

## IF-FILTERS OF PSEUDO-BL-ALGEBRAS

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### Abstract

Characterizations of IF-filters of a pseudo-BL-algebra are established. Some related properties are investigated. The notation of prime IF-filters and a characterization of a pseudo-BL-chain are given. Homomorphisms of IF-filters and direct product of IF-filters are studied.

**Keywords:** pseudo-BL-algebra, filter, IF-filter, prime IF-filters, pseudo-BL-chain, homomorphism, direct product.

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### 1. INTRODUCTION

In 1958, Chang [2] gave a notation and a characterization of MV-algebras. In 1998, Hájek [8] introduced BL-algebras, which contain the class of MV-algebras. Georgescu and Iorgulescu [5] and independently Rachůnek [10] introduced pseudo MV-algebras as a noncommutative extension of MV-algebras. Finally, in 2000 there were given a notion of pseudo-BL-algebras, which are a noncommutative extension of BL-algebras. Some important properties of pseudo-BL-algebras were studied in [3, 4] and [7].

Zadeh [14] introduced fuzzy sets. Fuzzy sets and filters of pseudo-BL-algebras were studied in [11] and anti fuzzy filters were investigated in [13]. In 1983, Atanassov [1] gave a notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Takeuti and Titants [12] introduced an intuitionistic fuzzy logic.

In this paper, we introduce a notation of intuitionistic fuzzy filters of pseudo-BL-algebras and study their properties. We introduce prime intuitionistic fuzzy filters and using them we give a characterization of a pseudo-BL-chain. We investigate a homomorphism of intuitionistic fuzzy filters. Finally, we study a direct product of intuitionistic fuzzy filters. We will write shortly IF-filters instead of intuitionistic fuzzy filters.

## 2. PRELIMINARIES

**Definition 1.** In [6], there were introduced a pseudo-BL-algebra  $A$  as an algebra  $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$  of type  $(2, 2, 2, 2, 2, 0, 0)$  satisfying the following axioms for all  $x, y, z \in A$ :

- (C1)  $(A, \vee, \wedge, 0, 1)$  is a bounded lattice;
- (C2)  $(A, \odot, 1)$  is a monoid;
- (C3)  $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z$ ;
- (C4)  $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$ ;
- (C5)  $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ .

**Lemma 1** ([7]). *Let  $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$  be a pseudo-BL-algebra. Then for all  $x, y, z \in A$ :*

- (i)  $y \leq x \rightarrow y$  and  $y \leq x \rightsquigarrow y$ ;
- (ii)  $x \odot y \leq x \wedge y$ ;
- (iii)  $x \odot y \leq x$  and  $x \odot y \leq y$ ;
- (iv)  $x \rightarrow 1 = x \rightsquigarrow 1 = 1$ ;
- (v)  $x \leq y \Leftrightarrow x \rightarrow y = x \rightsquigarrow y = 1$ ;
- (vi)  $x \rightarrow x = x \rightsquigarrow x = 1$ ;
- (vii)  $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$  and  $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$ .

We will write shortly  $A$  instead of  $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ .

**Definition 2.** A nonempty subset  $F$  of a pseudo-BL-algebra  $A$  is called a filter if it satisfies the following two conditions:

- (F1) if  $x, y \in F$ , then  $x \odot y \in F$ ;
- (F2) if  $x \in F$  and  $x \leq y$ , then  $y \in F$ .

A filter  $F$  of a pseudo-BL-algebra  $A$  is called *proper* if  $F \neq A$ . The proper filter  $F$  is prime if for all  $x, y \in A$

$$x \vee y \in F \text{ implies } (x \in F \text{ or } y \in F).$$

Now, we give definitions of a fuzzy filter and an anti fuzzy filter of a pseudo-BL-algebra  $A$  and their some properties.

Recall that a *fuzzy set* of  $A$  is a function  $\nu : A \rightarrow [0, 1]$ . For any fuzzy set  $\nu$  and real number  $\alpha \in [0, 1]$  there are defined two sets:

$$\begin{aligned} U(\nu, \alpha) &= \{x \in A : \nu(x) \geq \alpha\}; \\ L(\nu, \alpha) &= \{x \in A : \nu(x) \leq \alpha\}; \end{aligned}$$

which are called an upper and a lower  $\alpha$ -level set of  $\nu$ .

**Definition 3.** Let  $\nu$  be a fuzzy set of pseudo-BL-algebra  $A$ . A *complement* of  $\nu$  is the fuzzy set  $\nu^C$  defined as follows

$$\nu^C(x) = 1 - \nu(x)$$

for any  $x \in A$ .

A fuzzy set  $\mu$  is called:

1. a *fuzzy filter*, if for all  $x, y \in A$ 
  - (ff1)  $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$ ;
  - (ff2)  $x \leq y \Rightarrow \mu(x) \leq \mu(y)$ .
2. an *anti fuzzy filter*, if for all  $x, y \in A$ 
  - (af1)  $\mu(x \odot y) \leq \mu(x) \vee \mu(y)$ ;
  - (af2)  $x \leq y \Rightarrow \mu(y) \leq \mu(x)$ .

**Remark 1.** Let  $\mu$  and  $\nu$  be a fuzzy sets of a pseudo-BL-algebra  $A$ . Then:

- (i)  $\mu$  is a fuzzy filter of  $A$  iff  $\mu^C$  is an anti fuzzy filter of  $A$ ;
- (ii)  $\nu$  is an anti fuzzy filter of  $A$  iff  $\nu^C$  is a fuzzy filter of  $A$ .

**Definition 4** ([11]). Let  $F$  be a filter of a pseudo-BL-algebra  $A$  and  $\alpha, \beta \in [0, 1]$  such that  $\alpha > \beta$ . Let us define a fuzzy filter  $\mu_F(\alpha, \beta)$  as follows

$$\mu_F(\alpha, \beta)(x) = \begin{cases} \alpha & \text{if } x \in F, \\ \beta & \text{otherwise.} \end{cases}$$

**Remark 2** ([13]). A fuzzy set  $\mu_F^C(\alpha, \beta)$  is an anti fuzzy filter of  $A$ .

We denote by  $\chi_F$  the characteristic function of  $F$  and by  $\chi_F^C$  the complement of the characteristic function of  $F$ .

**Definition 5.** Let  $A$  be a pseudo-BL-algebra and  $\nu$  be a fuzzy filter of  $A$ . Then  $\nu$  is called a *fuzzy prime filter* if

$$\nu(x \vee y) = \nu(x) \vee \nu(y)$$

for all  $x, y \in A$ .

**Definition 6.** Let  $A$  be a pseudo-BL-algebra and  $\mu$  be an anti fuzzy filter of  $A$ . Then  $\mu$  is called an anti *fuzzy prime filter* if

$$\mu(x \vee y) = \mu(x) \wedge \mu(y)$$

for all  $x, y \in A$ .

For a fuzzy filter  $\nu$  of pseudo-BL-algebra  $A$  we define a set

$$M_\nu = \{x \in A : \nu(x) = \nu(1)\}$$

and similarly, for an anti fuzzy filter  $\mu$  we define a set

$$A_\mu = \{x \in A : \mu(x) = \mu(1)\}.$$

**Remark 3.** It is proved in [11] and [13] that a fuzzy filter  $\nu$  of  $A$  is a fuzzy prime filter (an anti fuzzy filter  $\mu$  of  $A$  is an anti fuzzy prime filter) iff  $M_\nu$  ( $A_\mu$ ) is a prime filter of  $A$ .

### 3. IF-FILTERS

**Definition 7.** A mapping  $\mathcal{B} : A \rightarrow [0, 1] \times [0, 1]$  such that  $\mathcal{B}(x) = (\nu_{\mathcal{B}}(x), \mu_{\mathcal{B}}(x))$ , in which  $\nu_{\mathcal{B}}(x) + \mu_{\mathcal{B}}(x) \leq 1$  for any  $x \in A$ , is called an IF-set of  $A$ .

In particular, we use  $0_\sim$  and  $1_\sim$  to denote the IF-empty set and the IF-whole set in a set  $A$  such that  $0_\sim(x) = (0; 1)$  and  $1_\sim(x) = (1; 0)$  for each  $x \in A$ , respectively.

For IF-sets  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  and  $\mathcal{C} = (\nu_{\mathcal{C}}, \mu_{\mathcal{C}})$  we define a relation  $\leq$  as follows:

$$\mathcal{B} \leq \mathcal{C} \Leftrightarrow (\nu_{\mathcal{B}}(x) < \nu_{\mathcal{C}}(x) \text{ or } (\nu_{\mathcal{B}}(x) = \nu_{\mathcal{C}}(x) \text{ and } \mu_{\mathcal{B}}(x) < \mu_{\mathcal{C}}(x)) \text{ for any } x \in A).$$

Now, we give the definition of an IF-filter of a pseudo-BL-algebra. From this place an IF-set  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  will be denoted by  $\mathcal{B}$ .

**Definition 8.** An IF-set  $\mathcal{B}$  of pseudo-BL-algebra  $A$  is an IF-filter of  $A$  if it satisfies the following conditions for all  $x, y \in A$ :

$$(IF1) \quad \nu_{\mathcal{B}}(x \odot y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y);$$

$$(IF2) \quad \mu_{\mathcal{B}}(x \odot y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y);$$

$$(IF3) \quad x \leq y \Rightarrow (\nu_{\mathcal{B}}(x) \leq \nu_{\mathcal{B}}(y) \text{ and } \mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(y)).$$

**Remark 4.** An IF-set  $\mathcal{B}$  of a pseudo-BL-algebra  $A$  is an IF-filter of  $A$  iff  $\nu_{\mathcal{B}}$  is a fuzzy filter and  $\mu_{\mathcal{B}}$  is an anti fuzzy filter of  $A$ .

It is easy to see, that (IF3) implies

(IF4)  $\nu_{\mathcal{B}}(x) \leq \nu_{\mathcal{B}}(1)$  and  $\mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(1)$  for every  $x \in A$ ;

(IF4')  $\nu_{\mathcal{B}}(0) \leq \nu_{\mathcal{B}}(x)$  and  $\mu_{\mathcal{B}}(0) \geq \mu_{\mathcal{B}}(x)$  for every  $x \in A$ .

**Proposition 1.** *Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$ . Then  $\mathcal{B}$  is an IF-filter of  $A$  iff  $\mathcal{B}_C = (\nu_{\mathcal{B}}, \nu_{\mathcal{B}}^C)$  and  ${}_C\mathcal{B} = (\mu_{\mathcal{B}}^C, \mu_{\mathcal{B}})$  are IF-filters of  $A$ .*

**Proof.**  $\Rightarrow$ : Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$ . By Remark 4  $\nu_{\mathcal{B}}$  is a fuzzy filter and  $\mu_{\mathcal{B}}$  is an anti fuzzy filter of  $A$ . Then  $\nu_{\mathcal{B}}^C$  is an anti fuzzy filter and  $\mu_{\mathcal{B}}^C$  is a fuzzy filter of  $A$ . Using Remark 4 once again we obtain that  $\mathcal{B}_C = (\nu_{\mathcal{B}}, \nu_{\mathcal{B}}^C)$  and  ${}_C\mathcal{B} = (\mu_{\mathcal{B}}^C, \mu_{\mathcal{B}})$  are IF-filters of  $A$ .

$\Leftarrow$ : By Remark 4. ■

**Example 1.** Let  $F$  be a filter of a pseudo-BL-algebra  $A$  and  $\mathcal{B}(F) = (\nu_{\mathcal{B}(F)}, \mu_{\mathcal{B}(F)})$  be an IF-set of  $A$  defined as follows

$$\nu_{\mathcal{B}(F)}(x) := \begin{cases} \alpha & \text{if } x \in F; \\ \beta & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}(F)}(x) := \begin{cases} \alpha_1 & \text{if } x \in F; \\ \beta_1 & \text{otherwise.} \end{cases}$$

where  $\alpha, \alpha_1, \beta, \beta_1 \in [0, 1]$ ,  $\alpha > \beta$ ,  $\alpha_1 < \beta_1$  and  $\alpha + \alpha_1, \beta + \beta_1 \leq 1$ .

By Definition 4 and Remark 2,  $\nu_{\mathcal{B}(F)}$  is a fuzzy filter of  $A$  and  $\mu_{\mathcal{B}(F)}$  is an anti fuzzy filter of  $A$ . Hence, by Remark 4,  $\mathcal{B}(F)$  is an IF-filter of  $A$ .

**Proposition 2.** *Let  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  be an IF-filter of a pseudo-BL-algebra  $A$ , then for all  $x, y \in A$ :*

- (i)  $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ ;
- (ii)  $\nu_{\mathcal{B}}(x \wedge y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ ;
- (iii)  $\nu_{\mathcal{B}}(x \odot y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ ;
- (iv)  $\mu_{\mathcal{B}}(x \wedge y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$ ;
- (v)  $\mu_{\mathcal{B}}(x \odot y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$ ;
- (vi)  $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$ .

**Proof.** By Lemma 1 (ii)  $x \odot y \leq x \wedge y \leq x \vee y$ . Then, by definition of an IF-filter,  $\nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y) \leq \nu_{\mathcal{B}}(x \odot y) \leq \nu_{\mathcal{B}}(x \wedge y) \leq \nu_{\mathcal{B}}(x \vee y)$  and  $\mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y) \geq \mu_{\mathcal{B}}(x \odot y) \geq \mu_{\mathcal{B}}(x \wedge y) \geq \mu_{\mathcal{B}}(x \vee y)$ . (i) and (vi) are proved. Applying Lemma 1 (iii), we have  $\nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y) \leq \nu_{\mathcal{B}}(x \odot y) \leq \nu_{\mathcal{B}}(x \wedge y) \leq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$  and  $\mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y) \geq \mu_{\mathcal{B}}(x \odot y) \geq \mu_{\mathcal{B}}(x \wedge y) \geq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$ . The proofs for (ii), (iii), (iv) and (v) are finished. ■

**Proposition 3.** *An IF-set  $\mathcal{B}$  of a pseudo-BL-algebra  $A$  is an IF-filter of  $A$  if and only if it satisfies (IF1), (IF2) and*

(IF5)  $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$  and  $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$  for all  $x, y \in A$ .

**Proof.**  $\Rightarrow$ : Let us suppose that  $\mathcal{B}$  is an IF-filter of  $A$ . Then, by (IF3),  $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$  and  $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$  for all  $x, y \in A$ .

$\Leftarrow$ : Conversely, let  $\mathcal{B}$  satisfies (IF1), (IF2) and (IF5). We need to show that  $\mathcal{B}$  satisfies (IF3). Let  $x, y \in A$  be such that  $x \leq y$ . By (IF5) we have  $\nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$  and  $\mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$ . Hence (IF3) is satisfied. ■

**Theorem 1.** *Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$ . The following are equivalent:*

(i)  $\mathcal{B}$  is an IF-filter;

(ii)  $\mathcal{B}$  satisfies (IF3) and for all  $x, y \in A$

$$(1) \quad \nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightarrow y),$$

$$(2) \quad \mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightarrow y),$$

(iii)  $\mathcal{B}$  satisfies (IF3) and for all  $x, y \in A$

$$(3) \quad \nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightsquigarrow y),$$

$$(4) \quad \mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightsquigarrow y).$$

**Proof.** Using Remark 4 of this paper, Proposition 3.3 and Corollary 3.4 of [13] and Theorem 3.3 of [11] we have the thesis. ■

**Proposition 4.** *Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$ . The following are equivalent:*

(i)  $\mathcal{B}$  is an IF-filter;

(ii) for all  $x, y, z \in A$

$$(5) \quad x \rightarrow (y \rightarrow z) = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$

$$(6) \quad x \rightarrow (y \rightarrow z) = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).$$

(iii) for all  $x, y, z \in A$

$$(7) \quad x \rightsquigarrow (y \rightsquigarrow z) = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$

$$(8) \quad x \rightsquigarrow (y \rightsquigarrow z) = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).$$

**Proof.** (i) $\Rightarrow$ (ii) Suppose that  $\mathcal{B}$  is an IF-filter of a pseudo-BL-algebra  $A$ . Let  $x, y, z \in A$  be such that  $x \rightarrow (y \rightarrow z) = 1$ . By Theorem 1 (ii)

$$(9) \quad \nu_{\mathcal{B}}(y \rightarrow z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightarrow (y \rightarrow z)) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(1) = \nu_{\mathcal{B}}(x),$$

$$(10) \quad \mu_{\mathcal{B}}(y \rightarrow z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightarrow (y \rightarrow z)) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(1) = \mu_{\mathcal{B}}(x).$$

Applying Theorem 1 (ii) the second time we obtain

$$(11) \quad \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(y) \wedge \nu_{\mathcal{B}}(y \rightarrow z),$$

$$(12) \quad \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(y) \vee \mu_{\mathcal{B}}(y \rightarrow z).$$

(9), (10), (11) and (12) force  $\nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$  and  $\mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$ .

(ii) $\Rightarrow$ (i) Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$  which satisfies (3). Let  $x, y \in A$  be such that  $x \leq y$ . By Lemma 1 (iv) and (v),

$$x \rightarrow (x \rightarrow y) = 1,$$

hence applying (5) and (6) we have

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x) = \nu_{\mathcal{B}}(x),$$

$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x) = \mu_{\mathcal{B}}(x),$$

that is, (IF3) holds.

Now we prove that (1) and (2) hold. By Lemma 1 (vi),  $(x \rightarrow y) \rightarrow (x \rightarrow y) = 1$ . Thus, applying (5) and (6) we get

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x \rightarrow y) \wedge \nu_{\mathcal{B}}(x) \text{ and}$$

$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x \rightarrow y) \vee \mu_{\mathcal{B}}(x).$$

Hence by Theorem 1,  $\mathcal{B}$  is an IF-filter.

(iii) $\Leftrightarrow$ (i) Analogously. ■

**Proposition 5.** *Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$ . The following are equivalent:*

(i)  $\mathcal{B}$  is an IF-filter;

(ii) for all  $x, y, z \in A$

$$(x \odot y) \rightarrow z = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$

$$(x \odot y) \rightarrow z = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y),$$

(iii) for all  $x, y, z \in A$

$$\begin{aligned} (x \odot y) \rightsquigarrow z = 1 &\Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y), \\ (x \odot y) \rightsquigarrow z = 1 &\Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y). \end{aligned}$$

**Proof.** By Proposition 4 and Lemma 1 (vii). ■

Let  $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$  be IF-filters of a pseudo-BL-algebra  $A$  for every  $i \in I$ . We define fuzzy sets  $\bigwedge_{i \in I} \nu_{\mathcal{B}_i}$  and  $\bigvee_{i \in I} \mu_{\mathcal{B}_i}$  as follows:

$$\begin{aligned} \left( \bigwedge_{i \in I} \nu_{\mathcal{B}_i} \right) (x) &= \bigwedge \{ \nu_{\mathcal{B}_i}(x) : i \in I \}, \\ \left( \bigvee_{i \in I} \mu_{\mathcal{B}_i} \right) (x) &= \bigvee \{ \mu_{\mathcal{B}_i}(x) : i \in I \}. \end{aligned}$$

For any IF-filters  $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$  for  $i \in I$ , of a pseudo-BL-algebra  $A$  we define the IF-set  $\bigcap_{i \in I} \mathcal{B}_i$  of  $A$  by

$$\bigcap_{i \in I} \mathcal{B}_i = \left( \bigwedge_{i \in I} \nu_{\mathcal{B}_i}, \bigvee_{i \in I} \mu_{\mathcal{B}_i} \right).$$

**Theorem 2.** Let  $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$  for  $i \in I$ , be IF-filters of a pseudo-BL-algebra  $A$ . Then  $\bigcap_{i \in I} \mathcal{B}_i$  is an IF-filter of  $A$ .

**Proof.** Let  $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$  for  $i \in I$ , be IF-filters of a pseudo-BL-algebra  $A$  and  $\mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ . We use Proposition 4 to show that  $\mathcal{B}$  is an IF-filter of  $A$ .

Let  $x, y, z \in A$  be such that  $x \rightarrow (y \rightarrow z) = 1$ . Hence

$$\begin{aligned} \nu_{\mathcal{B}}(z) &= \bigwedge_{i \in I} \nu_{\mathcal{B}_i}(z) \geq \bigwedge_{i \in I} (\nu_{\mathcal{B}_i}(x) \wedge \nu_{\mathcal{B}_i}(y)) = \bigwedge_{i \in I} \nu_{\mathcal{B}_i}(x) \wedge \bigwedge_{i \in I} \nu_{\mathcal{B}_i}(y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y), \\ \mu_{\mathcal{B}}(z) &= \bigvee_{i \in I} \mu_{\mathcal{B}_i}(z) \leq \bigvee_{i \in I} (\mu_{\mathcal{B}_i}(x) \vee \mu_{\mathcal{B}_i}(y)) = \bigvee_{i \in I} \mu_{\mathcal{B}_i}(x) \vee \bigvee_{i \in I} \mu_{\mathcal{B}_i}(y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y). \end{aligned}$$

The proof is closed. ■

**Remark 5.** The set of IF-filters of a pseudo-BL-algebra  $A$  forms a complete distributive lattice with relation  $\leq$ .



**Proof.** Since  $[0, 1]$  is a complete distributive lattice with usual ordering and by Theorem 2, the proof is completed. ■

**Theorem 3.** *A lattice of IF-filters of a pseudo-BL-algebra  $A$  is bounded.*

**Proof.** It is easily seen that  $0_\sim$  and  $1_\sim$  are IF-filters. Since  $0_\sim \leq \mathcal{B} \leq 1_\sim$  for every IF-filter  $\mathcal{B}$ , then a lattice of IF-filters is bounded. ■

**Theorem 4.** *The lattice of IF-filters of a pseudo-BL-algebra has no atoms.*

**Proof.** Let  $\mathcal{B}$  be an IF-filter of pseudo-BL-algebra  $A$  and  $\mathcal{B} \neq 0_\sim$ . Let us define an IF-set  $\mathcal{D}$  as follows

$$\mathcal{D} = \left( \frac{1}{2}\nu_{\mathcal{B}}, \frac{1}{2}\mu_{\mathcal{B}} \right).$$

It is obvious that  $\mathcal{D}$  is an IF-filter of  $A$  and  $0_\sim < \mathcal{D} < \mathcal{B}$ . Hence there are no atoms in a lattice of IF-filters of  $A$ . ■

Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$  and  $\alpha, \beta \in [0, 1]$  be such that  $\alpha + \beta \leq 1$ . Then we can define a set

$$A_{\mathcal{B}}^{(\alpha, \beta)} = \{x \in A : \nu_{\mathcal{B}}(x) \geq \alpha, \mu_{\mathcal{B}}(x) \leq \beta\}$$

called an  $(\alpha, \beta)$ -level of  $\mathcal{B}$ .

Let us notice that  $A_{\mathcal{B}}^{(\alpha, \beta)} = U(\nu_{\mathcal{B}}, \alpha) \cap L(\mu_{\mathcal{B}}, \beta)$ .

**Theorem 5.** Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$ . If  $\mathcal{B}$  is an IF-filter of  $A$ , then  $A_{\mathcal{B}}^{(\alpha, \beta)} = \emptyset$  or  $A_{\mathcal{B}}^{(\alpha, \beta)}$  is a filter of  $A$  for all  $\alpha \in [0, \nu_{\mathcal{B}}(1)]$ ,  $\beta \in [\mu_{\mathcal{B}}(1), 1]$  such that  $\alpha + \beta \leq 1$ .

**Proof.** By Theorem 3.10 of [13] and Theorem 3.6 of [11]  $\nu_{\mathcal{B}}$  is a fuzzy filter and  $\mu_{\mathcal{B}}$  is an anti fuzzy filter iff  $U(\nu_{\mathcal{B}}, \alpha)$  and  $L(\mu_{\mathcal{B}}, \beta)$  are filters or empty. According to fact that the intersection of filters is a filter and by Remark 4 we have the thesis. ■

**Corollary 1.** *If  $\mathcal{B}$  is an IF-filter of a pseudo-BL-algebra  $A$ , then the set*

$$A_b = \{x \in A : \nu_{\mathcal{B}}(x) \geq \nu_{\mathcal{B}}(b), \mu_{\mathcal{B}}(x) \leq \mu_{\mathcal{B}}(b)\}$$

*is a filter of  $A$  for every  $b \in A$  such that  $\nu_{\mathcal{B}}(b) + \mu_{\mathcal{B}}(b) \leq 1$ .*

## 4. PRIME IF-FILTERS

In this section we introduce and study prime IF-filters and their connection with pseudo-BL-chains.

**Definition 9.** An IF-filter  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  of a pseudo-BL-algebra  $A$  is said to be prime IF-filter if  $\nu_{\mathcal{B}}$  and  $\mu_{\mathcal{B}}$  are non-constant and satisfies following conditions for all  $x, y \in A$ :

$$\nu_{\mathcal{B}}(x \vee y) = \nu_{\mathcal{B}}(x) \vee \nu_{\mathcal{B}}(y) \text{ and } \mu_{\mathcal{B}}(x \vee y) = \mu_{\mathcal{B}}(x) \wedge \mu_{\mathcal{B}}(y).$$

**Remark 6.** An IF-filter  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  of a pseudo-BL-algebra  $A$  is said to be prime IF-filter iff  $\nu_{\mathcal{B}}$  is a fuzzy prime filter and  $\mu_{\mathcal{B}}$  is an anti fuzzy prime filter of  $A$ .

**Theorem 6.** Let  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  be a non-constant IF-filter of a pseudo-BL-algebra  $A$ . Then the following are equivalent:

- (i)  $\mathcal{B}$  is a prime IF-filter of  $A$ ;
- (ii) for all  $x, y \in A$ , if  $(\nu_{\mathcal{B}}(x \vee y) = \nu_{\mathcal{B}}(1) \text{ and } \mu_{\mathcal{B}}(x \vee y) = \mu_{\mathcal{B}}(1))$ , then

$$\begin{aligned} &(\nu_{\mathcal{B}}(x) = \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(1)) \text{ and} \\ &(\mu_{\mathcal{B}}(x) = \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(1)); \end{aligned}$$

- (iii) for all  $x, y \in A$ ,

$$\begin{aligned} &(\nu_{\mathcal{B}}(x \rightarrow y) = \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y \rightarrow x) = \nu_{\mathcal{B}}(1)) \text{ and} \\ &(\mu_{\mathcal{B}}(x \rightarrow y) = \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y \rightarrow x) = \mu_{\mathcal{B}}(1)); \end{aligned}$$

- (iv) for all  $x, y \in A$ ,

$$\begin{aligned} &(\nu_{\mathcal{B}}(x \rightsquigarrow y) = \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y \rightsquigarrow x) = \nu_{\mathcal{B}}(1)) \text{ and} \\ &(\mu_{\mathcal{B}}(x \rightsquigarrow y) = \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y \rightsquigarrow x) = \mu_{\mathcal{B}}(1)). \end{aligned}$$

**Proof.** By Theorem 4.1 of [11] and Theorem 4.3 of [13]. ■

**Theorem 7.** Let  $A$  be a pseudo-BL-algebra and  $\mathcal{B}$  be an IF-filter of  $A$ . Then  $\mathcal{B}$  is a prime IF-filter iff  $M_{\nu_{\mathcal{B}}}$  and  $A_{\mu_{\mathcal{B}}}$  are prime filters of  $A$ .

**Proof.** By Remark 3. ■

**Theorem 8.** Let  $A$  be a pseudo-BL-algebra,  $P$  be a filter of  $A$  and  $\alpha, \beta \in [0, 1]$  with  $\alpha > \beta$ . Then  $P$  is a prime filter of  $A$  if and only if  $\mathcal{B}(P) = (\mu_P(\alpha, \beta), \mu_P^C(1 - \alpha, 1 - \beta))$  define as in Example 1, is a prime IF-filter of  $A$ .

**Proof.** By Theorem 4.2 of [11] and Theorem 4.6 of [13]. ■

**Theorem 9.** *Let  $\mathcal{B}$  be an IF-set of a pseudo-BL-algebra  $A$  such that  $\nu_{\mathcal{B}}$  and  $\mu_{\mathcal{B}}$  are non-constant. Then the following are equivalent:*

- (i)  $\mathcal{B}$  is a prime IF-filter of  $A$ ;
- (ii) for every  $\alpha \in [0, 1]$ , if  $U(\nu_{\mathcal{B}}, \alpha), L(\mu_{\mathcal{B}}, \alpha) \neq \emptyset$  and  $U(\nu_{\mathcal{B}}, \alpha), L(\mu_{\mathcal{B}}, \alpha) \neq A$ , then  $U(\nu_{\mathcal{B}}, \alpha), L(\mu_{\mathcal{B}}, \alpha)$  are prime filters of  $A$ .

**Proof.** By Theorem 4.4 of [11] and Theorem 4.7 of [13]. ■

**Theorem 10.** *Let  $A$  be a non-trivial pseudo-BL-algebra. The following are equivalent:*

- (i)  $A$  is a pseudo-BL-chain;
- (ii) every IF-filter  $\mathcal{B}$  such that  $\nu_{\mathcal{B}}$  and  $\mu_{\mathcal{B}}$  are non-constant is a prime IF-filter of  $A$ ;
- (iii) every IF-filter  $\mathcal{B}$  such that  $\nu_{\mathcal{B}}$  and  $\mu_{\mathcal{B}}$  are non-constant,  $\nu_{\mathcal{B}}(1) = 1$  and  $\mu_{\mathcal{B}}(1) = 0$  is a prime IF-filter of  $A$ ;
- (iv) the IF-filter  $(\chi_{\{1\}}, \chi_{\{1\}}^C)$  is a prime IF-filter of  $A$ .

**Proof.** By Theorem 4.6 of [11] and Theorem 4.9 of [13]. ■

## 5. HOMOMORPHISM AND IF-FILTERS

Let  $A, B$  be pseudo-BL-algebras. Following [3] we define a homomorphism of pseudo-BL-algebras as a mapping  $h : A \rightarrow B$  such that the following conditions hold for all  $x, y \in A$  :

- (H1)  $h(x \odot y) = h(x) \odot h(y)$ ;
- (H2)  $h(x \rightarrow y) = h(x) \rightarrow h(y)$ ;
- (H3)  $h(x \rightsquigarrow y) = h(x) \rightsquigarrow h(y)$ ;
- (H4)  $h(0) = 0$ .

Recall that if  $h : A \rightarrow B$  is a homomorphism of pseudo-BL-algebras, then

- (H5)  $h(1) = 1$ ;
- (H6)  $h(x \wedge y) = h(x) \wedge h(y)$ ;

$$(H7) \quad h(x \vee y) = h(x) \vee h(y).$$

**Definition 10.** Let  $\mathcal{B}$  be an IF-filter of a pseudo-BL-algebra  $B$  and  $f : A \rightarrow B$  be a homomorphism of pseudo-BL-algebras. The preimage of  $\mathcal{B}$  is the IF-set  $\mathcal{B}^f = (\nu_{\mathcal{B}}^f, \mu_{\mathcal{B}}^f)$  defined by

$$\nu_{\mathcal{B}}^f(x) = \nu_{\mathcal{B}}(f(x)) \text{ and } \mu_{\mathcal{B}}^f(x) = \mu_{\mathcal{B}}(f(x))$$

for all  $x \in A$ .

**Theorem 11.** Let  $\mathcal{B}$  be an IF-filter of  $B$  and  $f : A \rightarrow B$  be a homomorphism of pseudo-BL-algebras. Then  $\mathcal{B}^f$  is an IF-filter of  $A$ .

**Proof.** Suppose that  $f : A \rightarrow B$  is a homomorphism of pseudo-BL-algebras and  $\mathcal{B}$  be an IF-filter of  $B$ . Let  $x, y \in A$ . Then

$$\begin{aligned} \nu_{\mathcal{B}}^f(x \odot y) &= \nu_{\mathcal{B}}(f(x \odot y)) = \nu_{\mathcal{B}}(f(x) \odot f(y)) \\ &\geq \nu_{\mathcal{B}}(f(x)) \wedge \nu_{\mathcal{B}}(f(y)) = \nu_{\mathcal{B}}^f(x) \wedge \nu_{\mathcal{B}}^f(y) \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathcal{B}}^f(x \odot y) &= \mu_{\mathcal{B}}(f(x \odot y)) = \mu_{\mathcal{B}}(f(x) \odot f(y)) \\ &\leq \mu_{\mathcal{B}}(f(x)) \vee \mu_{\mathcal{B}}(f(y)) = \mu_{\mathcal{B}}^f(x) \vee \mu_{\mathcal{B}}^f(y). \end{aligned}$$

Hence (IF1) and (IF2) hold.

Now let  $x, y \in A$  be such that  $x \leq y$ . Therefore,

$$\begin{aligned} \nu_{\mathcal{B}}^f(x) &= \nu_{\mathcal{B}}^f(x \wedge y) = \nu_{\mathcal{B}}(f(x \wedge y)) \\ &= \nu_{\mathcal{B}}(f(x) \wedge f(y)) \leq \nu_{\mathcal{B}}(f(y)) = \nu_{\mathcal{B}}^f(y) \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathcal{B}}^f(x) &= \mu_{\mathcal{B}}^f(x \wedge y) = \mu_{\mathcal{B}}(f(x \wedge y)) \\ &= \mu_{\mathcal{B}}(f(x) \wedge f(y)) \geq \mu_{\mathcal{B}}(f(y)) = \mu_{\mathcal{B}}^f(y). \end{aligned}$$

Thus, (IF3) holds.

Concluding,  $\mathcal{B}^f$  is an IF-filter of  $A$ . ■

**Theorem 12.** Let  $\mathcal{B}$  be an IF-set of  $B$ ,  $\mathcal{B}^f$  be an IF-filter of  $A$ , where  $f : A \rightarrow B$  is an epimorphism of pseudo-BL-algebras. Then  $\mathcal{B}$  is an IF-filter of  $A$ .

**Proof.** Let  $f : A \rightarrow B$  be an epimorphism of pseudo-BL-algebras. Then, for any  $x, y \in B$ , there exist  $a, b \in A$  such that  $x = f(a)$  and  $y = f(b)$ . Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}(x \odot y) &= \nu_{\mathcal{B}}(f(a) \odot f(b)) = \nu_{\mathcal{B}}(f(a \odot b)) \\ &= \nu_{\mathcal{B}}^f(a \odot b) \geq \nu_{\mathcal{B}}^f(a) \wedge \nu_{\mathcal{B}}^f(b) \\ &= \nu_{\mathcal{B}}(f(a)) \wedge \nu_{\mathcal{B}}(f(b)) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}(x \odot y) &= \mu_{\mathcal{B}}(f(a) \odot f(b)) = \mu_{\mathcal{B}}(f(a \odot b)) \\ &= \mu_{\mathcal{B}}^f(a \odot b) \leq \mu_{\mathcal{B}}^f(a) \vee \mu_{\mathcal{B}}^f(b) \\ &= \mu_{\mathcal{B}}(f(a)) \vee \mu_{\mathcal{B}}(f(b)) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).\end{aligned}$$

Hence (IF1) and (IF2) hold.

Now let  $x, y \in B$  be such that  $x \leq y$ . Then, there exist  $a, b \in A$  such that  $x = f(a)$  and  $y = f(b)$ . Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}(x) &= \nu_{\mathcal{B}}(x \wedge y) = \nu_{\mathcal{B}}(f(a) \wedge f(b)) = \nu_{\mathcal{B}}(f(a \wedge b)) \\ &= \nu_{\mathcal{B}}^f(a \wedge b) \leq \nu_{\mathcal{B}}^f(b) = \nu_{\mathcal{B}}(f(b)) = \nu_{\mathcal{B}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}(x) &= \mu_{\mathcal{B}}(x \wedge y) = \mu_{\mathcal{B}}(f(a) \wedge f(b)) = \mu_{\mathcal{B}}(f(a \wedge b)) \\ &= \mu_{\mathcal{B}}^f(a \wedge b) \geq \mu_{\mathcal{B}}^f(b) = \mu_{\mathcal{B}}(f(b)) = \mu_{\mathcal{B}}(y).\end{aligned}$$

Thus, (IF3) holds.

Concluding,  $\mathcal{B}$  is an IF-filter of  $B$ . ■

Now let us denote the set of all filters of pseudo-BL-algebra  $A$  by  $Fil(A)$  and the set of all IF-filters of  $A$  by  $IFil(A)$ . Let  $\alpha \in (0, 1)$ . We define maps  $f_{\alpha} : IFil(A) \rightarrow Fil(A) \cup \{\emptyset\}$  and  $g_{\alpha} : IFil(A) \rightarrow Fil(A) \cup \{\emptyset\}$  by

$$\begin{aligned}f_{\alpha}(\mathcal{B}) &= U(\nu_{\mathcal{B}}, \alpha), \\ g_{\alpha}(\mathcal{B}) &= L(\mu_{\mathcal{B}}, \alpha)\end{aligned}$$

for all  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}}) \in IFil(A)$ .

**Theorem 13.** *For any  $\alpha \in (0, 1)$ , the maps  $f_{\alpha}$  and  $g_{\alpha}$  are surjective from  $IFil(A)$  onto  $Fil(A) \cup \{\emptyset\}$ .*

**Proof.** It is obvious, that

$$f_\alpha(0_\sim) = U(0, \alpha) = \emptyset = L(1, \alpha) = g_\alpha(0_\sim).$$

Now let  $\emptyset \neq F \in \text{Fil}(A)$ . Then  $(\chi_F, \chi_F^C)$  is an IF-filter of  $A$ . Hence,

$$f_\alpha((\chi_F, \chi_F^C)) = U(\chi_F, \alpha) = F = L(\chi_F^C, \alpha) = g_\alpha((\chi_F, \chi_F^C)).$$

Therefore,  $f_\alpha$  and  $g_\alpha$  are surjective. ■

## 6. DIRECT PRODUCT OF IF-FILTERS

Let us define a direct product  $\prod_{i \in I}^n A_i$  of pseudo-BL-algebras as usually.

**Definition 11.** Let  $A$  be a pseudo-BL-algebra. Then we define an IF-relation on  $A$  as a mapping  $\mathcal{R} = (\nu'_\mathcal{R}, \mu'_\mathcal{R}) : A \times A \rightarrow [0, 1] \times [0, 1]$  such that  $\nu'_\mathcal{R}(x, y) + \mu'_\mathcal{R}(x, y) \leq 1$  for all  $x, y \in A$ .

Now define a direct product of IF-sets of pseudo-BL-algebra  $A$ .

**Definition 12.** Let  $\mathcal{B} = (\nu_\mathcal{B}, \mu_\mathcal{B})$  and  $\mathcal{G} = (\nu_\mathcal{G}, \mu_\mathcal{G})$  be IF-sets of  $A$ . We define a direct product  $\mathcal{B} \times \mathcal{G}$  by

$$\mathcal{B} \times \mathcal{G} = (\nu_\mathcal{B}, \mu_\mathcal{B}) \times (\nu_\mathcal{G}, \mu_\mathcal{G}) = (\nu_\mathcal{B} \times \nu_\mathcal{G}, \mu_\mathcal{B} \times \mu_\mathcal{G}),$$

where  $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) \wedge \nu_\mathcal{G}(y)$  and  $(\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \mu_\mathcal{B}(x) \vee \mu_\mathcal{G}(y)$  for all  $x, y \in A$ .

**Proposition 6.** Let  $\mathcal{B} = (\nu_\mathcal{B}, \mu_\mathcal{B})$  and  $\mathcal{G} = (\nu_\mathcal{G}, \mu_\mathcal{G})$  be IF-sets of a pseudo-BL-algebra  $A$ , then  $\mathcal{B} \times \mathcal{G}$  is an IF-set of  $A \times A$ .

**Proof.** Let  $\mathcal{B}, \mathcal{G}$  be IF-sets of  $A$ . Then for every  $x \in A$  we have  $\nu_\mathcal{B}(x) + \mu_\mathcal{B}(x) \leq 1$  and  $\nu_\mathcal{G}(x) + \mu_\mathcal{G}(x) \leq 1$ . Suppose that  $\nu_\mathcal{B}(x) \leq \nu_\mathcal{G}(y)$  for some  $x, y \in A$ . Then  $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) \wedge \nu_\mathcal{G}(y) = \nu_\mathcal{B}(x)$ . Let us consider two cases:

*Case 1.*  $\mu_\mathcal{B}(x) \leq \mu_\mathcal{G}(y)$

Hence  $(\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \mu_\mathcal{B}(x) \vee \mu_\mathcal{G}(y) = \mu_\mathcal{G}(y)$  and then  $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) + (\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) + \mu_\mathcal{G}(y) \leq \nu_\mathcal{G}(y) + \mu_\mathcal{G}(y) \leq 1$ .

*Case 2.*  $\mu_\mathcal{B}(x) > \mu_\mathcal{G}(y)$

Therefore  $(\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \mu_\mathcal{B}(x) \vee \mu_\mathcal{G}(y) = \mu_\mathcal{B}(x)$  and then  $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) + (\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) + \mu_\mathcal{B}(x) \leq 1$ . Hence  $\mathcal{B} \times \mathcal{G}$  is an IF-set of  $A \times A$ .

Analogously when  $\nu_\mathcal{B}(x) > \nu_\mathcal{G}(y)$ . ■

Now we give a trivial Proposition without a proof:

**Proposition 7.** *Let  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  and  $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$  be IF-sets of a pseudo-BL-algebra  $A$ , then*

- (i)  $\mathcal{B} \times \mathcal{G}$  is an IF-relation of  $A$ ;
- (ii)  $U(\nu_{\mathcal{B} \times \mathcal{G}}; \alpha) = U(\nu_{\mathcal{B}}; \alpha) \times U(\nu_{\mathcal{G}}; \alpha)$  and  $L(\mu_{\mathcal{B} \times \mathcal{G}}; \alpha) = L(\mu_{\mathcal{B}}; \alpha) \times L(\mu_{\mathcal{G}}; \alpha)$  for all  $\alpha \in [0, 1]$ .

**Theorem 14.** *Let  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  and  $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$  be IF-filters of a pseudo-BL-algebra  $A$ . Then  $\mathcal{B} \times \mathcal{G}$  is an IF-filter of  $A \times A$ .*

**Proof.** Let  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  and  $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$  be IF-filters of a pseudo-BL-algebra  $A$ . Suppose that  $x, y \in A$ . Then by (IF1) and (IF2),  $\nu_{\mathcal{B}}(x \odot y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ ,  $\nu_{\mathcal{G}}(x \odot y) \geq \nu_{\mathcal{G}}(x) \wedge \nu_{\mathcal{G}}(y)$  and  $\mu_{\mathcal{B}}(x \odot y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$ ,  $\mu_{\mathcal{G}}(x \odot y) \leq \mu_{\mathcal{G}}(x) \vee \mu_{\mathcal{G}}(y)$ . Let  $(x_1, x_2), (y_1, y_2) \in A \times A$ . Then,

$$\begin{aligned} (\nu_{\mathcal{B} \times \mathcal{G}})((x_1, x_2) \odot (y_1, y_2)) &= (\nu_{\mathcal{B} \times \mathcal{G}})(x_1 \odot y_1, x_2 \odot y_2) \\ &= \nu_{\mathcal{B}}(x_1 \odot y_1) \wedge \nu_{\mathcal{G}}(x_2 \odot y_2) \\ &\geq \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(x_2) \wedge \nu_{\mathcal{G}}(y_2) \\ &= (\nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{G}}(x_2)) \wedge (\nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(y_2)) \\ &= (\nu_{\mathcal{B} \times \mathcal{G}})(x_1, x_2) \wedge (\nu_{\mathcal{B} \times \mathcal{G}})(y_1, y_2). \end{aligned}$$

Similarly, we can prove that  $(\mu_{\mathcal{B} \times \mathcal{G}})((x_1, x_2) \odot (y_1, y_2)) \leq (\mu_{\mathcal{B} \times \mathcal{G}})(x_1, x_2) \vee (\mu_{\mathcal{B} \times \mathcal{G}})(y_1, y_2)$ .

It is proved that (IF1) and (IF2) hold.

Now let  $(x_1, x_2), (y_1, y_2) \in A \times A$  be such that  $(x_1, x_2) \leq (y_1, y_2)$ . Then

$$\begin{aligned} (\nu_{\mathcal{B} \times \mathcal{G}})(x_1, x_2) &= (\nu_{\mathcal{B} \times \mathcal{G}})((x_1, x_2) \wedge (y_1, y_2)) \\ &= (\nu_{\mathcal{B} \times \mathcal{G}})(x_1 \wedge y_1, x_2 \wedge y_2) \\ &= \nu_{\mathcal{B}}(x_1 \wedge y_1) \wedge \nu_{\mathcal{G}}(x_2 \wedge y_2) \\ &\leq \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(y_2) \\ &= (\nu_{\mathcal{B} \times \mathcal{G}})(y_1, y_2). \end{aligned}$$

and similarly  $(\mu_{\mathcal{B} \times \mathcal{G}})(x_1, x_2) \geq (\mu_{\mathcal{B} \times \mathcal{G}})(y_1, y_2)$ .

The proof is completed. ■

**Theorem 15.** *Let  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  be IF-set of a pseudo-BL-algebra  $A$ . Then  $\mathcal{B}$  is an IF-filter of  $A$  if and only if  $\mathcal{B} \times \mathcal{B}$  is an IF-filter of  $A \times A$ .*

**Proof.**  $\Rightarrow$ : By Theorem 14.

$\Leftarrow$ : Let  $\mathcal{B} \times \mathcal{B}$  be an IF-filter of  $A \times A$ . Let  $(x_1, x_2), (y_1, y_2) \in A \times A$ . Hence

$$\begin{aligned} \nu_{\mathcal{B}}(x_1 \odot y_1) \wedge \nu_{\mathcal{B}}(x_2 \odot y_2) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x_1 \odot y_1, x_2 \odot y_2) \\ &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})((x_1, x_2) \odot (y_1, y_2)) \\ &\geq (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x_1, x_2) \wedge (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(y_1, y_2) \\ &= \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(x_2) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{B}}(y_2). \end{aligned}$$

Putting  $x_1 = x_2$  and  $y_1 = y_2$  we have

$$\nu_{\mathcal{B}}(x_1 \odot y_1) \geq \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{B}}(y_1) = \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1).$$

Similarly,  $\mu_{\mathcal{B}}(x_1 \odot y_1) \leq \mu_{\mathcal{B}}(x_1) \vee \mu_{\mathcal{B}}(y_1)$ .

Let  $x, y \in A$  be such that  $x \leq y$ . Then by (IF3),

$$\nu_{\mathcal{B}}(x) = (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x, x) \leq (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(y, y) = \nu_{\mathcal{B}}(y).$$

Analogously,  $\mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(y)$ .

Hence  $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$  is an IF-filter of  $A$ . ■

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