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ON THE CONNECTIVITY OF THE ANNIHILATING-IDEAL GRAPHS

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Abstract

Let R be a commutative ring with identity and $\mathbb{A}^*(R)$ the set of non-zero ideals with non-zero annihilators. The *annihilating-ideal graph* of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1I_2 = (0)$. In this paper, we examine the presence of cut vertices and cut sets in the annihilating-ideal graph of a commutative Artinian ring and provide a partial classification of the rings in which they appear. Using this, we obtain the vertex connectivity of some annihilating-ideal graphs.

Keywords: annihilating-ideal graph, local ring, nilpotency, cut vertex.

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1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups. Let R be a commutative ring with identity. In [1], D.F. Anderson and P.S. Livingston associate a graph called *zero-divisor graph*, $\Gamma(R)$ to R with vertices $Z(R)^*$, the set of non-zero zero-divisors of R, and for two distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$ in R. Recently M. Behboodi and Z. Rakeei [4, 5] have introduced and investigated the annihilating-ideal graph of a commutative ring. We call an ideal I_1 of R, an *annihilating-ideal* if there exists a non-zero ideal I_2 of R such that $I_1 I_2 = (0)$. For a non-domain commutative ring R, let $J(R)$ be the Jacobson radical of R, $\langle x \rangle$ be the ideal of R generated by x and $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. The *annihilating-ideal graph* of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I_1 and I_2 are adjacent if and only if $I_1 I_2 = (0)$.

An ideal I of R is called *nil-ideal* if there exists a positive integer n such that $I^n = 0$ and $I^{n-1} \neq (0)$. This integer n is called the nilpotency of the ideal. The *annihilator* of $a \in R$ is the set of all elements x in R such that $ax = 0$ and is denoted by $ann(a)$. Let I be a non-zero ideal in R, $ann(I) = \{x \in R : xa = 0 \text{ for all } a \in I\}$. For basic definitions on rings, one may refer [2, 8].

Let $G = (V, E)$ be a simple connected graph. For a vertex $v \in V(G)$, the *neighborhood* (*degree*) of v, denoted by $N_G(v)$ ($deg_G(v)$), is the set (number) of vertices other than v which are adjacent to v. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G. The *distance* between two vertices x and y, denoted $d(x, y)$, is the length of the shortest path from x to y. The *diameter* of a connected graph G is the maximum distance between two distinct vertices of G. For any vertex x of a connected graph G, the *eccentricity* of x, denoted $e(x)$, is the maximum of the distances from x to the other vertices of G. The set of vertices with minimum eccentricity is called the *center* of the graph G. A set $\Omega \subset V(G)$ is said to be a *cut set* if there exist distinct $c, d \in V(G) \setminus \Omega$ such that every path in G from c to d involves at least one element of Ω, and no proper subset of Ω satisfies the same condition. A cut set consisting of only one element is called *cut vertex*. The *connectivity* or *vertex connectivity* $\kappa(G)$ is the size of a cut set with minimum cardinality. The *edge cut* of G is a set of edges whose removal render the graph disconnected. The *edge connectivity* $\kappa'(G)$ is the size of an edge cut with minimum cardinality in G. For basic definitions on graphs, one may refer [6, 11]. The following theorems are useful for further reference in this paper.

Theorem 1.1 [3]. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If x is a cut vertex of $\Gamma(R)$, then $\mathfrak{m}^{k-1} = \{0, x\}$.*

Theorem 1.2 [3]. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If x is a cut vertex of $\Gamma(R)$ for some $x \in Z(R)^*$, then $|R| = 2^{n+1}$ and $|\mathfrak{m}| = 2^n$ for some $n \in \mathbb{N}$. Hence R has characteristic 2^t for some $t \in \mathbb{N}$.*

Theorem 1.3 [7]**.** *Let n be a positive integer such that $n \neq p, 2p, p^2$ for any prime p. Then A is a cut set of $\Gamma(\mathbb{Z}_n)$ if and only if $A = ann^*(p)$ for some $p|n$.*

Theorem 1.4 [7]**.** *Let $R = \prod_{i=1}^{n} R_i$ with $n \geq 2$. If A is a cut vertex of $\Gamma(R)$, then there exists some i, $1 \leq i \leq n$, such that if $a = (a_1, a_2, \ldots, a_n) \in A$, then $a_k = 0$ for all $k \neq i$.*

Theorem 1.5 [7]**.** *Let $R = \prod_{i=1}^{n} R_i$ be a ring, where each R_i is a field and $n \geq 2$. If A is a cut set of $\Gamma(R)$, then $A = \{(0, \ldots, 0, x_i, 0, \ldots, 0) : x_i \in R_i^*\}$.*

In this paper, we examine the presence of cut vertices and cut sets in the annihilating-ideal graph of a finite commutative ring and provide a partial classification of the rings in which they appear. Using this, we obtain the vertex connectivity of some annihilating-ideal graphs. Also we study some connections between the graph-theoretic properties of this graph and some algebraic properties of a commutative ring.

2. Cut vertex and cut set of $\mathbb{AG}(R)$: the local case

In this section, we examine the presence of cut vertices in the annihilating-ideal graph of a finite commutative local rings and provide a partial classification of the rings in which they appear. Let I be an ideal in R.

Throughout, we assume that R is a finite commutative ring with identity, $Z(R)$ its set of zero-divisors, $R^* = R \setminus \{0\}$ and $I^* = I \setminus \{0\}$, where I is an ideal in R.

Remark 2.1. Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If $k = 2$, then $\mathbb{AG}(R)$ is complete.

Hereafter, if $(R, \mathfrak{m})$ is a finite local ring, then we take the unique maximal ideal as $\mathfrak{m}$ and its nilpotency as $k > 3$.

Proposition 2.2. *Let R be a finite commutative ring. If I is a cut vertex of $\mathbb{AG}(R)$, then $ann(I)$ is a maximal ideal of R.*

Proof. Suppose that $ann(I)$ is not maximal. Then there exists a maximal ideal M in R such that $ann(I) \subset M$. Since $\mathbb{AG}(R)$ is connected, there exists an ideal $J \neq I$ and J is adjacent to M, i.e., $JM = (0)$. This implies that $M \subset ann(J) \subset R$ and so $ann(J) = M$. Thus $ann(I) \subset ann(J) \subset R$. Let $I_1 - \cdots - I - \cdots - I_2$ be a path between two ideals I_1 and I_2 in $\mathbb{AG}(R)$ containing I. Then there exist ideals K_1, K_2 and $K_1 \neq K_2$ such that K_1 and K_2 are adjacent to I. i.e., $K_1 I = (0)$ and $K_2 I = (0)$. Therefore $K_1, K_2 \subseteq ann(I) \subset ann(J)$ and so K_1 and K_2 are also adjacent to J. Thus there exists a path $I_1 - \cdots - J - \cdots - I_2$ in $\mathbb{AG}(R)$ without containing I, a contradiction. ∎

Remark 2.3. Converse of the Proposition 2.2 is not true. For example, let $R = \frac{\mathbb{Z}_2[x,y]}{\langle x,y\rangle^2}$ be a local ring. Then $Z(R) = \{0, x, y, x+y\}$, $I_1 = \{0, x\}$, $I_2 = \{0, y\}$, $I_3 = \{0, x+y\}$ are nonzero proper ideals of R and $\mathbb{AG}(R) \cong K_4$. Here $ann(I_1)$ is maximal ideal of R. However I_1 is a not a cut vertex of $\mathbb{AG}(R)$.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If I is a cut vertex in $\mathbb{AG}(R)$, then $I = \mathfrak{m}^{k-1}$.*

Proof. Note that any ideal J in R is contained in $\mathfrak{m}$ and $\mathfrak{m}\mathfrak{m}^{k-1} = (0)$. This gives that $J\mathfrak{m}^{k-1} = (0)$ and so $\mathfrak{m}^{k-1}$ is adjacent to all the other vertices of $\mathbb{AG}(R)$. Thus if I is a cut vertex, then it should be $\mathfrak{m}^{k-1}$. ∎

Corollary 1. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. Then $\mathfrak{m}^{k-1}$ is a cut vertex of $\mathbb{AG}(R)$ if and only if $deg_{\mathbb{AG}(R)}(\mathfrak{m}) = 1$.*

Proof. Suppose $\mathfrak{m}^{k-1}$ is a cut vertex of $\mathbb{AG}(R)$. Note that $\mathfrak{m}^{k-1}\mathfrak{m} = (0)$. If $deg_{\mathbb{AG}(R)}(\mathfrak{m}) > 1$, then there exists a nonzero ideal $I' \neq \mathfrak{m}^{k-1}$ in R such that $I'\mathfrak{m} = (0)$. Clearly $I'J = (0)$ for all nonzero ideal J in R and so I' is adjacent to all the other vertices of $\mathbb{AG}(R)$. Thus $\mathfrak{m}^{k-1}$ is not a cut vertex of $\mathbb{AG}(R)$, a contradiction. Conversely, if $deg_{\mathbb{AG}(R)}(\mathfrak{m}) = 1$, then $\mathfrak{m}$ is only adjacent to $\mathfrak{m}^{k-1}$ and so $\mathfrak{m}^{k-1}$ is a cut vertex of $\mathbb{AG}(R)$. ∎

Corollary 2. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If $\mathbb{AG}(R)$ has a cut vertex, then $\mathbb{AG}(R)$ is neither Eulerian nor Hamiltonian.*

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If $x \in Z(R)^*$, then x is a cut vertex of $\Gamma(R)$ if and only if $\mathfrak{m}^{k-1} = \{0, x\}$ is a cut vertex of $\mathbb{AG}(R)$. Hence in this case, R has characteristic 2^t for $t \in \mathbb{N}$.*

Proof. If x is a cut vertex of $\Gamma(R)$, then by Theorem 1.1, $\mathfrak{m}^{k-1} = \{0, x\}$. Since $\mathfrak{m}$ is maximal and $\mathfrak{m}\mathfrak{m}^{k-1} = (0)$, $I' \subseteq \mathfrak{m}$ and so $I'\mathfrak{m}^{k-1} = (0)$ for every nonzero ideal I' in R. Since $\mathfrak{m}^{k-1} = \{0, x\}$ and $\mathfrak{m}^k = 0$, $\mathfrak{m} = ann(x)$. Suppose $\mathfrak{m}^{k-1}$ is not a cut vertex of $\mathbb{AG}(R)$. Then the subgraph induced by $\mathbb{A}^*(R) \setminus \{\mathfrak{m}^{k-1}\}$ in $\mathbb{AG}(R)$ is connected and so there exist a nonzero ideal $I_1 \neq \mathfrak{m}^{k-1}$ such that $I_1\mathfrak{m} = (0)$. From this, we get $ann(I_1) = \mathfrak{m}$ and so $ann(y) = \mathfrak{m}$ for all $y \in I_1^*$. Since $I_1 \neq \mathfrak{m}^{k-1}$, $y \notin \mathfrak{m}^{k-1}$ for some $y \in I_1^*$. Thus y is adjacent to all other vertices of $\Gamma(R)$ and so x is not a cut vertex of $\Gamma(R)$, contradiction.

Converse follows from Theorem 1.1. The later part of the statement follows from Theorem 1.2. ∎

Theorem 2.6. *Let $(R, \mathfrak{m})$ be a finite local ring and let k be the nilpotency of $\mathfrak{m}$. If I is a cut vertex in $\mathbb{AG}(R)$, then I^* is the center of $\Gamma(R)$.*

Proof. By Proposition 2.4, $I = \mathfrak{m}^{k-1}$ and so $ann(x) = \mathfrak{m}$ for all $x \in I^*$. Thus for each $x \in I^*$, $d(x, y) = 1$ for all $y \in Z(R)^*$, $y \neq x$ and so $I^* \subseteq \mathscr{C}$ where $\mathscr{C}$ is the center of $\Gamma(R)$. Suppose $I^* \subset \mathscr{C}$. Then there exist an element $y \in \mathscr{C}$ such that $y \notin I^*$. From this, we get $ann(y) = \mathfrak{m}$, $\langle y \rangle \neq I$, $\mathfrak{m}\langle y \rangle = (0)$ and so $deg_{\mathbb{AG}(R)}(\mathfrak{m}) \geq 2$, a contradiction. Hence $I^* = \mathscr{C}$. ■

Proposition 2.7. *Let $n = p^k$ where p is prime number, $k > 4$ and I be any non-zero ideal in $\mathbb{Z}_n$. Then I is a maximal ideal of $\mathbb{Z}_n$ if and only if $deg_{\mathbb{AG}(\mathbb{Z}_n)}(I) = 1$. Hence $ann(I)$ is a cut vertex of $\mathbb{AG}(\mathbb{Z}_n)$.*

Proof. If I is a maximal ideal of $\mathbb{AG}(\mathbb{Z}_n)$, then $I = \langle p \rangle$ and so I is only adjacent to $\langle p^{k-1} \rangle$. Conversely, assume that $deg_{\mathbb{AG}(\mathbb{Z}_n)}(I) = 1$. If I is not maximal, then $I = \langle p^i \rangle$ for some i, $2 \leq i \leq k-1$. From this, $deg_{\mathbb{AG}(\mathbb{Z}_n)}(I) \geq 2$, a contradiction. ■

The converse of the Proposition 2.2 is true in the following theorem.

Theorem 2.8. *Let $n = p^k$ where p is prime number, $k > 4$ and I be any non-zero ideal in $\mathbb{Z}_n$. Then the following are equivalent:*

(i) *I is unique cut vertex of $\mathbb{AG}(\mathbb{Z}_n)$,*

(ii) *$ann(I)$ is maximal,*

(iii) *$I = ann(\langle p \rangle)$,*

(iv) *I^* is a cut set of $\Gamma(\mathbb{Z}_n)$.*

Proof. (i)⇒(ii) If I is unique cut vertex of $\mathbb{AG}(\mathbb{Z}_n)$, then $I = \langle p^{k-1} \rangle$ and so I is adjacent to every other vertices of $\mathbb{AG}(\mathbb{Z}_n)$. Thus $ann(I)$ is maximal.

(ii)⇒(iii) Suppose $ann(I)$ is maximal. Then $ann(I) = \langle p \rangle$, $I = \langle p^{k-1} \rangle$ and so $I = ann(\langle p \rangle)$.

(iii)⇒(iv) Suppose $I = ann(\langle p \rangle)$. Then $I = \langle p^{k-1} \rangle$. Clearly $p \in Z^*(\mathbb{Z}_n)$ and p is only adjacent to every elements of I^*. Also no proper subset of I^* can act as a cut set of $\Gamma(\mathbb{Z}_n)$. Hence I^* is a cut set of $\Gamma(\mathbb{Z}_n)$.

(iv)⇒(i) Suppose I^* is a cut set of $\Gamma(\mathbb{Z}_n)$. Then by Theorem 1.3, $I = ann(\langle p \rangle)$ and so $I = \langle p^{k-1} \rangle$ is the unique cut vertex of $\mathbb{AG}(\mathbb{Z}_n)$ ■

3. CUT VERTEX AND CUT SET OF $\mathbb{AG}(R)$: THE NON-LOCAL CASE

In this section, first we prove that all cut sets in $\mathbb{AG}(\mathbb{Z}_n)$ are nothing but cut vertices. Next we classify cut vertices and cut sets of the annihilating-ideal graph of a finite commutative non-local ring.

Remark 3.1. If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ be an integer, where $r \geq 2$, $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \cdots < p_r$ and $n \neq pq$, where p, q are distinct primes. Let

I be any non-zero ideal in $\mathbb{Z}_n$. Then I is a maximal ideal of $\mathbb{Z}_n$ if and only if $deg_{\mathbb{AG}(\mathbb{Z}_n)}(I) = 1$.

In view of Proposition 2.7 and Remark 3.1, we have the following: Let $n > 3$ be an integer and not a prime number. Then $\mathbb{AG}(\mathbb{Z}_n)$ is neither Eulerian nor Hamiltonian.

The converse of the Proposition 2.2 is true in the following theorem.

Theorem 3.2. *If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ be an integer, where $r \geq 2$, $p_1, p_2, \dots, p_r$ are primes with $p_1 < p_2 < \cdots < p_r$ and $n \neq pq$, where p, q are distinct primes. Let I be any non-zero ideal in $\mathbb{Z}_n$. Then the following are equivalent:*

(i) *I is a cut vertex of $\mathbb{AG}(R)$,*

(ii) *$ann(I) = \langle p_i \rangle$ for some i,*

(iii) *$ann(\langle p_i \rangle) = I$ for some i,*

(iv) *I^* is a cut set of $\Gamma(\mathbb{Z}_n)$.*

Proof. (i)$\Rightarrow$(ii) If I is a cut vertex of $\mathbb{AG}(\mathbb{Z}_n)$, then by Proposition 2.2, $ann(I) = \langle p_i \rangle$ for some i.

(ii)$\Rightarrow$(iii) Suppose $ann(I) = \langle p_i \rangle$ for some i. By Remark 3.1, $deg_{\mathbb{AG}(\mathbb{Z}_n)}(\langle p_i \rangle) = 1$ and so $\langle p_i \rangle$ is only adjacent to $\left\langle \frac{n}{p_i} \right\rangle$ in $\mathbb{AG}(\mathbb{Z}_n)$. Thus $ann(\langle p_i \rangle) = \left\langle \frac{n}{p_i} \right\rangle = I$ for some i.

(iii)$\Rightarrow$(iv) If $ann(\langle p_i \rangle) = I$ for some i, then $ann(p_i) = I$. By Theorem 1.3, I^* is a cut set of $\Gamma(\mathbb{Z}_n)$.

(iv)$\Rightarrow$(i) Suppose I^* is a cut set of $\Gamma(\mathbb{Z}_n)$. Then by Theorem 1.3, $I^* = ann^*(p_i)$ for some i. Since $ann(p_i) = ann(\langle p_i \rangle)$, $I^* = ann^*(\langle p_i \rangle)$. Since $deg_{\mathbb{AG}(\mathbb{Z}_n)}(\langle p_i \rangle) = 1$, $I = \left\langle \frac{n}{p_i} \right\rangle$ and so I is a cut vertex of $\mathbb{AG}(\mathbb{Z}_n)$. ∎

Theorem 3.3. *Let $R = \prod_{i=1}^n R_i$ be a finite commutative non-local ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. If $I = \prod_{i=1}^n I_i$ is a cut vertex of $\mathbb{AG}(R)$, then $I = (0) \times \cdots \times (0) \times \mathfrak{m}_i^{n_i - 1} \times (0) \times \cdots \times (0)$ for some i, $1 \leq i \leq n$.*

Proof. Note that $Max(R) = \{M_i : M_i = R_1 \times \cdots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in R. Since I is a cut vertex of $\mathbb{AG}(R)$ and by Proposition 2.2, $ann(I)$ is a maximal ideal in R and so $ann(I) = M_i$ for some i. Thus $I_i \neq (0)$, $I_i \mathfrak{m}_i = (0)$ and $I_j = (0)$ for all $j \neq i$. Clearly $I_i K_i = (0)$ for every non-zero proper ideal K_i in R_i.

Let $\Omega = \{R_1 \times \cdots \times R_{i-1} \times K_i \times R_{i+1} \times \cdots \times R_n : K_i \text{ is a proper ideal in } R_i\}$. Then $IK = (0)$ for all $K \in \Omega$. Suppose that I_i' is any non-zero ideal in R_i with $I_i' \neq I_i$, $I_i' \mathfrak{m}_i = (0)$ and $A = (0) \times \cdots \times (0) \times I_i' \times (0) \times \cdots \times (0) \neq I$, then $AK = (0)$

for all $K \in \Omega$ and so I is not a cut vertex of $\mathbb{AG}(R)$, a contradiction. Thus there exist no non-zero proper ideal I_i' in R_i such that $I_i' \neq I_i$ and $\mathfrak{m}_i I_i' = (0)$. From this, I_i is the only element adjacent to $\mathfrak{m}_i$ in $\mathbb{AG}(R_i)$ and so $deg_{\mathbb{AG}(R_i)}(\mathfrak{m}_i) = 1$. Since $\mathfrak{m}_i^{n_i-1}\mathfrak{m}_i = (0)$, $I_i = \mathfrak{m}_i^{n_i-1}$. Hence $I = (0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$. ■

In view of Theorem 3.3, we have the following corollary.

Corollary 3. *Let $R = \prod_{i=1}^n R_i$ be a finite commutative non-local ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. If I is a cut vertex of $\mathbb{AG}(R)$, then $I = ann(M)$ for some maximal ideal M in R.*

Theorem 3.4. *Let $R = \prod_{i=1}^n R_i$ be a finite commutative non-local ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then $\mathbb{AG}(R)$ has a cut vertex if and only if $\mathbb{AG}(R_i)$ has a cut vertex for some i.*

Proof. Note that $Max(R) = \{M_i : M_i = R_1\times\cdots\times R_{i-1}\times\mathfrak{m}_i\times R_{i+1}\times\cdots\times R_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in R. Let I be a cut vertex of $\mathbb{AG}(R_i)$ for some i. By Proposition 2.4, $I = \mathfrak{m}_i^{n_i-1}$. By Corollary 1, $deg_{\mathbb{AG}(R_i)}(\mathfrak{m}_i) = 1$. Thus $deg_{\mathbb{AG}(R)}(M_i) = 1$ where $M_i \in Max(R)$ and $M_iI = (0)$ where $I = (0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$. Thus $(0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$ is a cut vertex of $\mathbb{AG}(R)$.

Conversely, let I'' be a cut vertex of $\mathbb{AG}(R)$. By Theorem 3.3, $I'' = (0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$ for some i and so $\mathfrak{m}_i$ is only adjacent to $\mathfrak{m}_i^{n_i-1}$ in $\mathbb{AG}(R_i)$. Hence $\mathfrak{m}_i^{n_i-1}$ is a cut vertex of $\mathbb{AG}(R_i)$. ■

The converse of the Proposition 2.2 is true in the following theorem.

Theorem 3.5. *Let $R = \prod_{i=1}^n R_i$ be a finite commutative non-local ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Suppose $\mathbb{AG}(R_i)$ has a cut vertex for all i, $1 \leq i \leq n$. Then the following are equivalent:*

(i) *I is a cut vertex of $\mathbb{AG}(R)$,*

(ii) *$ann(I)$ is a maximal ideal in R,*

(iii) *$I = ann(M)$ for some $M \in Max(R)$.*

Proof. By Proposition 2.4, $\mathfrak{m}_i^{n_i-1}$ is the unique cut vertex of $\mathbb{AG}(R_i)$ for all i. Therefore $deg_{\mathbb{AG}(R_i)}(\mathfrak{m}_i) = 1$ and so $deg_{\mathbb{AG}(R)}(M_i) = 1$ for all $M_i \in Max(R)$.

(i)⇔(ii) Suppose I is a cut vertex of $\mathbb{AG}(R)$. By Proposition 2.2, $ann(I)$ is maximal. Conversely, assume that $ann(I)$ is maximal ideal in R and $ann(I) = M_i$ for some $M_i \in Max(R)$. Since $deg_{\mathbb{AG}(R)}(M_i) = 1$, $(0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$ is the only vertex adjacent to M_i in $\mathbb{AG}(R)$. Thus $I = (0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$ is a cut vertex of $\mathbb{AG}(R)$.

(i)⇔(iii) Suppose I is a cut vertex of $\mathbb{AG}(R)$. By Theorem 3.4, $I = (0)\times\cdots\times(0)\times\mathfrak{m}_i^{n_i-1}\times(0)\times\cdots\times(0)$. Since $deg_{\mathbb{AG}(R)}(M_i) = 1$, $ann(M_i) = I$. Conversely, suppose $I = ann(M)$ for some $M \in Max(R)$. Since $deg_{\mathbb{AG}(R)}(M) = 1$ for all $M \in Max(R)$, I is a cut vertex of $\mathbb{AG}(R)$. ■

Theorem 3.6. *Let $R = \prod_{i=1}^{n} R_i$ be a finite commutative non-local ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then for each i $(1 \leq i \leq n)$, $\Omega_i = \{(0) \times \cdots \times (0) \times I_i \times (0) \times \cdots \times (0) : I_i \subseteq ann(\mathfrak{m}_i)\}$ is a cut set in $\mathbb{AG}(R)$ and hence $ann(M_i) = \Omega_i$ for $M_i \in Max(R)$.*

Proof. Let $\Omega'_i = \{R_1 \times \cdots \times R_{i-1} \times K_i \times R_{i+1} \times \cdots \times R_n : K_i$ is a proper ideal in $R_i\}$ for $1 \leq i \leq n$. Then for any $I' \in \Omega'_i$, $I'I = (0)$ for all $I \in \Omega_i$. When Ω_i is removed in $\mathbb{AG}(R)$, $\mathbb{AG}(R) \setminus \Omega_i$ is disconnected, since Ω'_i is isolated in $\mathbb{AG}(R) \setminus \Omega_i$. Also, no proper subset of Ω_i is a cut set of $\mathbb{AG}(R)$. Hence Ω_i is a cut set in $\mathbb{AG}(R)$. ■

Theorem 3.7. *Let $R = \prod_{i=1}^{n} R_i$ be a finite commutative non-local ring, where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then*

$$\kappa(\mathbb{AG}(R)) = \kappa'(\mathbb{AG}(R)) = \min_{1 \leq i \leq n} deg_{\mathbb{AG}(R_i)}(\mathfrak{m}_i) = \delta(\mathbb{AG}(R)).$$

Proof. Note that $Max(R) = \{M_i : M_i = R_1 \times \cdots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in R. Clearly $N_{\mathbb{AG}(R)}(M_i) = \{(0) \times \cdots \times (0) \times I_i \times (0) \times \cdots \times (0) : I_i\mathfrak{m}_i = (0), I_i$ is a non-zero ideal in $R_i\}$ and hence $deg_{\mathbb{AG}(R)}(M_i) = deg_{\mathbb{AG}(R_i)}(\mathfrak{m}_i)$ for $1 \leq i \leq n$.

Let $I = \prod_{i=1}^{n} I_i$ be any non-zero proper ideal in R, but not maximal, where each I_i is an ideal in R_i. Then $I \subset M_i$ for some i. Clearly $I_i \subseteq \mathfrak{m}_i$, $I_j \subset R_j$ for some $j \neq i$ and so $deg_{\mathbb{AG}(R)}(M_i) < deg_{\mathbb{AG}(R)}(I)$. Hence, for each maximal ideal M in R, $deg_{\mathbb{AG}(R)}(M) < deg_{\mathbb{AG}(R)}(I)$ for all non-zero proper ideal I in R and I is not maximal.

Let $t_i = deg_{\mathbb{AG}(R)}(M_i)$ with $t_1 \leq t_2 \leq \cdots \leq t_n$. Then $\delta(\mathbb{AG}(R)) = \min_{1 \leq i \leq n}\{t_i\} = t_1$. By Theorem 3.6, $ann(M_i)$ is a cut set of $\mathbb{AG}(R)$ for all i. Thus $ann(M_1)$ is a cut set of $\mathbb{AG}(R)$ with minimum cardinality and so $\kappa(\mathbb{AG}(R)) = t_1$. Let Ω be the set of all edges incident with M_1 in $\mathbb{AG}(R)$. Then Ω is an edge cut of $\mathbb{AG}(R)$ with minimum cardinality. Hence $\kappa'(\mathbb{AG}(R)) = t_1$. ■

Theorem 3.8. *Let $R = \prod_{i=1}^{n} R_i$ be a commutative ring, where each R_i is a field and $n \geq 2$. Let I be any non-zero ideal in R. Then the following are equivalent:*

(i) *I is a cut vertex of $\mathbb{AG}(R)$,*

(ii) *$ann(I)$ is a maximal ideal in R,*

(iii) *$I = ann(M)$ for some maximal ideal M in R,*

(iv) *I^* is a cut vertex of $\Gamma(R)$.*

Proof. Note that $Max(R) = \{M_i' = R_1 \times \cdots \times R_{i-1} \times (0) \times R_{i+1} \times \cdots \times R_n : 1 \leq i \leq n\}$ is the set of maximal ideals in R and $deg_{\mathbb{AG}(R)}(M_i') = 1$ for all i. Let $D_i = (0) \times \cdots \times (0) \times R_i \times (0) \times \cdots \times (0)$, for $1 \leq i \leq n$.

(i)⇔(ii) Suppose I is a cut vertex of $\mathbb{AG}(R)$. By Proposition 2.2, $ann(I)$ is a maximal ideal in R. Conversely, assume that $ann(I)$ is maximal ideal in R. Then $ann(I) = M_i'$ for some i. Since $deg_{\mathbb{AG}(R)}(M_i') = 1$, M_i' is only adjacent to D_i in $\mathbb{AG}(R)$. Thus $I = D_i$ and so I is a cut vertex of $\mathbb{AG}(R)$.

(i)⇔(iii) Suppose I is a cut vertex of $\mathbb{AG}(R)$. By (ii), $ann(I) = M_i$ for some i and so $I = D_i$. Thus $ann(M_i) = D_i = I$. Conversely, let $I = ann(M)$ for some maximal ideal M in R. Since $deg_{\mathbb{AG}(R)}(M) = 1$, I is a cut vertex of $\mathbb{AG}(R)$.

(i)⇔(iv) Suppose I is a cut vertex of $\mathbb{AG}(R)$. Then $I = D_i$ for some i and so $I^* = \{(0, \ldots, 0, x, 0, \ldots, 0) : 0 \neq x \in R_i\}$. When I^* is removed in $\Gamma(R)$, $\Gamma(R) \setminus \{I^*\}$ is disconnected, since $(1, \ldots, 1, 0, 1, \ldots, 1)$ is isolated in $\Gamma(R) \setminus \{I^*\}$. Also no proper subset of I^* is a cut set of $\Gamma(R)$. Hence I^* is a cut set $\Gamma(R)$. Conversely, suppose I^* is a cut set of $\Gamma(R)$. By Theorem 1.5, $I = \{(0, \ldots, 0, x, 0, \ldots, 0) : x \in R_i\} = (0) \times \cdots \times (0) \times R_i \times (0) \times \cdots \times (0)$. Hence I is a cut vertex of $\mathbb{AG}(R)$. ■

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