

ON THE ASSOCIATED PRIME IDEALS OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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Abstract

Let I and J be two ideals of a commutative Noetherian ring R and M be an R -module. For a non-negative integer n it is shown that, if the sets $\text{Ass}_R(\text{Ext}_R^n(R/I, M))$ and $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \leq n+1$ and all $j < n$, then so is $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$. We also study the finiteness of $\text{Ass}_R(\text{Ext}_R^i(R/I, H_{I,J}^n(M)))$ for $i = 1, 2$.

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1. INTRODUCTION

Let R be a commutative Noetherian ring, I and J be two ideals of R and M be an R -module. For all $i \in \mathbb{N}_0$ the i -th local cohomology functor with respect to (I, J) , denoted by $H_{I,J}^i(-)$, defined by Takahashi *et al.* in [14] as the i -th right derived functor of the (I, J) -torsion functor $\Gamma_{I,J}(-)$, where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor $H_I^i(-)$ when $J = 0$, see [5].

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules $H_I^i(M)$ ([12]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [7, 8] and [14].

Hartshorne in [9] proposed the following conjecture: “Let M be a finitely generated R -module and \mathfrak{a} be an ideal of R . Then $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$ is finitely generated for all $i \geq 0$ and $j \geq 0$.”

Also, Huneke in [10] raised some crucial problems on local cohomology modules. One of them was about the finiteness of the set of associated prime ideals of the local cohomology modules $H_I^i(M)$.

Although there are some counterexamples to these conjectures, see [13], but there are some partial positive answers in some special cases too, see for example [3] or [4].

In this paper, we consider these two problems for local cohomology modules defined by a pair of ideals over not necessary finitely generated modules. In particular, we investigate certain conditions on $H_{I,J}^j(M)$ such that the set of associated prime ideals of $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$ is finite.

More precisely, let $n \in \mathbb{N}_0$ and assume that the sets $\text{Ass}_R(\text{Ext}_R^n(R/I, M))$ and $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \leq n+1$ and all $j < n$ then, we use a spectral sequence argument to show that $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$ is finite, too (Theorem 2.3). Moreover, it is shown that if the sets $\text{Ass}_R(\text{Ext}_R^{n+1}(R/I, M))$ and $\text{Supp}(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \leq n+2$ and all $j < n$ then, so is $\text{Ass}_R(\text{Ext}_R^1(R/I, H_{I,J}^n(M)))$ (Theorem 2.7).

We also present a necessary and sufficient condition for the finiteness of the set $\text{Ass}_R(\text{Ext}_R^2(R/I, H_{I,J}^n(M)))$ (Theorem 2.8). These, also, generalize some known results concerning ordinary local cohomology modules.

Moreover, we study the grade $\mathfrak{p} := \inf_M \{i \in \mathbb{N}_0 : H_{\mathfrak{p}}^i(M) \neq 0\}$ for $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$, where $t = \inf \{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$ (Theorem 2.11).

2. ASSOCIATED PRIME IDEALS

In this section, first, we are going to study the set of associated prime ideals of some Ext-modules of local cohomology modules defined by a pair of ideals.

The following relation between associated prime ideals of modules in an exact sequence, which can be proved easily, is frequently used in our results.

Lemma 2.1. *Let $M \rightarrow N \rightarrow K \rightarrow 0$ be an exact sequence of R -modules. Then $\text{Ass}(K) \subseteq \text{Supp}(M) \cup \text{Ass}(N)$.*

Next lemma describes a convergence of Grothendieck spectral sequences.

Lemma 2.2. *Let M be an R -module. Then the following convergence of spectral sequences exists*

$$\text{Ext}_R^i(R/I, H_{I,J}^j(M)) \xrightarrow{i} \text{Ext}_R^{i+j}(R/I, M).$$

Proof. It is easy to see that $\text{Hom}_R(R/I, \Gamma_{I,J}(M)) = \text{Hom}_R(R/I, M)$. Also, for any injective R -module E , $\Gamma_{I,J}(E)$ is an injective R -module, by [14, 3.2] and [5, 2.1.4]. Now, in view of [11, 10.47], the assertion follows. ■

The following theorem, which concerns with Hartshorne's problem mentioned in the introduction, is one of the main results in this paper.

Theorem 2.3. *Let n be a non-negative integer and M be an R -module such that $\text{Ass}_R(\text{Ext}_R^n(R/I, M))$ and $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \leq n+1$ and all $j < n$. Then so is $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$.*

Proof. Consider the convergence of spectral sequences in Lemma 2.2 and note that $E_2^{i,j} = 0$ for all $i < 0$. Therefore, for all $2 \leq r \leq n+1$ there exists an exact sequence

$$(2.1) \quad 0 \rightarrow E_{r+1}^{0,n} \rightarrow E_r^{0,n} \xrightarrow{d_r^{0,n}} E_r^{r,n+1-r}.$$

Since, $E_r^{r,n+1-r}$ is a subquotient of $E_2^{r,n+1-r} = \text{Ext}_R^r(R/I, H_{I,J}^{n+1-r}(M))$, $\text{Supp}_R(E_r^{r,n+1-r})$ is a finite set. So, the above exact sequence implies that $\sharp \text{Ass}_R(E_r^{0,n}) < \infty$ if $\sharp \text{Ass}_R(E_{r+1}^{0,n}) < \infty$. Also, from the fact that $E_2^{i,j} = 0$ for all $j < 0$, we have $E_\infty^{0,n} \cong E_{n+2}^{0,n}$. Therefore, to prove the assertion it is enough to show that $\text{Ass}_R(E_\infty^{0,n})$ is a finite set.

Using the concept of the convergence of spectral sequences, there exists a bounded filtration

$$0 = \varphi^{n+1}H^n \subseteq \varphi^n H^n \subseteq \cdots \subseteq \varphi^1 H^n \subseteq \varphi^0 H^n = \text{Ext}_R^n(R/I, M)$$

of submodules of $\text{Ext}_R^n(R/I, M)$ such that

$$E_\infty^{i,n-i} \cong \varphi^i H^n / \varphi^{i+1} H^n \text{ for all } i = 0, \dots, n.$$

Therefore, $E_{n+1}^{n,0} \cong E_\infty^{n,0} \cong \varphi^n H^n$ is a subquotient of $E_2^{n,0} = \text{Ext}_R^n(R/I, \Gamma_{I,J}(M))$. So, by the assumption, $\text{Supp}_R(\varphi^n H^n)$ is a finite set. Now, assume inductively that $\#\text{Supp}_R(\varphi^i H^n) < \infty$ for all $1 < i \leq n$. Then, since

$$E_{n+1}^{1,n-1} \cong E_\infty^{1,n-1} \cong \varphi^1 H^n / \varphi^2 H^n$$

is a subquotient of $E_2^{1,n-1} = \text{Ext}_R^1(R/I, H_{I,J}^{n-1}(M))$, we deduce that $\text{Supp}_R(\varphi^1 H^n)$ is finite. But,

$$E_\infty^{0,n} \cong \text{Ext}_R^n(R/I, M) / \varphi^1 H^n$$

and Lemma 2.1 implies that $\#\text{Ass}_R(E_\infty^{0,n}) < \infty$, as desired. \blacksquare

As some immediate consequences of Theorem 2.3, we obtain the following results.

Corollary 2.4. *Let M be a finite R -module. Suppose that there is an integer n such that for all $i < n$ the set $\text{Supp}_R(H_{I,J}^i(M))$ is finite. Then $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$ is finite.*

Proof. Using the fact that $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M))) \subseteq V(I) \cap \text{Supp}_R(H_{I,J}^i(M))$ for all i and j , the result follows from Theorem 2.3. \blacksquare

Corollary 2.5. *Let M be a finite R -module. Suppose that $q = \inf\{i : H_{I,J}^i(M) \text{ is not Artinian}\}$ is an integer, then $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^q(M)))$ is finite.*

Proof. By [2, IV, p. 275, Proposition 7], $\text{Supp}_R(H_{I,J}^i(M))$ is finite for all $i < q$. Now, the result follows from Corollary 2.4. \blacksquare

For an R -module M and an ideal \mathfrak{a} of R , the grade of \mathfrak{a} on M is defined by

$$\text{grade } \mathfrak{a} := \inf_M \{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M) \neq 0\},$$

if this infimum exists, and ∞ otherwise. If M is a finite R -module and $\mathfrak{a}M \neq M$, this definition coincides with the length of a maximal M -sequence in \mathfrak{a} (cf. [5, 6.2.7]).

Corollary 2.6. *Let M be a finite R -module and $t = \inf\{i | H_{I,J}^i(M) \neq 0\}$ be an integer. Then $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M)))$ is finite. If in addition, $\text{grade } I = t$, then for a maximal M -sequence x_1, \dots, x_t in I , we have*

$$\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M))) = \{\mathfrak{p} \in \text{Ass}_R(M/(x_1, \dots, x_t)M) \cap V(I); \text{grade } \mathfrak{p} = t\}.$$

Proof. In view of Theorem 2.3, $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M)))$ is finite. In the case where grade $I = t$, using [1, 2.4(i)] and [6, 1.2.27], we have

$$\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M))) = \text{Ass}_R(\text{Hom}_R(R/I, H_I^t(M))) = \text{Ass}_R(H_I^t(M)).$$

Now, the assertion follows by [15, 3.10]. ■

In the rest of this paper we consider the set of associated prime ideals of some Ext modules of local cohomology modules defined by a pair of ideals.

Theorem 2.7. *Let n be a non-negative integer and M be an R -module such that $\text{Ass}_R(\text{Ext}_R^{n+1}(R/I, M))$ and $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \leq n+2$ and all $j < n$. Then so is $\text{Ass}_R(\text{Ext}_R^1(R/I, H_{I,J}^n(M)))$.*

Proof. Considering the convergence of the spectral sequences of Lemma 2.2, we have to show that $\text{Ass}_R(E_2^{1,n})$ is a finite set. Using similar arguments as used in Theorem 2.3, one can see that it is enough to show that $\text{Ass}_R(E_\infty^{1,n}) = \text{Ass}_R(E_{n+2}^{1,n})$ is a finite set.

By the concept of convergence of spectral sequences, there exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq \dots \subseteq \varphi^1H^{n+1} \subseteq \varphi^0H^{n+1} = \text{Ext}_R^{n+1}(R/I, M)$$

of submodules of $\text{Ext}_R^{n+1}(R/I, M)$ such that $E_\infty^{i,n+1-i} \cong \varphi^iH^{n+1}/\varphi^{i+1}H^{n+1}$ for all $i = 0, \dots, n+1$. Using the fact that $\sharp \text{Supp}_R(E_2^{i,j}) < \infty$ for all $i \leq n+2$ and all $j < n$ one can see that $\text{Supp}_R(\varphi^iH^{n+1})$ is a finite set for all $i = 2, \dots, n+2$. Also, $\sharp \text{Ass}_R(\varphi^1H^{n+1}) < \infty$. Now, since

$$E_{n+2}^{1,n} \cong E_\infty^{1,n} \cong \varphi^1H^{n+1}/\varphi^2H^{n+1},$$

using Lemma 2.1, we have $\sharp \text{Ass}_R(E_\infty^{1,n}) < \infty$, and the result follows. ■

The following theorem presents a necessary and sufficient condition for the finiteness of the set $\text{Ass}_R(\text{Ext}_R^i(R/I, H_{I,J}^n(M)))$ when $i = 0, 2$.

Theorem 2.8. *Let n be a non-negative integer and M be an R -module such that the sets $\text{Supp}_R(\text{Ext}_R^{n+1}(R/I, M))$ and $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \leq n+2$ and all $j < n$. Then $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^{n+1}(M)))$ is finite if and only if $\text{Ass}_R(\text{Ext}_R^2(R/I, H_{I,J}^n(M)))$ is finite.*

Proof. (\Leftarrow) Again, consider the convergence of spectral sequences of Lemma 2.2 and assume that $\text{Ass}_R(E_2^{2,n})$ is finite. Since $E_2^{i,j} = 0$ for all $i < 0$ or $j < 0$, using similar arguments as used in Theorem 2.3, one can see that $E_\infty^{0,n+1} \cong E_{n+3}^{0,n+1}$ and in order to prove that $\sharp \text{Ass}_R(E_2^{0,n+1}) < \infty$ we have to show that $\sharp \text{Ass}_R(E_\infty^{0,n+1}) < \infty$.

There exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq \dots \subseteq \varphi^1H^{n+1} \subseteq \varphi^0H^{n+1} = \text{Ext}_R^{n+1}(R/I, M)$$

of submodules of $\text{Ext}_R^{n+1}(R/I, M)$ such that $E_\infty^{0,n+1} \cong \text{Ext}_R^{n+1}(R/I, M)/\varphi^1H^{n+1}$. Since $\# \text{Supp}_R(\text{Ext}_R^{n+1}(R/I, M)) < \infty$ we have $\# \text{Ass}_R(E_\infty^{0,n+1}) < \infty$, as desired.

(\Rightarrow) Now, assume that $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^{n+1}(M))) < \infty$ and consider the exact sequence

$$0 \rightarrow \text{Ker } d_2^{0,n+1} \rightarrow E_2^{0,n+1} \xrightarrow{d_2^{0,n+1}} \text{Im } d_2^{0,n+1} \rightarrow 0.$$

Since $\text{Ker } d_2^{0,n+1} = E_3^{0,n+1}$ and $\# \text{Supp}_R(E_3^{0,n+1}) < \infty$, in view of Lemma 2.1, we have $\# \text{Ass}_R(\text{Im } d_2^{0,n+1}) < \infty$. Now, using the exact sequence

$$0 \rightarrow \text{Im } d_2^{0,n+1} \rightarrow E_2^{2,n} \xrightarrow{d_2^{2,n}} E_2^{4,n-1}$$

and the fact that $E_2^{4,n-1} = \text{Ext}_R^4(R/I, H_{I,J}^{n-1}(M))$ has finite support, we have $\# \text{Ass}_R(E_2^{2,n}) < \infty$, as desired. \blacksquare

Theorem 2.9. *Let n be a non-negative integer and M be an R -module of dimension d , such that $\text{Ass}_R(\text{Ext}_R^{n+d}(R/I, M))$ and $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$ are finite for all $i \geq n+1$ and all $j < d$. Then $\text{Ass}_R(\text{Ext}_R^n(R/I, H_{I,J}^d(M)))$ is finite.*

Proof. The method of the proof is similar to the Theorem 2.7, considering [14, 4.7]. \blacksquare

In the rest of this paper, we study the grade of prime ideals $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$ on M . For that, we shall use the following notations introduced in [14].

$$W(I, J) := \{\mathfrak{p} \in \text{Spec}(R) : I^n \subseteq \mathfrak{p} + J \text{ for some integer } n \geq 1\},$$

and

$$\widetilde{W}(I, J) := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R \text{ and } I^n \subseteq \mathfrak{a} + J \text{ for some integer } n \geq 1\}.$$

The following lemma can be proved using [14, 3.2].

Lemma 2.10. *For any non-negative integer i and any R -module M ,*

- (i) $\text{Supp}_R(H_{I,J}^i(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Supp}(H_{\mathfrak{a}}^i(M)).$
- (ii) $\text{Supp}_R(H_{I,J}^i(M)) \subseteq \text{Supp}_R(M) \cap W(I, J).$

In [15, 3.6] the authors study the grade \mathfrak{p} for $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$, where

$$t = \inf_M \{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$$

in the case where M is a finitely generated R -module. But their proof is not correct. Actually, they use the equality $\text{Supp}_R(M_x) = \{\mathfrak{p} \in \text{Supp}_R(M) : x \notin \mathfrak{p}\}$ which is not true. Here, we also made a correction to this result for not necessary finite modules.

Theorem 2.11. *Let M be an R -module and $t = \inf\{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$ be a non-negative integer. Then for all $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$, $\text{grade } \mathfrak{p} = t$.*

Proof. We use induction on t . Let $t = 0$ and $\mathfrak{p} \in \text{Ass}_R(\Gamma_{I,J}(M))$. Then $\mathfrak{p} = (0 :_R x)$ for some $x \in \Gamma_{I,J}(M)$. Hence $x \in \Gamma_{\mathfrak{p}}(M)$ and so $\Gamma_{\mathfrak{p}}(M) \neq 0$.

Now suppose that $t > 0$ and the case $t - 1$ is settled. Let $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$ and consider the exact sequence $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$, where $E = E_R(M)$ is the injective envelope of M . Therefore, using [15, 2.2], $H_{I,J}^i(L) \cong H_{I,J}^{i+1}(M)$ for all $i \geq 0$ and we get

$$\inf \{i \in \mathbb{N}_0 : H_{I,J}^i(L) \neq 0\} = \inf \{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\} - 1 = t - 1$$

and that $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^{t-1}(L))$. Thus, by inductive hypothesis, $\text{grade } \mathfrak{p} = t - 1$.

Now, consider the long exact sequence

$$H_{\mathfrak{p}}^{i-1}(M) \rightarrow H_{\mathfrak{p}}^{i-1}(E) \rightarrow H_{\mathfrak{p}}^{i-1}(L) \rightarrow H_{\mathfrak{p}}^i(M).$$

If $t > 1$, then $H_{\mathfrak{p}}^i(M) \cong H_{\mathfrak{p}}^{i-1}(L) = 0$ for all $i < t$ and $H_{\mathfrak{p}}^t(M) \cong H_{\mathfrak{p}}^{t-1}(L) \neq 0$. Thus $\text{grade } \mathfrak{p} = t$.

Let $t = 1$. Then $\Gamma_{\mathfrak{p}}(L) \neq 0$. By the above exact sequence, it is enough to show that $\Gamma_{\mathfrak{p}}(E) = 0$. On the contrary, assume that $\Gamma_{\mathfrak{p}}(E) \neq 0$. Then there exists a non-zero element $x \in E$ and $n \in \mathbb{N}$ such that $\mathfrak{p}^n x = 0$. We may assume that $\mathfrak{p}^n x = 0$ and $\mathfrak{p}^{n-1} x \neq 0$. So, there exists $r \in \mathfrak{p}^{n-1}$ such that $rx \neq 0$. Thus $\mathfrak{p} \subseteq (0 :_R rx)$. On the other hand, by Lemma 2.10,

$$\mathfrak{p} \in \text{Ass}_R(H_{I,J}^1(M)) \subseteq \text{Supp}_R(H_{I,J}^1(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Supp}_R(H_{\mathfrak{a}}^1(M)).$$

So that there exists $\mathfrak{a} \in \widetilde{W}(I,J)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Let $m \in \mathbb{N}$ with $I^m \subseteq \mathfrak{a} + J \subseteq \mathfrak{p} + J \subseteq (0 :_R rx) + J$. Hence $rx \in \Gamma_{I,J}(M)$ which contradicts with hypothesis and the choice of rx . Therefore $\Gamma_{\mathfrak{p}}(E) = 0$ and so $\text{grade } \mathfrak{p} = 1$. \blacksquare

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