# POINTED PRINCIPALLY ORDERED REGULAR SEMIGROUPS 

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#### Abstract

An ordered semigroup $S$ is said to be principally ordered if, for every $x \in S$ there exists $x^{\star}=\max \{y \in S \mid x y x \leqslant x\}$. Here we investigate those principally ordered regular semigroups that are pointed in the sense that the classes modulo Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{D}$ have biggest elements which are idempotent. Such a semigroup is necessarily a semiband. In particular we describe the subalgebra of $\left(S ;^{\star}\right)$ generated by a pair of comparable idempotents that are $\mathcal{D}$-related. We also prove that those $\mathcal{D}$-classes which are subsemigroups are ordered rectangular bands.


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An ordered regular semigroup $S$ is said to be principally ordered [3] if, for every $x \in S$ there exists $x^{\star}=\max \{y \in S \mid x y x \leqslant x\}$. The basic properties of the unary operation $x \mapsto x^{\star}$ in such a semigroup were established in [3] and are listed in [1, Theorem 13.26]. In particular, we recall for the reader's convenience that in such a semigroup the following properties hold and will be used throughout what follows:
$\left(P_{1}\right)(\forall x \in S) x=x x^{\star} x ;$
$\left(P_{2}\right)$ every $\mathcal{L}$-class $[x]_{\mathcal{L}}$ contains a biggest idempotent, namely $x^{\star} x$;
$\left(P_{3}\right)$ every $\mathcal{R}$-class $[x]_{\mathcal{R}}$ contains a biggest idempotent, namely $x x^{\star}$;
$\left(P_{4}\right)(\forall x \in S) x^{\star \star \star}=x^{\star}$;
$\left(P_{5}\right)$ every $x \in S$ has a biggest inverse, namely $x^{\circ}=x^{\star} x x^{\star}$;
$\left(P_{6}\right)(\forall x \in S) x^{\circ} \leqslant x^{\star}$;
$\left(P_{7}\right)(\forall x \in S) x \leqslant x^{\star \star}=x^{\circ \star}=x^{\star \circ}$.
The point of departure for our investigation here is the following observation.
Theorem 1. If $S$ is a principally ordered regular semigroup then the following statements are equivalent:
(1) every $\mathcal{L}$-class has a biggest element which is idempotent;
(2) $(\forall x \in S) x^{\star} x=\max [x]_{\mathcal{L}}$;
(3) every $\mathcal{R}$-class has a biggest element which is idempotent;
(4) $(\forall x \in S) x x^{\star}=\max [x]_{\mathcal{R}}$;
(5) $(\forall x \in S) x^{2} \leqslant x$;
(6) $(\forall x \in S) x^{\star} \in E(S)$.

Moreover, if $S$ satisfies any of the above conditions then
(7) $(\forall x \in S) \quad \max \left[x^{\star}\right]_{\mathcal{R}}=x^{\star}=x^{\star \star}=\max \left[x^{\star}\right]_{\mathcal{L}}$;
(8) $S$ is a semiband and Green's relation $\mathcal{H}$ is equality;
(9) $x \in S$ is completely regular if and only if $x \in E(S)$.

Proof. (1) $\Leftrightarrow$ (2): If (1) holds and $e=e^{2}=\max [x]_{\mathcal{L}}$ then, by $\left(P_{2}\right), e=e^{\star} e=$ $x^{\star} x$ whence (2) holds. The converse is clear.
$(3) \Leftrightarrow(4)$ : This is dual to $(1) \Leftrightarrow(2)$.
(2) $\Rightarrow$ (5): If (2) holds then $x \leqslant x^{\star} x$ gives, by $\left(P_{1}\right), x^{2} \leqslant x x^{\star} x=x$.
(5) $\Rightarrow$ (2): If (5) holds then $x^{3} \leqslant x^{2} \leqslant x$, so $x \leqslant x^{\star}$. Then $x x^{\star} \leqslant x^{\star 2} \leqslant x^{\star}$ and consequently $x=x x^{\star} x \leqslant x^{\star} x$ for every $x \in S$. Every $y \in[x]_{\mathcal{L}}$ is then such that $y \leqslant y^{\star} y=x^{\star} x$, whence it follows that $x^{\star} x=\max [x]_{\mathcal{L}}$.
$(4) \Leftrightarrow(5)$ : This is dual to $(2) \Leftrightarrow(5)$.
(5) $\Rightarrow(6)$ : Suppose that (5) holds. Then by the above so do (2) and (4). Now if $y \mathcal{R} x$ then, by (4), we have $y \leqslant x x^{\star}$ whence $y x \leqslant x x^{\star} x=x$. It follows by (5) that $x y x \leqslant x^{2} \leqslant x$ and so $y \leqslant x^{\star}$. In particular, on taking $y=x x^{\star}$ we obtain $x x^{\star} \leqslant x^{\star}$ for every $x \in S$. Replacing $x$ by $x^{\star \star}$ in this, we obtain $x^{\star \star} x^{\star} \leqslant x^{\star}$ and it follows by (2) that $x^{\star}=x^{\star \star} x^{\star} \in E(S)$.
(6) $\Rightarrow(2)$ : Clearly, every $e \in E(S)$ is such that $e \leqslant e^{\star}$. Thus, if (6) holds then $x^{\star} \leqslant x^{\star \star}$ and $x^{\star \star} \leqslant x^{\star \star \star}=x^{\star}$. Consequently $x^{\star}=x^{\star \star}$ for every $x \in S$.

We now observe that

$$
y \equiv x(\mathcal{R}) \Longrightarrow y^{\star}=x^{\star}
$$

Indeed, if $y \equiv x(\mathcal{R})$ then, by (6), $x x^{\star} y^{\star} x=y y^{\star} y^{\star} x=y y^{\star} x=x x^{\star} x=x$ whence $x^{\star} y^{\star} \leqslant x^{\star}$. Then $x^{\star} y^{\star} x^{\star} \leqslant x^{\star 2}=x^{\star}$ and consequently $y^{\star} \leqslant x^{\star}=x^{\star}$. Interchanging $x$ and $y$ produces the reverse inequality and therefore $y^{\star}=x^{\star}$.

Taking in particular $y=x x^{\star}$ we then have $x x^{\star} \leqslant\left(x x^{\star}\right)^{\star}=x^{\star}$ whence $x=$ $x x^{\star} x \leqslant x^{\star} x$ for every $x \in S$. If now $z \in[x]_{\mathcal{L}}$ then it follows that $z \leqslant z^{\star} z=x^{\star} x$ and therefore $x^{\star} x=\max [x]_{\mathcal{L}}$, which is (2).

Suppose now that the above conditions are satisfied.
(7) As shown in (6) $\Rightarrow(2), x^{\star}=x^{\star \star} \in E(S)$, and therefore $x^{\star}=x^{\star} x^{\star}$. Then, by (4) and $\left(P_{4}\right), x^{\star}=\max \left[x^{\star \star}\right]_{\mathcal{R}}=\max \left[x^{\star}\right]_{\mathcal{R}}$. Dually, we see that also $x^{\star}=\max \left[x^{\star}\right]_{\mathcal{L}}$.
(8) Since, by (6), each $x^{\star}$ is idempotent, we have $x=x x^{\star} x=x x^{\star} \cdot x^{\star} x$ and so every $x \in S$ is a product of two idempotents, whence $S$ is a semiband. Moreover, if $x \mathcal{H} y$ then $x=x x^{\star} x=x x^{\star} \cdot x^{\star} x=y y^{\star} \cdot y^{\star} y=y y^{\star} y=y$ whence $\mathcal{H}$ reduces to equality.
(9) If $x \in S$ is completely regular then there exists $x^{\prime} \in V(x)$ such that $x x^{\prime}=x^{\prime} x$. Then, by (5), $x^{\prime}=x^{\prime} x x^{\prime}=x^{\prime 2} x \leqslant x^{\prime} x$ from which it follows that $x=x x^{\prime} x \leqslant x x^{\prime} x x=x^{2}$ and consequently $x \in E(S)$. The converse is clear.

Definition. We shall say that a principally ordered regular semigroup is pointed whenever it satisfies any of the six equivalent properties of Theorem 1.

By way of providing a source of examples, we recall that the natural order $\leqslant_{n}$ on the idempotents of a regular semigroup is defined by

$$
e \leqslant_{n} f \Longleftrightarrow e=e f=f e,
$$

and that an ordered regular semigroup $(T ; \leqslant)$ is said to be naturally ordered if the order $\leqslant$ extends the natural order, in the sense that if $e \leqslant_{n} f$ then $e \leqslant f$. In this case, a fundamental property is that if $e \leqslant f$ then $e=e f e$; see, for example, [1, Theorem 13.11].

Theorem 2. If $T$ is a naturally ordered regular semigroup with a biggest idempotent $\xi$ then the semiband $\langle E(T)\rangle$ is a pointed principally ordered regular semigroup.

Proof. If $\bar{e}=e_{1} \cdots e_{n} \in\langle E(T)\rangle$ then, since $\xi$ is the biggest element of $\langle E(T)\rangle$, we have that $e \xi e=e$ for every $e \in E(T)$, and consequently

$$
\bar{e} \xi \bar{e}=e_{1} \cdots e_{n} \xi e_{1} \cdots e_{n} \leqslant e_{1} \xi e_{1} \cdots e_{n}=e_{1} \cdots e_{n}=\bar{e}
$$

It follows that the regular subsemigroup $\langle E(T)\rangle$ is principally ordered with $\bar{e}^{\star}=\xi$ for every $\bar{e} \in\langle E(T)\rangle$. Furthermore, $\bar{e}^{2}=\bar{e} e_{1} \bar{e} \leqslant \bar{e} \xi \bar{e} \leqslant \bar{e}$ and it follows by Theorem $1(5)$ that $\langle E(T)\rangle$ is pointed.

To avoid unnecessary repetition throughout what follows, $S$ will always denote a pointed principally ordered regular semigroup.

As we have seen above, a characteristic property of $S$ is that the classes modulo Green's relations $\mathcal{R}$ and $\mathcal{L}$ have biggest elements which are idempotent. We now show that the same is true for Green's relation $\mathcal{D}$.

Theorem 3. Green's relation $\mathcal{D}$ on $S$ is given by

$$
(x, y) \in \mathcal{D} \Longleftrightarrow x^{\circ}=y^{\circ}
$$

Moreover, every $\mathcal{D}$-class has a biggest element which is idempotent. Specifically,

$$
(\forall x \in S) \quad x^{\circ}=x^{\circ \circ}=\max \left[x^{\star} x\right]_{\mathcal{R}}=\max \left[x x^{\star}\right]_{\mathcal{L}}=\max [x]_{\mathcal{D}} \in E(S) .
$$

Proof. As observed in the proof of Theorem 1, we have $\left(x x^{\star}\right)^{\star}=x^{\star}$ and therefore, by Theorem 1(4),

$$
x^{\circ}=x^{\star} x x^{\star}=x^{\star} x\left(x^{\star} x\right)^{\star}=\max \left[x^{\star} x\right]_{\mathcal{R}} \in E(S),
$$

and dually for $\mathcal{L}$. Moreover, by $\left(P_{7}\right)$ and Theorem $1(6,7)$,

$$
x^{\circ \circ}=x^{\circ \star} x^{\circ} x^{\circ \star}=x^{\star \star} x^{\star} x x^{\star} x^{\star \star}=x^{\star} x x^{\star}=x^{\circ} .
$$

If now $x \mathcal{D} y$ then there exists $z \in S$ such that $x \mathcal{L} z \mathcal{R} y$. Then $x^{\star} x=z^{\star} z$ and $z z^{\star}=y y^{\star}$. It follows from the above that $x^{\circ}=z^{\circ}=y^{\circ}$. On the other hand, $x \mathcal{L} x^{\star} x \mathcal{R} x^{\circ}$ gives $x \mathcal{D} x^{\circ}$. Consequently $x \mathcal{D} y \Longleftrightarrow x^{\circ}=y^{\circ}$. Finally, by Theorem $1(2,4)$ we see that $x \leqslant x x^{\star} \leqslant x^{\star} x x^{\star}=x^{\circ}$ whence it follows that $x^{\circ}=\max [x]_{\mathcal{D}} \in E(S)$.

Theorem 4. (1) $x \in S$ is a maximal idempotent if and only if it is a maximal element;
(2) $S$ contains at most one maximal element.

Proof. (1) Suppose that $e$ is a maximal idempotent of $S$. If $x \in S$ is such that $e \leqslant x$ then we have $e \leqslant x \leqslant x^{\star} \in E(S)$, whence the hypothesis that $e$ is maximal in $E(S)$ gives $e=x$. Thus $e$ is a maximal element of $S$. Conversely, if $x \in S$ is a maximal element then $x \leqslant x^{\star}$ gives $x=x^{\star}$ whence, by Theorem $1(6), x \in E(S)$.
(2) Let $e$ and $f$ be maximal elements of $S$. By (1), each is then idempotent. Now, by Theorem 1(5), ef $\cdot e \cdot e f=(e f)^{2} \leqslant e f$ and gives $e \leqslant(e f)^{\star}$. It follows that $e=(e f)^{\star}$ and likewise $e=(f e)^{\star}$. Similarly, $f=(f e)^{\star}=(e f)^{\star}$ and consequently $e=f$.

By [4, Theorem 3.3], a principally ordered regular semigroup is naturally ordered if and only if the assignment $x \mapsto x^{\star}$ is antitone. In this case, as shown in [1, Theorem 13.29], each $\left(x x^{\star}\right)^{\star}$ is a maximal idempotent. Using this fact in the case where $S$ is pointed, we obtain the following characterisation.

Theorem 5. The following statements are equivalent:
(1) $S$ is naturally ordered;
(2) $S$ has a biggest element $\xi$ and $x^{\star}=\xi$ for every $x \in S$.

Proof. (1) $\Rightarrow$ (2): If (1) holds then each $\left(x x^{\star}\right)^{\star}=x^{\star}$ is a maximal idempotent and $x \leqslant x^{\star}$. Then property (2) follows immediately from Theorem 4.
$(2) \Rightarrow(1)$ : Suppose conversely that (2) holds and let $e, f \in E(S)$ be such that $e \leqslant n f$. By (2), $e^{\star}=\xi=f^{\star}$ and consequently $e=e f=f e f \leqslant f e^{\star} f=f f^{\star} f=f$. Thus $S$ is naturally ordered.

Corollary. If $S$ is naturally ordered then Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide.
Proof. By Theorem 5, $\left(x^{2}\right)^{\star}=\xi=x^{\star}$. Consequently, $x^{2}=x^{2}\left(x^{2}\right)^{\star} x^{2}=$ $x^{2} x^{\star} x^{2}=x^{3}$. Then $x^{2} \in E(S)$ and so $S$ is group bound. It follows by [6, Theorem 1.2.20] that $\mathcal{D}$ and $\mathcal{J}$ coincide.

Consider now the subset $S^{\star}=\left\{x^{\star} \mid x \in S\right\}$. This is related to the subset $S^{\circ}$ and to the set $C=\left\{x \in S \mid x^{\star}=x^{\circ}\right\}$ of compact elements as follows.

Theorem 6. $S^{\star}=C \cap S^{\circ}$.
Proof. The identity $x^{\star \star}=x^{\star 0}$ shows that $S^{\star} \subseteq C$. Similarly, $x^{\star}=x^{\star \star \star}=x^{\star \star \circ}=$ $x^{\star \circ \circ}$ shows that $S^{\star} \subseteq S^{\circ}$. Thus $S^{\star} \subseteq C \cap S^{\circ}$. Conversely, if $x \in C \cap S^{\circ}$ then $x^{\star}=x^{\circ}$ and $x=x^{\circ \circ}$, whence $x=x^{\circ \circ}=x^{\star \circ}=x^{\star \star} \in S^{\star}$.

As the following example shows, $S^{\star}$ is not in general a subsemigroup of $S$.
Example 1. Let $L$ be a lattice and consider the cartesian ordered set

$$
L^{[2]}=\{(x, y) \in L \times L \mid y \leqslant x\} .
$$

With respect to the multiplication defined by

$$
(x, y)(a, b)=(x \vee a, y \wedge b)
$$

it is clear that $L^{[2]}$ is an ordered band. It is readily verified that $L^{[2]}$ is principally ordered with $(x, y)^{\star}=(x, x)$. By Theorem $1(5), L^{[2]}$ is pointed with $\left(L^{[2]}\right)^{\star}=$ $\{(x, x) \mid x \in L\}$. Now $\left(L^{[2]}\right)^{\star}$ is not a subsemigroup, for clearly $(x, y)^{\star}(a, b)^{\star}=$ $(x \vee a, x \wedge a)$ and this belongs to $\left(L^{[2]}\right)^{\star}$ if and only if $x=a$. The particular case of $\mathbb{N}^{[2]}$ is illustrated as follows:


However, in the presence of an identity element 1 the subset $S^{\star}$ has a particular description.

Theorem 7. If $S$ has an identity element 1 , then $S^{\star}=\{x \in S \mid 1 \leqslant x\}$ and is a join semilattice in which $x \vee y=x y$.

Proof. If $x \in S$, then since $x 1 x=x^{2} \leqslant x$ we have $1 \leqslant x^{\star}$. Conversely, let $1 \leqslant x$. Then

$$
x^{\star}= \begin{cases}1 x^{\star} \leqslant x x^{\star} \leqslant x^{\star} x^{\star}=x^{\star} & \text { whence } x^{\star}=x x^{\star} \\ x^{\star} 1 \leqslant x^{\star} x \leqslant x^{\star} x^{\star}=x^{\star} & \text { whence } x^{\star}=x^{\star} x\end{cases}
$$

Hence $x^{\star} \mathcal{H} x$ and so $x^{\star}=x$ by Theorem 1(8). Thus we see that $S^{\star}=\{x \in$ $\langle E(S)\rangle \mid 1 \leqslant x\}$ and is a sub-band. Now if $x, y \in S^{\star}$ then $x=x 1 \leqslant x y$ and $y=1 y \leqslant x y$, so that $x y$ is an upper bound for $\{x, y\}$. Furthermore, if $z \in S$ is any upper bound for $\{x, y\}$ then necessarily $z \in S^{\star}$ whence $x y \leqslant z^{2}=z$. Consequently, $S^{\star}$ is a join semilattice in which $x \vee y=x y$.

Example 2. Let 3 denote the 3 -element chain $0<1<2$ and consider the ordered regular semigroup consisting of those isotone mappings $f$ on $\mathbf{3}$ which are such that $f(0)=0$. Equivalently, this is the semigroup Res $\mathbf{3}$ of residuated mappings on 3 [2]. It has the following Hasse diagram and Cayley table, in which $[0 a b]$ denotes the mapping $f$ such that $f(0)=0, f(1)=a, f(2)=b$.


This semiband is principally ordered and pointed, with identity element $e$. Here we have $x^{\star}=u$ for $x \neq e$ and $e^{\star}=e$, so that $S^{\star}=\{e, u\}$.

Example 3. Consider, for $n \geqslant 2$, the ordered semigroup $\mathbf{B}_{n}$ of $n \times n$ matrices with entries in a boolean algebra $\mathbf{B}$. For the basic operations in $\mathbf{B}$ we use the notation $a+b$ (for $a \vee b$ ) and $a b$ (for $a \wedge b$ ).

As shown in [1], this semigroup is regular if and only if $n=2$. Moreover, as is established in [5], $\mathbf{B}_{2}$ is principally ordered with

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]^{\star}=\left[\begin{array}{ll}
b^{\prime}+c^{\prime}+d & a^{\prime}+d^{\prime}+b \\
a^{\prime}+d^{\prime}+c & b^{\prime}+c^{\prime}+a
\end{array}\right]
$$

The set of idempotents is

$$
E\left(\mathbf{B}_{2}\right)=\left\{\left.\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \right\rvert\, b+c \leqslant a+d, b c \leqslant a d\right\},
$$

and the regular subsemigroup they generate is

$$
\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle=\left\{\left.\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \right\rvert\, b c \leqslant a d\right\} .
$$

The semiband $\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle$ is also principally ordered and pointed. This follows from Theorem 1 and the observation that $b c \leqslant a d$ gives $b^{\prime}+c^{\prime} \geqslant a^{\prime}+d^{\prime}$ whence

$$
\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]^{\star}=\left[\begin{array}{cc}
1 & a^{\prime}+d^{\prime}+b \\
a^{\prime}+d^{\prime}+c & 1
\end{array}\right] \in E\left(\mathbf{B}_{2}\right) .
$$

Since $\mathbf{B}_{2}$ has an identity element it follows from Theorem 7 that $\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle^{\star}$ is the join semilattice

$$
\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle^{\star}=\left\{X \in \mathbf{B}_{2} \mid I_{2} \leqslant X\right\}=\left\{\left.\left[\begin{array}{ll}
1 & x \\
y & 1
\end{array}\right] \right\rvert\, x, y \in \mathbf{B}\right\} .
$$

We can also identify the compact elements of $\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle$. For this we recall from $\left(P_{5}\right)$ that every $X=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ in $\mathbf{B}_{2}$ has a biggest inverse, namely

$$
X^{\circ}=X^{\star} X X^{\star}=\left[\begin{array}{ll}
b^{\prime}(a+c)+c^{\prime}(a+b)+d & a^{\prime}(c+d)+d^{\prime}(a+c)+b \\
a^{\prime}(b+d)+d^{\prime}(a+b)+c & b^{\prime}(c+d)+c^{\prime}(b+d)+a
\end{array}\right] .
$$

In particular, if $X \in\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle$ then the inequality $b c \leqslant a d$ gives $d=d+b c$ and $a=a+b c$, so that we obtain

$$
X^{\circ}=\left[\begin{array}{cc}
a+b+c+d & a^{\prime}(c+d)+d^{\prime}(a+c)+b \\
a^{\prime}(b+d)+d^{\prime}(a+b)+c & a+b+c+d
\end{array}\right] .
$$

Thus, if $X \in\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle$ is compact then necessarily $a+b+c+d=1$. Conversely, if the property $a+b+c+d=1$ holds then

$$
\begin{aligned}
a^{\prime}(c+d)+d^{\prime}(a+c)+b & \geqslant a^{\prime}(a+b)^{\prime}+d^{\prime}(b+d)^{\prime}+b \\
& =a^{\prime} b^{\prime}+b^{\prime} d^{\prime}+b \\
& =a^{\prime}+d^{\prime}+b .
\end{aligned}
$$

Clearly, the reverse inequality holds, so that $a^{\prime}(c+d)+d^{\prime}(a+c)+b=a^{\prime}+d^{\prime}+b$. Likewise, we see that $a^{\prime}(b+d)+d^{\prime}(a+b)+c=a^{\prime}+d^{\prime}+c$ and consequently $X^{\circ}=X^{\star}$. Hence the set of compact elements of $\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle$ is

$$
C=\left\{\left.\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle \right\rvert\, a+b+c+d=1\right\} .
$$

We now turn attention to the $\mathcal{D}$-classes of $S$. For idempotents $e, f$ with $e \leqslant f$ and $(e, f) \in \mathcal{D}$ we first focus on the structure of the subalgebra of $\left(S ;{ }^{\star}\right)$ generated by $\{e, f\}$. In this connection the following observation is important.
Theorem 8. Any two comparable $\mathcal{D}$-related idempotents of $S$ are mutually inverse.

Proof. Let $e, f \in E(S)$ be such that $e \leqslant f$ and $e \mathcal{D} f$. Then, by Theorem 3, $e^{\circ}=f^{\circ}$. Consequently, by Theorem 1(7) and ( $P_{7}$ ), $e^{\star}=e^{\star \star}=e^{\circ \star}=f^{\circ \star}=f^{\star \star}=$ $f^{\star}$. Moreover, the idempotents $e^{\circ}$ and $e^{\star}$ are such that $e^{\circ} e^{\star}=e^{\circ}=e^{\star} e^{\circ}$, whence $e^{\circ} \leqslant n e^{\star}$.

We first observe that $e=e e e \leqslant e f e \leqslant e f^{\star} e=e e^{\star} e=e$ so that $e=e f e$.
Consider now $f e e^{\star}$. That $f e e^{\star} \in E(S)$ follows from the inequalities

$$
f e e^{\star}=f e e^{\star} e e e^{\star} \leqslant f e e^{\star} \cdot f e e^{\star} \leqslant f f f^{\star} f e e^{\star}=f e e^{\star} .
$$

Now $f e e^{\star} \cdot e^{\circ}=f e e^{\circ}=f e e^{\star}$ and

$$
e^{\circ} \cdot f e e^{\star}\left\{\begin{array}{l}
\leqslant f^{\circ} f f e^{\star}=f^{\circ} f f^{\star}=f^{\circ} f f^{\circ}=f^{\circ}=e^{\circ} ; \\
\geqslant e^{\circ} e e^{\star}=e^{\circ} e e^{\circ}=e^{\circ},
\end{array}\right.
$$

so that $e^{\circ} \cdot f e e^{\star}=e^{\circ}$. Consequently $f e e^{\star} \mathcal{L} e^{\circ}=f^{\circ} \mathcal{L} f f^{\star}$.
Furthermore, $f e e^{\star} \cdot f f^{\star}=f e f^{\star} f f^{\star}=f e f^{\circ}=f e e^{\circ}=f e e^{\star}$ and $f f^{\star} \cdot f e e^{\star}=$ $f e e^{\star}$ show that $f e e^{\star} \leqslant n f f^{\star}$. Since these idempotents are also $\mathcal{L}$-equivalent it follows that $f e e^{\star}=f f^{\star}$.

Using the above observations, we see that $f e f \cdot f f^{\star}=f e f f^{\star}=f e f e e^{\star}=$ $f e e^{\star}=f f^{\star}$ whence fef $\mathcal{R} f f^{\star} \mathcal{R} f$. Since $f e f \in E(S)$ with $f e f \leqslant_{n} f$ it follows that $f=f e f$.

Thus $e$ and $f$ are mutually inverse.
Corollary. The following statements are equivalent:
(1) $S$ is completely simple;
(2) $S$ is compact and naturally ordered.

Proof. (1) $\Rightarrow$ (2): If $S$ is completely simple then, since $\leqslant_{n}$ reduces to equality, $S$ is trivially naturally ordered. Since the idempotents $x^{\circ}, x^{\star}$ are such that $x^{\circ} \leqslant n$ $x^{\star}$, it follows that $x^{\circ}=x^{\star}$ and therefore $S$ is compact.
$(2) \Rightarrow(1)$ : Suppose that (2) holds and that $e, f \in E(S)$ are such that $e \leqslant_{n} f$. By Theorem $5, S$ has a biggest element $\xi$ and $f^{\star}=\xi=e^{\star}$. Compactness now gives $f^{\circ}=e^{\circ}$ whence, by Theorem $3,(e, f) \in \mathcal{D}$. Since also the natural order implies that $e \leqslant f$, it follows by Theorem 8 that the idempotents $e$ and $f$ are mutually inverse. Consequently, $f=f e f=e$. Thus $\leqslant_{n}$ reduces to equality and $S$ is completely simple.

Theorem 9. Let $e, f$ be idempotents of $S$ such that $e \leqslant f$ and $e \mathcal{D} f$. If $T$ is the subalgebra of $\left(S ;^{*}\right)$ generated by $\{e, f\}$ then $T$ is a band having at most 10 elements. In the case where $T$ has precisely 10 elements it is represented by the Hasse diagram

in which elements joined by lines of positive gradient are $\mathcal{\mathcal { R }}$-related, those joined by lines of negative gradient are $\mathcal{L}$-related, and the vertical line also indicates $\leqslant_{n}$.

Proof. Since $e \mathcal{D} f$ it follows from Theorem 3 that $e^{\circ}=f^{\circ}$ whence $e^{\star}=f^{\star}$. The elements of $T$ are then finite products of the elements $e, f$ and $e^{\star}\left[=f^{\star}\right]$. Moreover, since $e \leqslant f$, every $x \in T$ is such that $e \leqslant x \leqslant e^{\star}$. By Theorem 8 , $e$ and $f$ are mutually inverse, so for every $x \in T$ we have

$$
f=f e f \leqslant f x f \leqslant f e^{\star} f=f f^{\star} f=f
$$

whence $f=f T f$. In a similar way we see that $e=e T e$ and likewise

$$
e e^{\star}=e T e^{\star}, \quad e^{\star} e=e^{\star} T e, \quad f f^{\star}=f T f^{\star}, \quad f^{\star} f=f^{\star} T f, \quad e f=e T f, \quad f e=f T e .
$$

For example, $e e^{\star}=e e e^{\star} \leqslant e x e^{\star} \leqslant e e^{\star} e^{\star}=e e^{\star}$ gives $e e^{\star}=e T e^{\star}$. It follows from this that $e e^{\star}=e f e^{\star}=e f f^{\star}$ whence $e e^{\star} f=e f$ and then $e f=e e f \leqslant$ $e x f \leqslant e e^{\star} f=e f$ and consequently $e f=e T f$. Finally, it is readily seen from the above that $e^{\star} T e^{\star}=\left\{e^{0}, e^{\star}\right\}$. It now follows from these observations that $T$ is a band which consists of at most 10 elements, has precisely two $\mathcal{D}$-classes, and is as described in the above Hasse diagram.

Example 4. In the semigroup $\left\langle E\left(\mathbf{B}_{2}\right)\right\rangle$ of Example 3, let $|\mathbf{B}| \geqslant 8$ and consider the idempotents

$$
e=\left[\begin{array}{ll}
a & 0 \\
0 & 0
\end{array}\right], \quad f=\left[\begin{array}{ll}
a & b \\
b & b
\end{array}\right] \quad \text { where } 0<b<a<1 .
$$

Simple calculations which use the expressions for $X^{\star}$ and $X^{\circ}$ given in Example 3 reveal that $e^{\star}=\left[\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right]=f^{\star}$, and that $e^{\circ}=\left[\begin{array}{ll}a & a \\ a & a\end{array}\right]=f^{\circ}$ whence $e \mathcal{D} f$ with $e<f$.

Furthermore,

$$
e e^{\star}=\left[\begin{array}{ll}
a & a \\
0 & 0
\end{array}\right], f f^{\star}=\left[\begin{array}{ll}
a & a \\
b & b
\end{array}\right], e^{\star} e=\left[\begin{array}{ll}
a & 0 \\
a & 0
\end{array}\right], f^{\star} f=\left[\begin{array}{ll}
a & b \\
a & b
\end{array}\right], e f=\left[\begin{array}{ll}
a & b \\
0 & 0
\end{array}\right], f e=\left[\begin{array}{ll}
a & 0 \\
b & 0
\end{array}\right] .
$$

Consequently we have a copy of the band depicted in Theorem 9.
We now proceed to describe the structure of those $\mathcal{D}$-classes that are subsemigroups of $S$ (which is the case for $D_{e}$ in Theorem 9, but not so for $D_{u}$ in Example 2 since $g f=0$ ).

Theorem 10. Given $e \in E(S)$, suppose that $D_{e}$ is a subsemigroup of $S$. Then $L_{e^{\circ}}$ is a left zero semigroup, $R_{e} \circ$ is a right zero semigroup, and $D_{e}$ is isomorphic to the ordered rectangular band $L_{e^{\circ}} \times R_{e^{\circ}}$.

Proof. We observe first that, since $x^{\circ}=e^{\circ}$ for every $x \in D_{e}$,

$$
\begin{aligned}
& x \in L_{e^{\circ}} \Longleftrightarrow x^{\circ} x=e^{\circ} \Longleftrightarrow x=x x^{\circ} \in D_{e} ; \\
& x \in R_{e^{\circ}} \Longleftrightarrow x x^{\circ}=e^{\circ} \Longleftrightarrow x=x^{\circ} x \in D_{e} .
\end{aligned}
$$

If therefore $x, y \in L_{e^{\circ}}$ we have $x y=x x^{\circ} y=x e^{\circ} y=x y^{\circ} y=x e^{\circ}=x x^{\circ}=x$ and consequently $L_{e^{\circ}}$ is a left zero semigroup. Likewise, $R_{e^{\circ}}$ is a right zero semigroup. Then

$$
L_{e^{\circ}} \times R_{e^{\circ}}=\left\{\left(x e^{\circ}, e^{\circ} y\right) \mid x, y \in D_{e}\right\}
$$

is a rectangular band. Consider therefore the mapping $\vartheta: D_{e} \rightarrow L_{e^{\circ}} \times R_{e^{\circ}}$ given by the prescription $\vartheta(x)=\left(x e^{\circ}, e^{\circ} x\right)$, which is clearly isotone.

Now if ( $a, b$ ) $\in L_{e^{\circ}} \times R_{e}$ 。 then, since $a b \in D_{e}$ by the hypothesis with

$$
\vartheta(a b)=\left(a b e^{\circ}, e^{\circ} a b\right)=\left(a b b^{\circ}, a^{\circ} a b\right)=\left(a e^{\circ}, e^{\circ} b\right)=(a, b),
$$

we see that $\vartheta$ is surjective. Moreover,

$$
\begin{aligned}
\vartheta(x) \leqslant \vartheta(y) & \Longleftrightarrow x e^{\circ} \leqslant y e^{\circ}, e^{\circ} x \leqslant e^{\circ} y \\
& \Longleftrightarrow x=x e^{\circ} x \leqslant y e^{\circ} y=y .
\end{aligned}
$$

It follows from these observations that $\vartheta$ is an order isomorphism.
We now observe that if $e, f$ are $\mathcal{D}$-equivalent idempotents such that $e \leqslant_{n} f$ then $e=e f=f e \leqslant f e^{\circ}=f f^{\circ}$ and consequently $e=e f \leqslant f f^{\circ} f=f$. Thus $D_{e}$ is a naturally ordered regular semigroup with a biggest idempotent $e^{0}$. Since $(x y)^{\circ}=e^{\circ}=y^{\circ} x^{\circ}$ for all $x, y \in D_{e}$, it follows by [1, Theorem 13.18] that $e^{\circ}$ is a middle unit of $D_{e}$. Using this fact, we see that

$$
\vartheta(x) \vartheta(y)=\left(x e^{\circ}, e^{\circ} x\right)\left(y e^{\circ}, e^{\circ} y\right)=\left(x e^{\circ} y e^{\circ}, e^{\circ} x e^{\circ} y\right)=\left(x y e^{\circ}, e^{\circ} x y\right)=\vartheta(x y),
$$

whence we conclude that $\vartheta$ defines an ordered semigroup isomorphism $D_{e} \simeq$ $L_{e^{\circ}} \times R_{e^{\circ}}$.

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