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SUPERIOR SUBALGEBRAS AND IDEALS OF BCK/BCI-ALGEBRAS

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Abstract

The notions of superior subalgebras and (commutative) superior ideals are introduced, and their relations and related properties are investigated. Conditions for a superior ideal to be commutative are provided.

Keywords: superior mapping, superior subalgebra, (commutative) superior ideal.

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1. Introduction

Algebras have played an important role in pure and applied mathematics and have its comprehensive applications in many aspects including dynamical systems and genetic code of biology (see [1, 2, 6], and [11]). Starting from the four DNA bases order in the Boolean lattice, Sáanchez *et al.* [10] proposed a novel Lie Algebra of the genetic code which shows strong connections among algebraic

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relationship, codon assignments and physicochemical properties of amino acids. A BCK/BCI-algebra (see [3, 4, 9]) is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers. Jun and Song [5] introduced the notion of BCK-valued functions and investigated several properties. They established block-codes by using the notion of BCK-valued functions, and shown that every finite BCK-algebra determines a block-code.

In this paper, we first introduce the notion of superior mapping by using partially ordered sets. Using the superior mapping, we introduce the concept of superiorsubalgebras and (commutative) superiorideals in BCK/BCI-algebras, and investigate related properties. We discuss relations among a superiorsubalgebra, a superiorideal and a commutative superiorideal.

2. Preliminaries

We display basic definitions and properties of BCK/BCI-algebras that will be used in this paper. For more details of BCK/BCI-algebras, we refer the reader to [3, 7, 8] and [9].

An algebra $\mathcal{L} := (L; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x, y, z \in L)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in L)$ $((x * (x * y)) * y = 0)$,

(III) $(\forall x \in L)$ $(x * x = 0)$,

(IV) $(\forall x, y \in L)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra $\mathcal{L}$ satisfies the following identity:

(V) $(\forall x \in L)$ $(0 * x = 0)$,

then $\mathcal{L}$ is called a BCK-algebra.

Any BCK/BCI-algebra $\mathcal{L}$ satisfies the following conditions:

(2.1) $$(\forall x \in L)\,(x * 0 = x)\,,$$

(2.2) $$(\forall x, y, z \in L)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,,$$

(2.3) $$(\forall x, y, z \in L)\,((x * y) * z = (x * z) * y)\,,$$

(2.4) $$(\forall x, y, z \in L)\,((x * z) * (y * z) \leq x * y)$$

where $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra $\mathcal{L}$ is said to be commutative if $x \wedge y = y \wedge x$ for all $x, y \in L$ where $x \wedge y = y * (y * x)$.

A nonempty subset S of a BCK/BCI-algebra $\mathcal{L}$ is called a subalgebra of $\mathcal{L}$ if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra $\mathcal{L}$ is called an ideal of $\mathcal{L}$ if it satisfies:

$$0 \in A, \tag{2.5}$$

$$(\forall x, y \in L)\,(x * y \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.6}$$

A subset A of a BCK-algebra $\mathcal{L}$ is called a commutative ideal of $\mathcal{L}$ if it satisfies (2.5) and

$$(\forall x, y, z \in L)\,((x * y) * z \in A,\ z \in A \ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.7}$$

3. Superior mappings

Let L be a set of parameters and let U be a partially ordered set with the partial ordering $\preceq$ and the first element e. For a mapping $\tilde{f} : L \to \mathcal{P}(U)$, we consider the mapping

$$||\tilde{f}|| : L \to U,\ x \mapsto \begin{cases} \sup \tilde{f}(x) & \text{if } \ \exists \sup \tilde{f}(x), \\ e & \text{otherwise}, \end{cases} \tag{3.1}$$

which is called the superiormapping of L with respect to $(\tilde{f}, L)$. In this case, we say that $(\tilde{f}, L)$ is a pair on $(U, \preceq)$.

Example 3.1. Let $U = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the relation "x divides y". The Hasse diagram of U appears in Figure 1.

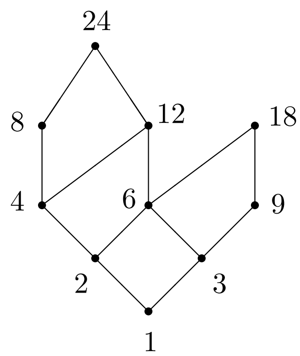

Figure 1

For a set $L = \{a, b, c, d\}$ of parameters, let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,4,6\} & \text{if } x = a, \\ \{8,12,18\} & \text{if } x = b, \\ \{1,3,6,9\} & \text{if } x = c, \\ \{4,6,8,12\} & \text{if } x = d. \end{cases}$$

Then the superiormapping of L with respect to $(\tilde{f}, L)$ is described as follows: $||\tilde{f}||(a) = 12$, $||\tilde{f}||(c) = 18$ and $||\tilde{f}||(d) = 24$, but $||\tilde{f}||(b) = 1$ because there does not exist the supremum of $\tilde{f}(b)$.

Example 3.2. For any positive integer m, we will let $\mathbf{D}_m$ denote the set of divisors of m ordered by divisibility. The Hasse diagram of

$$\mathbf{D}_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

appears in Figure 2.

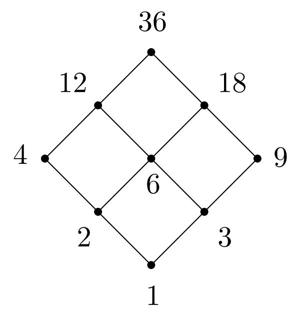

Figure 2

For a set $L = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ of parameters, let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ with $U = \mathbf{D}_{36}$ in which $\tilde{f}$ is defined as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1,2,3\} & \text{if } x = a_1, \\ \{2,3,6\} & \text{if } x = a_2, \\ \{2,3,4,6\} & \text{if } x = a_3, \\ \{12,36\} & \text{if } x = a_4, \\ \{4,6,9\} & \text{if } x = a_5, \\ \{3,4,6,9\} & \text{if } x = a_6. \end{cases}$$

Then the superiormapping of L with respect to $(\tilde{f}, L)$ is described as follows: $||\tilde{f}||(a_1) = ||\tilde{f}||(a_2) = 6$, $||\tilde{f}||(a_3) = 12$, and $||\tilde{f}||(a_4) = ||\tilde{f}||(a_5) = ||\tilde{f}||(a_6) = 36$.

4. SUPERIOR SUBALGEBRAS AND IDEALS

Definition 4.1. Let $\mathcal{L} := (L, *, 0)$ be a BCK/BCI-algebra and let $\big(\tilde{f}, L\big)$ be a pair on $(U, \preceq)$. By a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$, we mean the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\big(\tilde{f}, L\big)$ which satisfies:

$$(\forall x, y \in L)\Big(||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(x), ||\tilde{f}||(y)\}\Big) \tag{4.1}$$

whenever there exists $\sup\{||\tilde{f}||(x), ||\tilde{f}||(y)\}$ for any $x, y \in L$.

Example 4.2. Let $L = \{0, a, b, c\}$ be a set with a binary operation '$*$' shown in Table 1.

Table 1. Cayley table for the binary operation '$*$'.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [9]). Consider the poset $(U, \preceq)$ which is given in Example 3.1.

(1) Let $\big(\tilde{f}, L\big)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given by

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1, 2\} & \text{if } x = 0, \\ \{4, 6, 8\} & \text{if } x = a, \\ \{2, 3, 4, 6\} & \text{if } x = b, \\ \{1, 2, 3, 6\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of $\mathcal{L}$ with respect to $\big(\tilde{f}, L\big)$ is described as follows: $||\tilde{f}||(0) = 2$, $||\tilde{f}||(a) = 24$, $||\tilde{f}||(b) = 12$ and $||\tilde{f}||(c) = 6$, and it is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.

(2) Let $\big(\tilde{g}, L\big)$ be a pair on $(U, \preceq)$ in which $\tilde{g}$ is provided as follows:

$$\tilde{g} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2, 4, 6\} & \text{if } x \in \{0, a\}, \\ \{1, 3, 6, 9\} & \text{if } x = b, \\ \{4, 6, 8, 12\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of $\mathcal{L}$ with respect to $\big(\tilde{g}, L\big)$ is described as follows: $||\tilde{g}||(0) = ||\tilde{g}||(a) = 12$, $||\tilde{g}||(b) = 18$ and $||\tilde{g}||(c) = 24$, and it is not a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$ since $||\tilde{g}||(b * b) = ||\tilde{g}||(0) = 12$ and $\sup\{||\tilde{g}||(b), ||\tilde{g}||(b)\} = 18$ are noncomparable.

(3) Let $\left(\tilde{h}, L\right)$ be a pair on $(U, \preceq)$ in which $\tilde{h}$ is given as follows:

$$\tilde{h} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,4,6\} & \text{if } x = 0, \\ \{8,12,18\} & \text{if } x = a, \\ \{1,3,6,9\} & \text{if } x = b, \\ \{2,3,9\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of $\mathcal{L}$ with respect to $\left(\tilde{h}, L\right)$ is described as follows: $||\tilde{g}||(0) = 12$, $||\tilde{g}||(a) = 1$, and $||\tilde{g}||(b) = ||\tilde{g}||(c) = 18$. Since

$$||\tilde{g}||(a * a) = ||\tilde{g}||(0) = 12 \not\preceq 1 = \sup\{||\tilde{g}||(a), ||\tilde{g}||(a)\},$$

$||\tilde{f}||$ is not a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.

Example 4.3. Let $L = \{0, 1, 2, a, b\}$ be a set with a binary operation '$*$' shown in Table 2.

Table 2. Cayley table for the binary operation '$*$'.

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then $\mathcal{L} := (L, *, 0)$ is a BCI-algebra (see [9]). Consider the poset $(U, \preceq)$ which is given in Example 3.2. Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is defined by

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1\} & \text{if } x = 0, \\ \{4,6,9,12\} & \text{if } x \in \{1, b\}, \\ \{2,3\} & \text{if } x = 2, \\ \{3,6,9\} & \text{if } x = a. \end{cases}$$

Then the superiormapping of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ is described as follows: $||\tilde{f}||(0) = 1$, $||\tilde{f}||(a) = 18$, $||\tilde{f}||(b) = ||\tilde{f}||(1) = 36$ and $||\tilde{f}||(2) = 6$, and it is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.

Definition 4.4. Let $\mathcal{L} := (L, *, 0)$ be a BCK/BCI-algebra and let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$. By a superiorideal on $(\mathcal{L}, \tilde{f})$, we mean the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\left(\tilde{f}, L\right)$ which satisfies the following conditions:

(4.2) $$(\forall x \in L) \left(||\tilde{f}||(0) \preceq ||\tilde{f}||(x) \right),$$

$$(4.3) \qquad (\forall x, y \in L)\left(||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\}\right),$$

whenever there exists $\sup\{||\tilde{f}||(x), ||\tilde{f}||(y)\}$ for any $x, y \in L$.

Example 4.5.

(1) In Example 4.2(1), the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.

(2) In Example 4.2(2), the superiormapping $||\tilde{g}||$ of $\mathcal{L}$ with respect to $(\tilde{g}, L)$ is not a superiorideal on $(\mathcal{L}, \tilde{g})$.

(3) In Example 4.2(3), the superiormapping $||\tilde{g}||$ of $\mathcal{L}$ with respect to $(\tilde{h}, L)$ is not a superiorideal on $(\mathcal{L}, \tilde{h})$.

Proposition 4.6. *Let $\mathcal{L} := (L, *, 0)$ be a BCK/BCI-algebra. Then every superiorsubalgebra $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$ satisfies the condition* (4.2).

Proof. Since $x * x = 0$ for all $x \in L$, it is clear. ■

Theorem 4.7. *Let $\mathcal{L} := (L, *, 0)$ be a BCK/BCI-algebra. If $||\tilde{f}||$ is a superiorsubalgebra (ideal) on $(\mathcal{L}, \tilde{f})$, then the nonempty set*

$$||\tilde{f}||_\alpha := \{x \in L \mid ||\tilde{f}||(x) \preceq \alpha\}$$

is a subalgebra (ideal) of $\mathcal{L}$ for all $\alpha \in U$.

Proof. Assume that $||\tilde{f}||$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$. Let $\alpha \in U$ and suppose that $||\tilde{f}||_\alpha \neq \emptyset$. If $x, y \in ||\tilde{f}||_\alpha$, then $||\tilde{f}||(x) \preceq \alpha$ and $||\tilde{f}||(y) \preceq \alpha$. It follows from (4.1) that

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||(x), ||\tilde{f}||(y)\} \preceq \alpha$$

and that $x * y \in ||\tilde{f}||_\alpha$. Therefore $||\tilde{f}||_\alpha$ is a subalgebra of $\mathcal{L}$. Now, suppose that $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. Let $\alpha \in U$ be such that $||\tilde{f}||_\alpha \neq \emptyset$. Then there exists $x \in L$ such that $||\tilde{f}||(x) \preceq \alpha$, and so $||\tilde{f}||(0) \preceq ||\tilde{f}||(x) \preceq \alpha$. Thus $0 \in ||\tilde{f}||_\alpha$. Let $x, y \in L$ be such that $x * y \in ||\tilde{f}||_\alpha$ and $y \in ||\tilde{f}||_\alpha$. Then $||\tilde{f}||(x * y) \preceq \alpha$ and $||\tilde{f}||(y) \preceq \alpha$. It follows from (4.3) that

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\} \preceq \alpha.$$

Thus $x \in ||\tilde{f}||_\alpha$, and therefore $||\tilde{f}||_\alpha$ is an ideal of $\mathcal{L}$. ■

The following example illustrates Theorem 4.7.

Example 4.8. (1) Consider the BCK-algebra $\mathcal{L}$ and the poset $(U, \preceq)$ which are given in Examples 4.2 and 3.1, respectively. Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1,3\} & \text{if } x = 0, \\ \{2,3,6\} & \text{if } x = a, \\ \{1,3,9\} & \text{if } x = b, \\ \{2,3,6,9\} & \text{if } x = c. \end{cases}$$

Then the superiormapping of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $||\tilde{f}||(0) = 3$, $||\tilde{f}||(a) = 6$, $||\tilde{f}||(b) = 9$ and $||\tilde{f}||(c) = 18$, and it is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$. It is routine to verify that $||\tilde{f}||_\alpha$ is a subalgebra of $\mathcal{L}$ for all $\alpha \in U$.

(2) Consider the BCI-algebra $\mathcal{L}$ and the poset $(U, \preceq)$ which are given in Examples 4.3 and 3.1, respectively. Let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is defined by

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1\} & \text{if } x = 0, \\ \{4,6\} & \text{if } x \in \{1, b\}, \\ \{1,3\} & \text{if } x = 2, \\ \{2,3,6\} & \text{if } x = a. \end{cases}$$

Then the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ is described as follows: $||\tilde{f}||(0) = 1$, $||\tilde{f}||(1) = 12$, $||\tilde{f}||(2) = 3$, $||\tilde{f}||(a) = 6$, and $||\tilde{f}||(b) = 12$. It is routine to verify that $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. Thus $||\tilde{f}||_8 = ||\tilde{f}||_4 = ||\tilde{f}||_2 = \emptyset$, and $||\tilde{f}||_{24} = ||\tilde{f}||_{12} = L$, $||\tilde{f}||_{18} = ||\tilde{f}||_6 = \{0, 2, a\}$, $||\tilde{f}||_9 = ||\tilde{f}||_3 = \{0, 2\}$, $||\tilde{f}||_1 = \{0\}$ which are ideals of $\mathcal{L}$.

Proposition 4.9. *If $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$, then $||\tilde{f}||(x) \preceq ||\tilde{f}||(y)$ for all $x, y \in L$ with $x \leq y$.*

Proof. Let $x, y \in L$ be such that $x \leq y$. Using (4.3) and (4.2), we have

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\} = \sup\{||\tilde{f}||(0), ||\tilde{f}||(y)\} = ||\tilde{f}||(y),$$

proving the result. ■

Theorem 4.10. *Let $\mathcal{L}$ be a BCK-algebra. Every superiorideal on $(\mathcal{L}, \tilde{f})$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$.*

Proof. Let $||\tilde{f}||$ be a superiorideal on $(\mathcal{L}, \tilde{f})$. Since $x * y \leq x$ for all $x, y \in L$, it follows from Proposition 4.9 that

$$||\tilde{f}||(x * y) \preceq ||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\} \preceq \sup\{||\tilde{f}||(x), ||\tilde{f}||(y)\}$$

and that $||\tilde{f}||$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$. ■

The converse of Theorem 4.10 may not be true as seen in the following example.

Example 4.11. Let $L = \{0, 1, 2, 3\}$ be a set with a binary operation '$*$' shown in Table 3.

Table 3. Cayley table for the binary operation '$*$'.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [9]). Let $U = \{a, b, c, d, e, f\}$ be ordered as pictured in Figure 3.

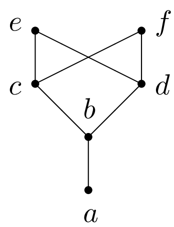

Figure 3

Let $\big(\tilde{f}, L\big)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{a, b\} & \text{if } x = 0, \\ \{a, b, c\} & \text{if } x = 1, \\ \{b, c, d, f\} & \text{if } x \in \{2, 3\}. \end{cases}$$

Then the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\big(\tilde{f}, L\big)$ is described as follows: $||\tilde{f}||(0) = b$, $||\tilde{f}||(1) = c$ and $||\tilde{f}||(2) = ||\tilde{f}||(3) = f$. By routine calculations, we know that $||\tilde{f}||$ is a superiorsubalgebra on $(\mathcal{L}, \tilde{f})$, but it is not a superiorideal on $(\mathcal{L}, \tilde{f})$ because

$$||\tilde{f}||(2) = f \npreceq c = \sup\{||\tilde{f}||(2 * 1), ||\tilde{f}||(1)\}.$$

Proposition 4.12. *Every superiorideal $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$ satisfies the following assertion.*

$$(\forall x, y, z \in L)\left(x * y \leq z \ \Rightarrow\ ||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(y), ||\tilde{f}||(z)\}\right). \tag{4.4}$$

Proof. Let $x, y, z \in L$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\} = \sup\{||\tilde{f}||(0), ||\tilde{f}||(z)\} = ||\tilde{f}||(z)$$

by (4.3) and (4.2). It follows that

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\} \preceq \sup\{||\tilde{f}||(z), ||\tilde{f}||(y)\}.$$

This completes the proof. ■

Theorem 4.13. *Let $||\tilde{f}||$ be the superiormapping of $\mathcal{L}$ with respect to $(\tilde{f}, L)$. If $||\tilde{f}||$ satisfies two conditions* (4.2) *and* (4.4)*, then $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.*

Proof. Since $x * (x * y) \leq y$ for all $x, y \in L$, it follows from (4.4) that

$$||\tilde{f}||(x) \preceq \sup\{||\tilde{f}||(x * y), ||\tilde{f}||(y)\}$$

for all $x, y \in L$. Therefore $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. ■

5. Commutative superiorideals

Definition 5.1. Let $\mathcal{L} := (L, *, 0)$ be a BCK-algebra and let $(\tilde{f}, L)$ be a pair on $(U, \preceq)$. By a commutative superiorideal on $(\mathcal{L}, \tilde{f})$, we mean the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $(\tilde{f}, L)$ which satisfies the condition (4.2) and

$$(\forall x, y, z \in L)\left(||\tilde{f}||(x * (y * (y * x))) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\}\right) \tag{5.1}$$

whenever there exists $\sup\{||\tilde{f}||(x), ||\tilde{f}||(y)\}$ for any $x, y \in L$.

Example 5.2. Let $U = \{1, 2, 3, \ldots, 8\}$ be ordered as pictured in Figure 4.

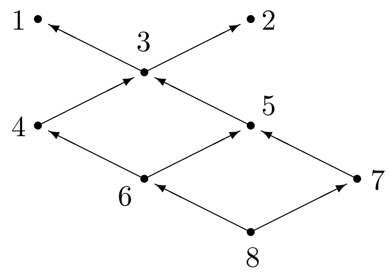

Figure 4

Table 4. Cayley table for the binary operation '$*$'.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Let $L = \{0, a, b, c, d\}$ be a set with a binary operation '$*$' shown in Table 4. Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [9]).

Let $\big(\tilde{f}, L\big)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{6, 8\} & \text{if } x \in \{0, b\}, \\ \{4, 6, 7\} & \text{if } x = d, \\ \{2, 3, 5, 6, 7\} & \text{if } x \in \{a, c\}. \end{cases}$$

Then the superiormapping $||\tilde{f}||$ of $\mathcal{L}$ with respect to $\big(\tilde{f}, L\big)$ is described as follows: $||\tilde{f}||(0) = ||\tilde{f}||(b) = 6$, $||\tilde{f}||(d) = 3$, and $||\tilde{f}||(a) = ||\tilde{f}||(c) = 2$. It is routine to check that $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$.

Theorem 5.3. *If $\mathcal{L}$ is a BCK-algebra, then every commutative superiorideal on $(\mathcal{L}, \tilde{f})$ is a superiorideal on $(\mathcal{L}, \tilde{f})$.*

Proof. Let $||\tilde{f}||$ be a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. Using (V), (2.1) and (5.1), we have

$$\begin{aligned} ||\tilde{f}||(x) &= ||\tilde{f}||(x * (0 * (0 * x))) \\ &\preceq \sup\{||\tilde{f}||((x * 0) * z), ||\tilde{f}||(z)\} \\ &= \sup\{||\tilde{f}||(x * z), ||\tilde{f}||(z)\} \end{aligned}$$

for all $x, z \in L$. Hence $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. ■

The following example shows that the converse of Theorem 5.3 is not true in general.

Example 5.4. Let $L = \{0, a, b, c, d\}$ be a set with a binary operation '$*$' shown in Table 5.

Table 5. Cayley table for the binary operation '$*$'.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Then $\mathcal{L} := (L, *, 0)$ is a BCK-algebra (see [9]). Consider the poset $(U, \preceq)$ which is given in Example 5.2. Let $\left(\tilde{f}, L\right)$ be a pair on $(U, \preceq)$ where $\tilde{f}$ is given as follows:

$$\tilde{f} : L \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{8\} & \text{if } x = 0, \\ \{5,6,7\} & \text{if } x = a, \\ \{3,4,5,7\} & \text{if } x \in \{b,c,d\}. \end{cases}$$

Then the superiormapping $||\tilde{f}||$ on $(\mathcal{L}, \tilde{f})$ is described as follows: $||\tilde{f}||(0) = 8$, $||\tilde{f}||(a) = 5$ and $||\tilde{f}||(b) = ||\tilde{f}||(c) = ||\tilde{f}||(d) = 3$. Routine calculations show that $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$. But it is not a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ since

$$||\tilde{f}||(b * (c * (c * b))) \not\preceq \sup\{||\tilde{f}||((b * c) * 0), ||\tilde{f}||(0)\}.$$

Proposition 5.5. *Let $||\tilde{f}||$ be a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. Then the following assertion is valid.*

$$(\forall x, y \in L)\left(||\tilde{f}||(x * (y * (y * x))) \preceq ||\tilde{f}||(x * y)\right). \tag{5.2}$$

Proof. Taking $z = 0$ in (5.1) and using (4.2) and (2.1), we have the desired result. ■

We provide conditions for a superiorideal to be commutative.

Theorem 5.6. *Let $||\tilde{f}||$ be a superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. If the condition (5.2) is valid, then $||\tilde{f}||$ is commutative.*

Proof. Assume that $||\tilde{f}||$ satisfies the condition (5.2). The condition (4.3) induces

$$||\tilde{f}||(x * y) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\} \tag{5.3}$$

for all $x, y, z \in L$. Combining (5.3) and (5.2), we know that

$$||\tilde{f}||(x * (y * (y * x))) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\}$$

for all $x, y, z \in L$. Therefore $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$. ∎

Combining Theorems 4.13 and 5.6, we have the following corollary.

Corollary 5.7. *Let $||\tilde{f}||$ be the superiormapping of a BCK-algebra $\mathcal{L}$ with respect to $(\tilde{f}, L)$. If $||\tilde{f}||$ satisfies* (4.2), (4.4) *and* (5.2), *then $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$.*

Theorem 5.8. *In a commutative BCK-algebra, every superiorideal is a commutative superiorideal.*

Proof. Let $||\tilde{f}||$ be a superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a commutative BCK-algebra. Note that

$$\begin{aligned}
&((x * (y * (y * x))) * ((x * y) * z)) * z \\
&= ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\leq (x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) = 0,
\end{aligned}$$

that is, $(x * (y * (y * x))) * ((x * y) * z) \leq z$ for all $x, y, z \in L$. It follows from Proposition 4.12 that

$$||\tilde{f}||(x * (y * (y * x))) \preceq \sup\{||\tilde{f}||((x * y) * z), ||\tilde{f}||(z)\}$$

for all $x, y, z \in L$. Therefore $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$. ∎

Corollary 5.9. *If a BCK-algebra $\mathcal{L}$ satisfies the following condition:*

$$(5.4) \qquad (\forall x, y \in L)\,(x * (x * y) \leq y * (y * x)),$$

then every superiorideal is a commutative superiorideal.

Lemma 5.10 [9]**.** *Let A be an ideal of a BCK-algebra $\mathcal{L}$. Then A is commutative if and only if the following assertion holds.*

$$(5.5) \qquad (\forall x, y \in A)\,(x * y \in A \;\Rightarrow\; x * (y * (y * x)) \in A).$$

Theorem 5.11. *If $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra, then the nonempty set*

$$||\tilde{f}||_\alpha := \{x \in L \mid ||\tilde{f}||(x) \preceq \alpha\}$$

is a commutative ideal of $\mathcal{L}$ for all $\alpha \in U$.

Proof. Assume that $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$ where $\mathcal{L}$ is a BCK-algebra. Then $||\tilde{f}||$ is a superiorideal on $(\mathcal{L}, \tilde{f})$ by Theorem 5.3. Hence if $||\tilde{f}||_\alpha \neq \emptyset$, then $||\tilde{f}||_\alpha$ is an ideal of $\mathcal{L}$ for all $\alpha \in U$. Let $x, y \in L$ be such that $x * y \in ||\tilde{f}||_\alpha$. Using (5.2), we have $||\tilde{f}||(x * (y * (y * x))) \preceq ||\tilde{f}||(x * y) \preceq \alpha$ and so $x * (y * (y * x)) \in ||\tilde{f}||_\alpha$. Hence, by Lemma 5.10, $||\tilde{f}||_\alpha$ is a commutative ideal of $\mathcal{L}$ for all $\alpha \in U$. ■

Theorem 5.12. *Let $||\tilde{f}||$ and $||\tilde{g}||$ be superiorideals on $(\mathcal{L}, \tilde{f})$ and $(\mathcal{L}, \tilde{g})$, respectively, where $\mathcal{L}$ is a BCK-algebra such that $||\tilde{f}||(0) = ||\tilde{g}||(0)$ and $||\tilde{g}||(x) \preceq ||\tilde{f}||(x)$ for all $x(\neq 0) \in L$. If $||\tilde{f}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{f})$, then $||\tilde{g}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{g})$.*

Proof. For any $x, y \in L$, let $u = x * y$. Using hypothesis, (5.2), (2.3) and (III), we have

$$\begin{aligned} ||\tilde{g}||((x * u) * (y * (y * (x * u)))) &\preceq ||\tilde{f}||((x * u) * (y * (y * (x * u)))) \\ &\preceq ||\tilde{f}||((x * u) * y) = ||\tilde{f}||((x * y) * u) \\ &= ||\tilde{f}||(0) = ||\tilde{g}||(0), \end{aligned}$$

and so $||\tilde{g}||((x * u) * (y * (y * (x * u)))) = ||\tilde{g}||(0)$. Note that

$$\begin{aligned} &(x * (y * (y * x))) * (x * (y * (y * (x * u)))) \\ &\leq (y * (y * (x * u))) * (y * (y * x)) \\ &\leq (y * x) * (y * (x * u)) \\ &\leq (x * u) * x = 0 * u = 0, \end{aligned}$$

and thus $(x * (y * (y * x))) * (x * (y * (y * (x * u)))) = 0$. It follows from (4.3), (4.2) and (2.3) that

$$\begin{aligned} ||\tilde{g}||(x * (y * (y * x))) &\preceq \sup\{||\tilde{g}||((x * (y * (y * x))) * (x * (y * (y * (x * u))))), \\ &\qquad ||\tilde{g}||(x * (y * (y * (x * u))))\} \\ &= \sup\{||\tilde{g}||(0), ||\tilde{g}||(x * (y * (y * (x * u))))\} \\ &= ||\tilde{g}||(x * (y * (y * (x * u)))) \\ &\preceq \sup\{||\tilde{g}||((x * (y * (y * (x * u)))) * u), ||\tilde{g}||(u)\} \\ &= \sup\{||\tilde{g}||((x * u) * (y * (y * (x * u)))), ||\tilde{g}||(u)\} \\ &= \sup\{||\tilde{g}||(0), ||\tilde{g}||(u)\} \\ &= ||\tilde{g}||(u) = ||\tilde{g}||(x * y). \end{aligned}$$

Therefore $||\tilde{g}||$ is a commutative superiorideal on $(\mathcal{L}, \tilde{g})$ by Theorem 5.6. ■

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