|$ of $\mathcal{L}$ with respect to $(\tilde{h}, L)$ is not a superiorideal on $(\mathcal{L}, \tilde{h})$.

Proposition 4.6. Let $\mathcal{L}:=(L, *, 0)$ be a $B C K / B C I-a l g e b r a$. Then every superiorsubalgebra $\|\tilde{f}\|$ on $(\mathcal{L}, \tilde{f})$ satisfies the condition (4.2).

Proof. Since $x * x=0$ for all $x \in L$, it is clear.
Theorem 4.7. Let $\mathcal{L}:=(L, *, 0)$ be a BCK/BCI-algebra. If $\|\tilde{f}\|$ is a superiorsubalgebra (ideal) on $(\mathcal{L}, \tilde{f})$, then the nonempty set

$$
\|\tilde{f}\|_{\alpha}:=\{x \in L \mid\|\tilde{f}\|(x) \preceq \alpha\}
$$

is a subalgebra (ideal) of $\mathcal{L}$ for all $\alpha \in U$.
Proof. Assume that $\|\tilde{f}\|$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$. Let $\alpha \in U$ and suppose that $\|\tilde{f}\|_{\alpha} \neq \emptyset$. If $x, y \in\|\tilde{f}\|_{\alpha}$, then $\|\tilde{f}\|(x) \preceq \alpha$ and $\|\tilde{f}\|(y) \preceq \alpha$. It follows from (4.1) that

$$
\|\tilde{f}\|(x * y) \preceq \sup \{\|\tilde{f}\|(x),\|\tilde{f}\|(y)\} \preceq \alpha
$$

and that $x * y \in\|\tilde{f}\|_{\alpha}$. Therefore $\|\tilde{f}\|_{\alpha}$ is a subalgebra of $\mathcal{L}$. Now, suppose that $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. Let $\alpha \in U$ be such that $\|\tilde{f}\|_{\alpha} \neq \emptyset$. Then there exists $x \in L$ such that $\|\tilde{f}\|(x) \preceq \alpha$, and so $\|\tilde{f}\|(0) \preceq\|\tilde{f}\|(x) \preceq \alpha$. Thus $0 \in\|\tilde{f}\|_{\alpha}$. Let $x, y \in L$ be such that $x * y \in\|\tilde{f}\|_{\alpha}$ and $y \in\|\tilde{f}\|_{\alpha}$. Then $\|\tilde{f}\|(x * y) \preceq \alpha$ and $\|\tilde{f}\|(y) \preceq \alpha$. It follows from (4.3) that

$$
\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(x * y),\|\tilde{f}\|(y)\} \preceq \alpha .
$$

Thus $x \in\|\tilde{f}\|_{\alpha}$, and therefore $\|\tilde{f}\|_{\alpha}$ is an ideal of $\mathcal{L}$.
The following example illustrates Theorem 4.7.

Example 4.8. (1) Consider the BCK-algebra $\mathcal{L}$ and the poset ( $U, \preceq$ ) which are given in Examples 4.2 and 3.1, respectively. Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{1,3\} & \text { if } x=0 \\ \{2,3,6\} & \text { if } x=a \\ \{1,3,9\} & \text { if } x=b, \\ \{2,3,6,9\} & \text { if } x=c\end{cases}
$$

Then the superiormapping of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(0)=3,\|\tilde{f}\|(a)=6,\|\tilde{f}\|(b)=9$ and $\|\tilde{f}\|(c)=18$, and it is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$. It is routine to verify that $\|\tilde{f}\|_{\alpha}$ is a subalgebra of $\mathcal{L}$ for all $\alpha \in U$.
(2) Consider the BCI-algebra $\mathcal{L}$ and the poset $(U, \preceq)$ which are given in Examples 4.3 and 3.1 , respectively. Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is defined by

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{1\} & \text { if } x=0, \\ \{4,6\} & \text { if } x \in\{1, b\}, \\ \{1,3\} & \text { if } x=2, \\ \{2,3,6\} & \text { if } x=a .\end{cases}
$$

Then the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(0)=1,\|\tilde{f}\|(1)=12,\|\tilde{f}\|(2)=3,\|\tilde{f}\|(a)=6$, and $\|\tilde{f}\|(b)=12$. It is routine to verify that $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. Thus $\|\tilde{f}\|_{\tilde{8}}=\|\tilde{f}\|_{4_{\sim}}=\|\tilde{f}\|_{2}=\emptyset$, and $\|\tilde{f}\|_{24}=\|\tilde{f}\|_{12}=L,\|\tilde{f}\|_{18}=\|\tilde{f}\|_{6}=\{0,2, a\},\|\tilde{f}\|_{9}=\|\tilde{f}\|_{3}=\{0,2\}$, $\|\tilde{f}\|_{1}=\{0\}$ which are ideals of $\mathcal{L}$.
Proposition 4.9. If $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$, then $\|\tilde{f}\|(x) \preceq\|\tilde{f}\|(y)$ for all $x, y \in L$ with $x \leq y$.

Proof. Let $x, y \in L$ be such that $x \leq y$. Using (4.3) and (4.2), we have

$$
\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(x * y),\|\tilde{f}\|(y)\}=\sup \{\|\tilde{f}\|(0),\|\tilde{f}\|(y)\}=\|\tilde{f}\|(y)
$$

proving the result.
Theorem 4.10. Let $\mathcal{L}$ be a BCK-algebra. Every superiorideal on $(\mathcal{L}, \tilde{f})$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.
Proof. Let $\|\tilde{f}\|$ be a superiorideal on $(\mathcal{L}, \tilde{f})$. Since $x * y \leq x$ for all $x, y \in L$, it follows from Proposition 4.9 that

$$
\|\tilde{f}\|(x * y) \preceq\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(x * y),\|\tilde{f}\|(y)\} \preceq \sup \{\|\tilde{f}\|(x),\|\tilde{f}\|(y)\}
$$

and that $\|\tilde{f}\|$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.

The converse of Theorem 4.10 may not be true as seen in the following example.
Example 4.11. Let $L=\{0,1,2,3\}$ be a set with a binary operation ' $*$ ' shown in Table 3.

Table 3. Cayley table for the binary operation ' $*$ '.

| $*$ | 0 | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

Then $\mathcal{L}:=(L, *, 0)$ is a BCK-algebra (see [9]). Let $U=\{a, b, c, d, e, f\}$ be ordered as pictured in Figure 3.


Figure 3
Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{a, b\} & \text { if } x=0, \\ \{a, b, c\} & \text { if } x=1, \\ \{b, c, d, f\} & \text { if } x \in\{2,3\} .\end{cases}
$$

Then the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(0)=b,\|\tilde{f}\|(1)=c$ and $\|\tilde{f}\|(2)=\|\tilde{f}\|(3)=f$. By routine calculations, we know that $\|\tilde{f}\|$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$, but it is not a superiorideal on $(\mathcal{L}, \tilde{f})$ because

$$
\|\tilde{f}\|(2)=f \npreceq c=\sup \{\|\tilde{f}\|(2 * 1),\|\tilde{f}\|(1)\} .
$$

Proposition 4.12. Every superiorideal $\|\tilde{f}\|$ on $(\mathcal{L}, \tilde{f})$ satisfies the following assertion.

$$
\begin{equation*}
(\forall x, y, z \in L)(x * y \leq z \Rightarrow\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(y),\|\tilde{f}\|(z)\}) . \tag{4.4}
\end{equation*}
$$

Proof. Let $x, y, z \in L$ be such that $x * y \leq z$. Then $(x * y) * z=0$, and so

$$
\|\tilde{f}\|(x * y) \preceq \sup \{\|\tilde{f}\|((x * y) * z),\|\tilde{f}\|(z)\}=\sup \{\|\tilde{f}\|(0),\|\tilde{f}\|(z)\}=\|\tilde{f}\|(z)
$$

by (4.3) and (4.2). It follows that

$$
\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(x * y),\|\tilde{f}\|(y)\} \preceq \sup \{\|\tilde{f}\|(z),\|\tilde{f}\|(y)\} .
$$

This completes the proof.
Theorem 4.13. Let $\|\tilde{f}\|$ be the superiormapping of $\mathcal{L}$ with respect to $(\tilde{f}, L)$. If $\|\tilde{f}\|$ satisfies two conditions (4.2) and (4.4), then $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.

Proof. Since $x *(x * y) \leq y$ for all $x, y \in L$, it follows from (4.4) that

$$
\|\tilde{f}\|(x) \preceq \sup \{\|\tilde{f}\|(x * y),\|\tilde{f}\|(y)\}
$$

for all $x, y \in L$. Therefore $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.

## 5. Commutative superiorideals

Definition 5.1. Let $\mathcal{L}:=(L, *, 0)$ be a BCK-algebra and let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$. By a commutative superiorideal on ( $\mathcal{L}, \tilde{f}$ ), we mean the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ which satisfies the condition (4.2) and

$$
\begin{equation*}
(\forall x, y, z \in L)(\|\tilde{f}\|(x *(y *(y * x))) \preceq \sup \{\|\tilde{f}\|((x * y) * z),\|\tilde{f}\|(z)\}) \tag{5.1}
\end{equation*}
$$

whenever there exists $\sup \{\|\tilde{f}\|(x),\|\tilde{f}\|(y)\}$ for any $x, y \in L$.
Example 5.2. Let $U=\{1,2,3, \ldots, 8\}$ be ordered as pictured in Figure 4.


Figure 4

Table 4. Cayley table for the binary operation ' $*$ '.

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | $a$ | 0 | $a$ |  |
| $b$ | $b$ | $b$ | 0 | $b$ | 0 |
| $c$ | $c$ | $c$ | 0 | $c$ |  |
| $d$ | $d$ | $d$ | $d$ | $d$ | 0 |

Let $L=\{0, a, b, c, d\}$ be a set with a binary operation ' $*$ ' shown in Table 4. Then $\mathcal{L}:=(L, *, 0)$ is a BCK-algebra (see [9]).

Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{6,8\} & \text { if } x \in\{0, b\} \\ \{4,6,7\} & \text { if } x=d \\ \{2,3,5,6,7\} & \text { if } x \in\{a, c\}\end{cases}
$$

Then the superiormapping $\|\tilde{f}\|$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $\|\tilde{f}\|(0)=\|\tilde{f}\|(b)=6,\|\tilde{f}\|(d)=3$, and $\|\tilde{f}\|(a)=\|\tilde{f}\|(c)=2$. It is routine to check that $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$.

Theorem 5.3. If $\mathcal{L}$ is a BCK-algebra, then every commutative superiorideal on $(\mathcal{L}, \tilde{f})$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.

Proof. Let $\|\tilde{f}\|$ be a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCKalgebra. Using (V), (2.1) and (5.1), we have

$$
\begin{aligned}
\|\tilde{f}\|(x) & =\|\tilde{f}\|(x *(0 *(0 * x))) \\
& \preceq \sup \{\|\tilde{f}\|((x * 0) * z),\|\tilde{f}\|(z)\} \\
& =\sup \{\|\tilde{f}\|(x * z),\|\tilde{f}\|(z)\}
\end{aligned}
$$

for all $x, z \in L$. Hence $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.
The following example shows that the converse of Theorem 5.3 is not true in general.

Example 5.4. Let $L=\{0, a, b, c, d\}$ be a set with a binary operation ' $*$ ' shown in Table 5.

Table 5. Cayley table for the binary operation ' $*$ '.

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | 0 | 0 |  |
| $b$ | $b$ | $b$ | 0 | 0 | 0 |
| $c$ | $c$ | $c$ | 0 | 0 |  |
| $d$ | $d$ | $d$ | $c$ | 0 |  |

Then $\mathcal{L}:=(L, *, 0)$ is a BCK-algebra (see [9]). Consider the poset $(U, \preceq)$ which is given in Example 5.2. Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$
\tilde{f}: L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases}\{8\} & \text { if } x=0 \\ \{5,6,7\} & \text { if } x=a \\ \{3,4,5,7\} & \text { if } x \in\{b, c, d\}\end{cases}
$$

Then the superiormapping $\|\tilde{\tilde{f}}\|$ on $(\mathcal{L}, \tilde{f})$ is described as follows: $\|\tilde{f}\|(0)=8$, $\|\tilde{f}\|(a)=5$ and $\|\tilde{f}\|(b)=\|\tilde{f}\|(c)=\|\tilde{f}\|(d)=3$. Routine calculations show that $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. But it is not a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ since

$$
\|\tilde{f}\|(b *(c *(c * b))) \npreceq \sup \{\|\tilde{f}\|((b * c) * 0),\|\tilde{f}\|(0)\} .
$$

Proposition 5.5. Let $\|\tilde{f}\|$ be a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. Then the following assertion is valid.

$$
\begin{equation*}
(\forall x, y \in L)(\|\tilde{f}\|(x *(y *(y * x))) \preceq\|\tilde{f}\|(x * y)) . \tag{5.2}
\end{equation*}
$$

Proof. Taking $z=0$ in (5.1) and using (4.2) and (2.1), we have the desired result.

We provide conditions for a superiorideal to be commutative.
Theorem 5.6. Let $\|\tilde{f}\|$ be a superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. If the condition (5.2) is valid, then $\|\tilde{f}\|$ is commutative.

Proof. Assume that $\|\tilde{f}\|$ satisfies the condition (5.2). The condition (4.3) induces

$$
\begin{equation*}
\|\tilde{f}\|(x * y) \preceq \sup \{\|\tilde{f}\|((x * y) * z),\|\tilde{f}\|(z)\} \tag{5.3}
\end{equation*}
$$

for all $x, y, z \in L$. Combining (5.3) and (5.2), we know that

$$
\|\tilde{f}\|(x *(y *(y * x))) \preceq \sup \{\|\tilde{f}\|((x * y) * z),\|\tilde{f}\|(z)\}
$$

for all $x, y, z \in L$. Therefore $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$.
Combining Theorems 4.13 and 5.6, we have the following corollary.
Corollary 5.7. Let $\|\tilde{f}\|$ be the superiormapping of a BCK-algebra $\mathcal{L}$ with respect to $(\tilde{f}, L)$. If $\|\tilde{f}\|$ satisfies (4.2), (4.4) and (5.2), then $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$.

Theorem 5.8. In a commutative BCK-algebra, every superiorideal is a commutative superiorideal.
Proof. Let $\|\tilde{f}\|$ be a superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a commutative BCKalgebra. Note that

$$
\begin{aligned}
& ((x *(y *(y * x))) *((x * y) * z)) * z \\
& =((x *(y *(y * x))) * z) *((x * y) * z) \\
& \leq(x *(y *(y * x))) *(x * y) \\
& =(x *(x * y)) *(y *(y * x))=0,
\end{aligned}
$$

that is, $(x *(y *(y * x))) *((x * y) * z) \leq z$ for all $x, y, z \in L$. It follows from Proposition 4.12 that

$$
\|\tilde{f}\|(x *(y *(y * x))) \preceq \sup \{\|\tilde{f}\|((x * y) * z),\|\tilde{f}\|(z)\}
$$

for all $x, y, z \in L$. Therefore $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$.
Corollary 5.9. If a BCK-algebra $\mathcal{L}$ satisfies the following condition:

$$
\begin{equation*}
(\forall x, y \in L)(x *(x * y) \leq y *(y * x)) \tag{5.4}
\end{equation*}
$$

then every superiorideal is a commutative superiorideal.
Lemma 5.10 [9]. Let $A$ be an ideal of a BCK-algebra $\mathcal{L}$. Then $A$ is commutative if and only if the following assertion holds.

$$
\begin{equation*}
(\forall x, y \in A)(x * y \in A \Rightarrow x *(y *(y * x)) \in A) . \tag{5.5}
\end{equation*}
$$

Theorem 5.11. If $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra, then the nonempty set

$$
\|\tilde{f}\|_{\alpha}:=\{x \in L \mid\|\tilde{f}\|(x) \preceq \alpha\}
$$

is a commutative ideal of $\mathcal{L}$ for all $\alpha \in U$.

Proof. Assume that $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. Then $\|\tilde{f}\|$ is a superiorideal on $(\mathcal{L}, \tilde{f})$ by Theorem 5.3. Hence if $\|\tilde{f}\|_{\alpha} \neq \emptyset$, then $\|\tilde{f}\|_{\alpha}$ is an ideal of $\mathcal{L}$ for all $\alpha \in U$. Let $x, y \in L$ be such that $x * y \in\|\tilde{f}\|_{\alpha}$. Using (5.2), we have $\|\tilde{f}\|(x *(y *(y * x))) \preceq\|\tilde{f}\|(x * y) \preceq \alpha$ and so $x *(y *(y * x)) \in\|\tilde{f}\|_{\alpha}$. Hence, by Lemma 5.10, $\|\tilde{f}\|_{\alpha}$ is a commutative ideal of $\mathcal{L}$ for all $\alpha \in U$.

Theorem 5.12. Let $\|\tilde{f}\|$ and $\|\tilde{g}\|$ be superiorideals on $(\mathcal{L}, \tilde{f})$ and $(\mathcal{L}, \tilde{g})$, respectively, where $\mathcal{L}$ is a BCK-algebra such that $\|\tilde{f}\|(0)=\|\tilde{g}\|(0)$ and $\|\tilde{g}\|(x) \preceq\|\tilde{f}\|(x)$ for all $x(\neq 0) \in L$. If $\|\tilde{f}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$, then $\|\tilde{g}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{g})$.

Proof. For any $x, y \in L$, let $u=x * y$. Using hypothesis, (5.2), (2.3) and (III), we have

$$
\begin{aligned}
\|\tilde{g}\|((x * u) *(y *(y *(x * u)))) & \preceq\|\tilde{f}\|((x * u) *(y *(y *(x * u)))) \\
& \preceq\|\tilde{f}\|((x * u) * y)=\|\tilde{f}\|((x * y) * u) \\
& =\|\tilde{f}\|(0)=\|\tilde{g}\|(0),
\end{aligned}
$$

and so $\|\tilde{g}\|((x * u) *(y *(y *(x * u))))=\|\tilde{g}\|(0)$. Note that

$$
\begin{aligned}
& (x *(y *(y * x))) *(x *(y *(y *(x * u)))) \\
& \leq(y *(y *(x * u))) *(y *(y * x)) \\
& \leq(y * x) *(y *(x * u)) \\
& \leq(x * u) * x=0 * u=0
\end{aligned}
$$

and thus $(x *(y *(y * x))) *(x *(y *(y *(x * u))))=0$. It follows from (4.3), (4.2) and (2.3) that

$$
\begin{aligned}
\|\tilde{g}\|(x *(y *(y * x))) \preceq & \sup \{\|\tilde{g}\|((x *(y *(y * x))) *(x *(y *(y *(x * u))))) \\
& \|\tilde{g}\|(x *(y *(y *(x * u))))\} \\
& =\sup \{\|\tilde{g}\|(0),\|\tilde{g}\|(x *(y *(y *(x * u))))\} \\
= & \|\tilde{g}\|(x *(y *(y *(x * u)))) \\
& \preceq \sup \{\|\tilde{g}\|((x *(y *(y *(x * u)))) * u),\|\tilde{g}\|(u)\} \\
= & \sup \{\|\tilde{g}\|((x * u) *(y *(y *(x * u)))),\|\tilde{g}\|(u)\} \\
= & \sup \{\|\tilde{g}\|(0),\|\tilde{g}\|(u)\} \\
= & \|\tilde{g}\|(u)=\|\tilde{g}\|(x * y)
\end{aligned}
$$

Therefore $\|\tilde{g}\|$ is a commutative superiorideal on $(\mathcal{L}, \tilde{g})$ by Theorem 5.6.

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