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A VARIATION OF ZERO-DIVISOR GRAPHS

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Abstract

In this paper, we define a new graph for a ring with unity by extending the definition of the usual *'zero-divisor graph'*. For a ring R with unity, $\Gamma_1(R)$ is defined to be the simple undirected graph having all non-zero elements of R as its vertices and two distinct vertices x, y are adjacent if and only if either $xy = 0$ or $yx = 0$ or $x + y$ is a unit. We consider the conditions of connectedness and show that for a finite commutative ring R with unity, $\Gamma_1(R)$ is connected if and only if R is not isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_2^k$ (for any $k \in \mathbb{N} - \{1\}$). Then we characterize the rings R for which $\Gamma_1(R)$ realizes some well-known classes of graphs, viz., complete graphs, star graphs, paths (i.e., P_n), or cycles (i.e., C_n). We then look at different graph-theoretical properties of the graph $\Gamma_1(F)$, where F is a finite field. We also find all possible $\Gamma_1(R)$ graphs with at most 6 vertices.

Keywords: rings, zero-divisor graphs, finite fields.

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1. Introduction

The study of zero-divisor graphs, which involves the association of a ring to a graph, helps in gaining insight about the algebraic properties of rings, especially the structure of the set of zero-divisors. The idea of zero-divisor graphs was first given by Beck [8], and was continued by D.D. Anderson and Naseer [1]. They related a commutative ring R to a graph by defining a graph which has as its vertices all elements of R and two distinct vertices x, y are adjacent if and only if $xy = 0$. Later on, D.F. Anderson and Livingston [3] modified the definition in order to get a more complete illustration of the zero-divisors and defined zero-divisor graphs as we know it today. They defined zero-divisor graphs in the following way: Let R be a commutative ring with 1. Then the zero-divisor graph of R, denoted by $\Gamma(R)$, is the simple undirected graph with all non-zero zero-divisors of R as vertices, and two distinct vertices x, y are adjacent if and only if $xy = 0$. The interrelation between the ring-theoretic structure of R and the graph-theoretic structure of $\Gamma(R)$ has brought out interesting results from the perspective of both algebra and graph theory (cf. [2, 3, 4, 6, 9], for example).

Zero-divisor graphs were initially defined for a commutative ring only. Later on, Redmond [11] generalized the concept of zero-divisor graphs to a non-commutative ring in the following way: Let R be a ring. Then the undirected zero-divisor graph of R, denoted by $\overline{\Gamma}(R)$, is the graph whose vertices are the non-zero zero-divisors of R, and there is an edge between two distinct vertices a and b if and only if either $ab = 0$ or $ba = 0$.

The following result, proved by Anderson and Livingston, will be used for deriving several results in this paper.

Theorem 1.1 (3, Theorem 2.3)**.** *Let R be a commutative ring with unity. Then $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$.*

A similar result, which takes into account the non-commutative case, was given by Redmond:

Theorem 1.2 (11, Theorem 3.2)**.** *Let R be a ring with unity. Then $\overline{\Gamma}(R)$ is connected with $diam(\overline{\Gamma}(R)) \leq 3$.*

The study of the set of zero-divisors of a ring often gets complicated due to a lack of nice algebraic structure (it need not be closed under addition).

Here we introduce a new type of graph $\Gamma_1(R)$ (cf. Definition 2.1) for a ring R, taking all non-zero elements of the ring as vertices and adding a new condition to the adjacency condition of a zero-divisor graph. Consequently, $\Gamma(R)$ (or $\overline{\Gamma}(R)$, in the non-commutative case) is a subgraph of $\Gamma_1(R)$ for any ring R.

In 2010, Ashrafi et al [5] defined unit graphs over a ring with unity in the following way: Let R be a ring. Then the unit graph $G(R)$ associated with the ring R is the simple undirected graph which has as its vertices all elements of R and distinct vertices x, y are adjacent if and only if $x + y$ is a unit in R.

Clearly, the vertex set of $\Gamma_1(R)$ is a subset of the vertex set of $G(R)$. Also, the second part of the adjacency condition of $\Gamma_1(R)$ (cf. Definition 2.1) is same with the adjacency condition of $G(R)$. However, $\Gamma_1(R)$ need not be a subgraph of $G(R)$. For example, $\Gamma_1(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is not a subgraph of $G(\mathbb{Z}_2 \times \mathbb{Z}_4)$ (cf. Figure 1, 2). It is also worth noting that $\Gamma_1(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is connected whereas $G(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is not. However, $\Gamma_1(D)$ is a subgraph of $G(D)$ for a division ring D. In fact, $\Gamma_1(D)$ is obtained by removing the vertex 0 (along with all the edges incident on it) from $G(D)$.

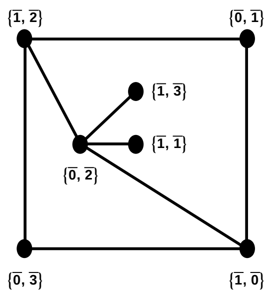

Figure 1. $\Gamma_1(\mathbb{Z}_2 \times \mathbb{Z}_4)$

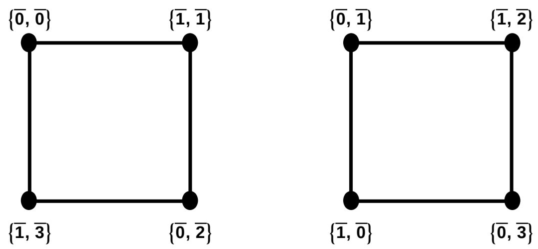

Figure 2. Unit graph $G(\mathbb{Z}_2 \times \mathbb{Z}_4)$

In this paper, we first consider the connectedness of $\Gamma_1(R)$. We first have several lemmas, ultimately leading to the main result which completely characterizes the finite commutative rings R (with unity) for which $\Gamma_1(R)$ is connected. For instance, the first lemma shows that for a commutative ring R with unity, if the number of zero-divisors is less than a number depending on the characteristic of R, then the graph $\Gamma_1(R)$ is connected. Another lemma shows that for a finite commutative ring R with unity, if the characteristic of R is not equal to 2 and $R \ncong \mathbb{Z}_3$, then $\Gamma_1(R)$ will be connected. In fact, if the characteristic of R is non-prime, then $\Gamma_1(R)$ is connected even when the ring is non-commutative. The main result in this section is that for a commutative ring R with unity $1\ (\neq 0)$, $\Gamma_1(R)$ is connected if and only if R is not isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_2^k$ for any $k \in \mathbb{N} - \{1\}$. Note that by $\mathbb{Z}_2^k$, we denote the ring $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{k\ times}$.

Then, we characterize the rings R for which $\Gamma_1(R)$ realizes some important classes of graphs, viz., complete graphs, star graphs, paths (i.e., P_n), and cycles (i.e., C_n). We also obtain some results regarding the existence of a cycle in $\Gamma_1(R)$. In the next section, we move to finite fields and observe that for a finite field F, the $\Gamma_1(F)$ graph is regular, connected (if the number of elements in F is greater than 3), and Eulerian. We then obtain a result showing that if $\Gamma_1(R)$ is regular with certain degree sequences, then R has to be a finite field. In the appendix, we give all possible $\Gamma_1(R)$ graphs with at most 6 vertices.

In this paper, $Char(R)$ denotes the characteristic of a ring R, $\mathbb{F}_n$ is the finite field of order n, $deg(v)$ denotes the degree of a vertex v, $\phi(n)$ is the Euler-phi function with argument n, and $a \leftrightarrow b$ denotes "a is adjacent to b" for two distinct vertices a, b. All the rings discussed in this paper have a unity 1 ($\neq 0$). For usual algebraic terms, we refer to any standard book on ring theory, and we refer to [12] for graph-theoretic terms and definitions.

2. The graph $\Gamma_1(R)$

Definition 2.1. Let R be a ring with unity. Let $G = (V, E)$ be a simple undirected graph in which $V = R - \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit. We denote this simple undirected graph G by $\Gamma_1(R)$.

Note that for a finite ring, an element is either a zero-divisor or a unit. So in the case of a finite ring R, $\Gamma_1(R) = (V, E)$, where $V = R - \{0\}$ and for any $a, b \in V$, $ab \in E$ if and only if $a \neq b$ and either $ab = 0$ or $ba = 0$ or $a + b \notin Z(R)$, where $Z(R)$ is the set of zero-divisors of R including 0.

Remark 2.2. In general, the graph $\overline{\Gamma}(R)$ is a subgraph of $\Gamma_1(R)$. If R is commutative, then the zero-divisor graph $\Gamma(R)$ is a subgraph of $\Gamma_1(R)$. It follows from Theorem 1.1 and Theorem 1.2 that the non-zero zero-divisors of R lie in the same component of $\Gamma_1(R)$, i.e., any two non-zero zero-divisors of R are connected by a path in $\Gamma_1(R)$.

We note that the unit graph $G(F)$ is connected for any finite field F (since all units are adjacent to the vertex 0), and $\Gamma_1(F)$ is a subgraph of $G(F)$. Now a subgraph of a connected graph need not be connected. In fact, $G(\mathbb{Z}_3)$ is connected whereas $\Gamma_1(\mathbb{Z}_3)$ is not (cf. Figure 4 in the Appendix). However, the following proposition shows that $\Gamma_1(F)$ is connected for any finite field F other than $\mathbb{Z}_3$.

Proposition 2.3. *Let F be a finite field such that $F \ncong \mathbb{Z}_3$. Then $\Gamma_1(F)$ is connected.*

Proof. Since $\Gamma_1(\mathbb{Z}_2)$ is a single vertex graph, it is connected. Suppose $|F| > 3$. We first note that for each $a \in F - \{0\}$, a is adjacent to b for all $b \in F - \{a, 0\}$ unless $a + b = 0$ (i.e., $b = -a$). Thus, if the characteristic of F is 2, then $\Gamma_1(F)$ is complete. If the characteristic of F is $p > 2$ (p is a prime), then $1 \neq -1$. Since $|F| > 3$, we have an element $a \in F - \{0, 1, -1\}$. Then $a + 1 \neq 0$ and $a - 1 \neq 0$.

So $a+1$ and $a-1$ are both units. Thus a is adjacent to both 1 and -1. This happens for any $a \in F - \{0, 1, -1\}$. Thus, $\Gamma_1(F)$ is connected. ∎

In Section 4, we will completely characterize $\Gamma_1(F)$. Next, we consider the connectedness problem for $\Gamma_1(R)$ graphs in general. The main result in this regard is the following:

Main Theorem for connectedness. *Let R be a finite commutative ring with unity. Then $\Gamma_1(R)$ is connected if and only if R is not isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_2^k$ (for any $k \in \mathbb{N} - \{1\}$).*

We now give a series of lemmas which ultimately lead towards proving the main theorem stated above.

Lemma 2.4. *Let R be a finite commutative ring (with unity) with more than three elements and $Char(R) = n$. If the number of non-zero zero-divisors in R is less than $(n-1)$, then $\Gamma_1(R)$ is connected.*

Proof. By Remark 2.2, there is a path between any two non-zero zero-divisors in $\Gamma_1(R)$.

Case I. Suppose there are no zero-divisors of R other than 0. Then R is a finite integral domain and hence, a field. Thus, by Proposition 2.3, $\Gamma_1(R)$ is connected.

Case II. Suppose that there are non-zero zero-divisors in R. We first show that there exists a non-zero zero-divisor which is adjacent to 1 in $\Gamma_1(R)$. Let a be a non-zero zero-divisor. Let m be the least positive integer for which $ma = 0$. Clearly, $m \leq n$. Now define $S_i = \{i \cdot 1 + ka \mid k = 1, 2, \ldots, m-1\}$ for each $i = 1, 2, \ldots, n-1$. First, we note that for each i, the elements of S_i are distinct, for if $i \cdot 1 + k_1 a = i \cdot 1 + k_2 a$ for some $k_1, k_2 \in 1, 2, \ldots, m-1$ with $k_1 > k_2$, then $(k_1 - k_2)a = 0$. But $1 \leq k_1 - k_2 < m$, which contradicts the minimality of m. Thus, $|S_i| = m-1$ for all $i = 1, 2, \ldots, n-1$. Now suppose that in S_1, at least one element is a unit. Then 1 is adjacent to some ra, which is a non-zero zero-divisor (ka is a zero-divisor for any integer k). If there is no unit in S_1, we consider S_2. If S_2 contains a unit, say $2 \cdot 1 + sa$, then 1 is adjacent to $1 + sa$, the latter being a non-zero zero-divisor, since we have assumed that there are no units in S_1. If S_2 contains no units, we move to S_3 (where finding a unit will ensure that 1 is adjacent to a non-zero zero-divisor of the form $2 + ka$) and continue this upto S_{n-1}, until and unless we find a non-zero zero-divisor adjacent to 1 (i.e., unless we get a unit in some S_t).

Now we show that if we reach S_{n-1} in this process, then the sets $S_1, S_2, \ldots, S_{n-1}$ are disjoint. First we note that $0 \notin S_1$. Now if S_1 does not contain a unit, then $0 \notin S_2$, for if $2 \cdot 1 + ka = 0$ for some k (clearly $k \neq 0$), then $1 + (1 + ka) = 0$, i.e., $1 + ka$ is a unit, which contradicts that S_1 does not contain a unit. The same argument (that $0 \in S_i$ implies existence of a unit in S_{i-1}) shows that if we reach S_{n-1} in the above process, then $0 \notin S_t$ for all $t = 1, 2, \ldots, n-1$. Now if S_p and S_q contain a common element for some $1 \leq p < q \leq n-1$, then $p + k_1 a = q + k_2 a$ for some $1 \leq k_1, k_2 \leq m-1$. This gives $(q-p) + (k_2 - k_1)a = 0$, which implies that $0 \in S_{q-p}$ (since $q - p < n - 1$). So it contradicts that none of S_i for $i = 1, 2, \ldots, n-1$ contains 0. Hence, $S_1, S_2, \ldots, S_{n-1}$ are all disjoint.

So if we reach $\{(n-1) + ka \mid k \in \mathbb{N}\}$ in this process, then we have at least $(n-1)(m-1)$ non-zero zero-divisors. Now since $m \geq 2$, we have $(n-1)(m-1) \geq (n-1)$. Thus we have at least $n-1$ non-zero zero-divisors, which is a contradiction to our assumption. Hence, there exists at least one non-zero zero-divisor r to which 1 is adjacent. Let $1 + r = u_1$, where u_1 is a unit. Let v and t be any two units in R. So $v + vr = vu_1$ and $t + tr = tu_1$. Note that $vr, tr \neq 0$. Also, vu_1, tu_1 are units, each being products of two units. So $v \leftrightarrow vr$ and $t \leftrightarrow tr$. Now vr, tr being non-zero zero-divisors in R, there is a path between them, say P, in $\Gamma_1(R)$. Thus we have a path $v - vr - P - tr - t$ between v and t.

So we see that in $\Gamma_1(R)$, there is a path between any two non-zero zero-divisors, there is a path between any two units, and we also have a path $v - vr - P_1 - z$ between any non-zero zero-divisor z and any unit v (where P_1 is a path between the non-zero zero-divisors vr and z). Hence, the graph $\Gamma_1(R)$ is connected. ■

Corollary 2.5. $\Gamma_1(\mathbb{Z}_n)$ *is connected for all* $n \geq 4$.

Proof. This follows directly from Lemma 2.4 since $\mathbb{Z}_n$ is a commutative ring with unity, its characteristic is n, and the number of non-zero zero-divisors is $n - 1 - \phi(n)$, which is less than $n - 1$. ■

Now the condition mentioned in Lemma 2.4 is not a necessary one. For example, the characteristic of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is 3, and it has 4 non-zero zero-divisors, viz., $(\bar{1}, \bar{0}), (\bar{2}, \bar{0}), (\bar{0}, \bar{1}), (\bar{0}, \bar{2})$. Therefore it does not satisfy the condition stated in Lemma 2.4. However, $\Gamma_1(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is connected (cf. Lemma 2.9).

Next, we have the following result:

Lemma 2.6. *Let R be a finite commutative ring (with unity) with at least one non-zero nilpotent element. Then $\Gamma_1(R)$ is connected.*

Proof. Let a ($\neq 0$) be a nilpotent element in R. Now in $\Gamma_1(R)$, there is a path between any two non-zero zero-divisors. Note that since the ring R is finite, an element is either a zero-divisor or a unit. Now the sum of a nilpotent element and a unit in R is also a unit. So in $\Gamma_1(R)$, all the units will be adjacent to a. Thus, between any two units, there is a path in $\Gamma_1(R)$. Now a being a non-zero zero-divisor, there is a path between a and any non-zero zero-divisor. So there is a path between any arbitrary non-zero zero-divisor and any arbitrary unit (through a). Hence, the graph is connected. ■

We now give two more results.

Lemma 2.7. *Let R be ring with unity. Then in $\Gamma_1(R)$, $x \leftrightarrow y$ if and only if $-x \leftrightarrow -y$.*

Proof. It is sufficient to prove that $x \leftrightarrow y$ implies $-x \leftrightarrow -y$. Now $x \leftrightarrow y$ if and only if either $xy = 0$ or $yx = 0$ or $x + y = u$ for some unit u. So in first two cases $(-x)(-y) = 0$ or $(-y)(-x) = 0$, and in the third case $(-x)+(-y) = -x-y = -u$, which is also a unit. Hence, $x \leftrightarrow y$ if and only if $-x \leftrightarrow -y$. ■

Lemma 2.8. *Let R be a finite ring with unity. Then in $\Gamma_1(R)$, there is a path between any two units.*

Proof. Note that R being a finite ring, any non-zero element in R is either a zero-divisor or a unit. Let a be a unit in R other than 1 and -1. We show that there is a path between a and 1. Now $1 + (a - 1) = a$. So $1 \leftrightarrow a - 1$. Again $(1 - a) + a = 1$. So $a \leftrightarrow 1 - a$. Now if $a - 1$ is a non-zero zero-divisor, then so is $1 - a$. Hence there is a path between $1 - a$ and $a - 1$, say P. So we have a path $a \leftrightarrow (1 - a) - P - (a - 1) \leftrightarrow 1$ between a and 1. Now let $a - 1$ be a unit. So $a \leftrightarrow -1$. Now we find a path between 1 and -1, which will then give a path between a and 1. If $Char(R) = 2$, then $1 = -1$, and we are done. So let $Char(R) \neq 2$. Now if $2 \cdot 1$ is a zero-divisor, then so is $-2 \cdot 1$, and hence there is a path between $2 \cdot 1$ and $-2 \cdot 1$, say P_1. Now $2 \cdot 1 + (-1) = 1$, so $2 \cdot 1 \leftrightarrow -1$, and hence by Lemma 2.7, $-2 \cdot 1 \leftrightarrow 1$. So we have a path $1 \leftrightarrow -2 \cdot 1 - P_1 - 2 \cdot 1 \leftrightarrow -1$ between 1 and -1. Again, if $2 \cdot 1$ is a unit, then so is $2a$, being the product of two units. So $1 \leftrightarrow a - 1 \leftrightarrow a + 1 \leftrightarrow -1$ is a path between 1 and -1. Thus, in all cases, there is a path between a and 1. This clearly shows that there is a path between any two units, through 1. ■

Now we come to the following result, which is, in a way, stronger than Lemma 2.4.

Lemma 2.9. *Let R be finite commutative ring with unity and $Char(R) \neq 2$. Then $\Gamma_1(R)$ is connected unless $R \cong \mathbb{Z}_3$.*

Proof. *Case* I. Let the characteristic of the ring R be a composite number n. Then R has non-zero zero-divisors, and there is a path between any two non-zero zero-divisors in $\Gamma_1(R)$. Now by Lemma 2.8, there is a path between any two units. Since the ring R is finite, any non-zero element is either a zero-divisor or a unit. So if we can show that at least one non-zero zero-divisor is adjacent to a unit, then we will get a path between any two vertices, i.e., the graph will be connected. Now we consider the set $S = \{m \cdot 1 \mid m = 0, 1, 2, \ldots, n - 1\}$. Then S is clearly a subring of R, and also S is isomorphic to $\mathbb{Z}_n$. Now a non-zero zero-divisor in S is a non-zero zero-divisor in R. Also, since $1 \in S$, a unit in S is a unit in R. Consequently, two vertices in $\Gamma_1(S)$ are adjacent if and only if they are adjacent in $\Gamma_1(R)$, i.e., $\Gamma_1(S)$ is an induced subgraph of $\Gamma_1(R)$. Now $\Gamma_1(S)$ is isomorphic to $\Gamma_1(\mathbb{Z}_n)$. By Corollary 2.5, $\Gamma_1(\mathbb{Z}_n)$ is connected. Moreover, n being composite, $\mathbb{Z}_n$ has non-zero zero-divisors. So in $\Gamma_1(\mathbb{Z}_n)$, at least one non-zero zero-divisor is connected to a unit. Thus, from the graph isomorphism, at least one non-zero zero-divisor is adjacent to a unit in $\Gamma_1(S)$, and hence, the same is true in $\Gamma_1(R)$. So $\Gamma_1(R)$ is connected.

Case II. Let the characteristic of R be p, where p is an odd prime. Then R contains a subring isomorphic to $\mathbb{Z}_p$, but we cannot proceed like we did in Case I because $\mathbb{Z}_p$ does not have non-zero zero-divisors. Now if R has no non-zero zero-divisors, then it is a finite integral domain, and hence a finite field. In this case, $\Gamma_1(R)$ is connected by Proposition 2.3, unless $R \cong \mathbb{Z}_3$. So assume that R contains a non-zero zero-divisor x. Now since R is finite, all of $\{x^n \mid n \in \mathbb{N}\}$ cannot be distinct. So we must have $x^r = x^s$ for some positive integers r, s. Let m be the least natural number such that $x^m = x^n$ for some $0 < n < m$. First, we consider the case when $n > 1$. In this case, let $g = x^{m-1} - x^{n-1}$. Clearly, $g \neq 0$, otherwise the minimality of m would be contradicted. Now $g^2 = (x^{m-1} - x^{n-1})^2 = (x(x^{m-2} - x^{n-2}))^2 = x^2(x^{m-2} - x^{n-2})(x^{m-2} - x^{n-2}) = (x^m - x^n)(x^{m-2} - x^{n-2}) = 0$. So we have a nonzero nilpotent element g in R. Hence, by Theorem 2.6, $\Gamma_1(R)$ is connected. Now let $n = 1$, i.e., $x^m = x$. In this case, we observe that $(x^{m-1})^2 = x^{2m-2} = x^m \cdot x^{m-2} = x \cdot x^{m-2} = x^{m-1}$, i.e., x^{m-1} is an idempotent element. So $(x^{m-1} + \frac{p-1}{2} \cdot 1)^2 = x^{m-1} + (p-1)x^{m-1} + \frac{p^2-2p+1}{4} \cdot 1 = \frac{p^2-2p+1}{4} \cdot 1$, as the characteristic of R is p. Now $\frac{p^2-2p+1}{4}$ cannot be a multiple of p. Hence $\frac{p^2-2p+1}{4} \cdot 1$ is a non-zero element belonging to a subring of R isomorphic to $\mathbb{Z}_p$. So it is a unit in R. Now $(x^{m-1} + \frac{p-1}{2} \cdot 1)^2$ being a unit, $x^{m-1} + \frac{p-1}{2} \cdot 1$ must also be a unit. So we have an edge between a non-zero zero-divisor x^{m-1} and a unit $\frac{p-1}{2} \cdot 1$. Now we already have that there is a path between any two non-zero zero-divisors, and also, there is a path between any two units. So the existence of an edge between a non-zero zero-divisor and a unit clearly shows that the graph $\Gamma_1(R)$ is connected. ∎

Note that when the characteristic of a finite ring R is composite, we can proceed exactly as in Case I of Lemma 2.9 even if R is non-commutative, because Case I does not require the commutativity of R. So we have the following result.

Corollary 2.10. *Let R be a finite ring with unity and with non-prime characteristic. Then $\Gamma_1(R)$ is connected.*

Next, we have the following result.

Lemma 2.11. Let $F_1, F_2, \ldots, F_k$ be finite fields, $k \geq 2$. Then $\Gamma_1(F_1 \times F_2 \times \cdots \times F_k)$ is connected if and only if at least one of the fields is not isomorphic to $\mathbb{Z}_2$.

Proof. Since there is always a path between any two non-zero zero-divisors by Theorem 1.1, we proceed by trying to find an adjacent zero-divisor for any arbitrary unit $(a_1, a_2, \ldots, a_k)$ in $F_1 \times F_2 \times \cdots \times F_k$. Note that here $(a_1, a_2, \ldots, a_k)$ is a unit if and only if $a_i \neq 0$ for all $i = 1, 2, \ldots, k$.

Case I. Assume at least one of the fields has characteristic not equal to 2. Then $F_1 \times F_2 \times \cdots \times F_k$ is a commutative ring with characteristic > 2. Hence, by Lemma 2.9, the graph $\Gamma_1(F_1 \times F_2 \times \cdots \times F_k)$ is connected.

Case II. Assume all the fields have Characteristics 2 and at least one of the fields has more than 2 elements (i.e., is not isomorphic to $\mathbb{Z}_2$). Let F_1 not be isomorphic to $\mathbb{Z}_2$ (without loss of generality). Then F_1 will have at least one unit x different from 1. Consider a unit $(a_1, a_2, \ldots, a_k)$.

Now we have

$$(a_1, a_2, \ldots, a_k) \leftrightarrow \begin{cases} (1 - a_1, 0, 0, \ldots, 0) & \text{if } a_1 \neq 1; \\ (x + 1, 0, 0, \ldots, 0) & \text{if } a_1 = 1. \end{cases}$$

So any unit is adjacent to some non-zero zero-divisor, and hence the graph is connected.

Case III. Assume $F_1, F_2, \ldots, F_k$ are all isomorphic to $\mathbb{Z}_2$. Here, the element $(\bar{1}, \bar{1}, \bar{1}, \ldots, \bar{1})$ is the only unit, and it has no adjacent zero-divisors in $\Gamma_1(F_1 \times F_2 \times \cdots \times F_k)$. So the graph cannot be connected.

Thus $\Gamma_1(F_1 \times F_2 \times \cdots \times F_k)$ is connected if and only if at least one of the fields is not isomorphic to $\mathbb{Z}_2$. ∎

Now, we finally proceed to prove the main theorem for connectedness with the help of the lemmas given so far. The main theorem, which we prove below, completely characterizes the finite commutative rings R with unity for which $\Gamma_1(R)$ is connected.

Theorem 2.12. *Let R be a finite commutative ring with unity. Then $\Gamma_1(R)$ is connected if and only if R is not isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_2^k$ (for any $k \in \mathbb{N} - \{1\}$).*

Proof. It is sufficient to show that $\Gamma_1(R)$ is disconnected if and only if $R \cong \mathbb{Z}_3$ or $\mathbb{Z}_2^k$ (for any $k \in \mathbb{N} - \{1\}$). We have already seen that $\Gamma_1(\mathbb{Z}_3)$ is disconnected. Also, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then the vertex $(\bar{1}, \bar{1}, \ldots, \bar{1})$ is isolated in $\Gamma_1(R)$. So $\Gamma_1(R)$ is disconnected. Conversely, let $\Gamma_1(R)$ be disconnected. By Lemma 2.9, either $R \cong \mathbb{Z}_3$ or $Char(R) = 2$. Let $R \ncong \mathbb{Z}_3$. Now if R has a non-zero nilpotent element, then $\Gamma_1(R)$ is connected by Lemma 2.6. So R must be a reduced ring, i.e., a ring with no non-zero nilpotent elements. Since R is finite and reduced, R is a direct product of finite fields. Now in this case, we have by Lemma 2.11 that $\Gamma_1(R)$ is disconnected if and only if R is isomorphic to the ring $\mathbb{Z}_2^k$ for some $k \in \mathbb{N} - \{1\}$. Hence the result. ∎

3. Characterization of rings with respect to the graph $\Gamma_1(R)$

We now aim to characterize the rings for which $\Gamma_1(R)$ realizes some well-known classes of graphs. Clearly, this also helps us to classify the particular graphs from those classes which are realizable as $\Gamma_1(R)$. We start with the following result.

Theorem 3.1. *Let R be a finite ring with unity. Then $\Gamma_1(R)$ is a complete graph if and only if R is a field of characteristic 2.*

Proof. If R is a field of characteristic 2, then $a+b \neq 0$ for any two distinct non-zero elements a, b in F. So $a+b$ is a unit. This shows that $a \leftrightarrow b$ in $\Gamma_1(R)$ for any two distinct vertices a, b. So $\Gamma_1(R)$ is a complete graph (in fact, $\Gamma_1(\mathbb{F}_{2^n}) \cong K_{2^n-1}$, as shown in Corollary 4.2). Alternatively, the unit graph $G(R)$ in this case is complete by [5, Theorem 3.4], and hence $\Gamma_1(R)$ is also complete (since $\Gamma_1(R)$ is the subgraph of $G(R)$ induced by the non-zero vertices if R is a division ring). Conversely, let $\Gamma_1(R)$ be complete. If possible, let $1 \neq -1$. Now 1 is not adjacent to the vertex -1, which contradicts that $\Gamma_1(R)$ is complete. So we must have $1=-1$, i.e., $Char(R)=2$. If possible, let R not be a field. Since R is a finite ring, this implies that R has a non-zero zero-divisor, say z. Now $z+1 \neq 0$. We have that $1+(z+1)=z$ and $1 \cdot (z+1) \neq 0$. So 1 is not adjacent to $z+1$. This contradicts that $\Gamma_1(R)$ is complete. So R must be a field. Hence the result. ■

Next, we consider the existence of cycles in $\Gamma_1(R)$. This, as we will see afterwards, helps significantly in the classification of $\Gamma_1(R)$ graphs. First, we have the following lemma.

Lemma 3.2. *$\Gamma_1(\mathbb{Z}_n)$ is acyclic if and only if $n=2,3,4,6$.*

Proof. It is easy to see that $\Gamma_1(\mathbb{Z}_n)$ is acyclic for $n=2,3,4,6$ (cf. Figure 3, 4, 6, 9 in the Appendix). Also, as shown in Figure 8, $\Gamma_1(\mathbb{Z}_5) \cong C_4$. Let us consider $n>6$. We look for cycles in $\Gamma_1(\mathbb{Z}_n)$ for different values of n. If $n=p$, where p is any odd prime, then we have the 3-cycle $\bar{1}-\bar{2}-\bar{3}-\bar{1}$. If $n=2p$, where p is any odd prime, then we have the 4-cycle $\bar{1}-\bar{2}-\bar{p}-(\overline{-2})-\bar{1}$. If $n=pq$, where p, q are distinct odd primes, we have the 3-cycle $\bar{1}-\bar{3}-(\overline{-2})-\bar{1}$. Again, if $n=p_1p_2\cdots p_k$, where the p_i's are distinct primes and $k>2$, then we have the 3-cycle $\overline{\frac{n}{p_1}}-\overline{\frac{n}{p_2}}-\overline{\frac{n}{p_3}}-\overline{\frac{n}{p_1}}$. Now consider the case when n is not square-free, i.e., when $\mathbb{Z}_n$ contains a non-zero nilpotent. First, let n be odd. It is easy to see that there exists a non-zero nilpotent $\bar{x}$ such that $\bar{x}^2=\bar{0}$. Now noting that the sum of a unit and a nilpotent is a unit in $\mathbb{Z}_n$, we have the 3-cycle $\bar{1}-\bar{x}-\overline{2x}-\bar{1}$ in $\Gamma_1(\mathbb{Z}_n)$. Again, let n be even. If $\overline{\frac{n}{2}}$ is nilpotent, we have the 3-cycle $\bar{1}-(\overline{-2})-\overline{\frac{n}{2}}-\bar{1}$. If $\overline{\frac{n}{2}}$ is not nilpotent, then $n=2lp^k$ for some odd prime p, odd l and $k>2$. In this case we have the 3-cycle $\bar{1}-\overline{2lp^{k-1}}-\overline{4lp^{k-1}}-\bar{1}$. So, having considered all the cases, we have that $\Gamma_1(\mathbb{Z}_n)$ is acyclic if and only if $n=2,3,4,6$. ■

Next, we have the following result.

Lemma 3.3. *Let R be a finite commutative ring (with unity) distinct from $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ such that $Char(R) = 3, 4$ or 6. Then $\Gamma_1(R)$ contains a cycle.*

Proof. First, let $Char(R) = 3$. If R contains a unit u distinct from $1, -1$, then we have the 3-cycle $u+1 \leftrightarrow u+2 \leftrightarrow -u \leftrightarrow u+1$ in $\Gamma_1(R)$. Again, let all elements of $R - \{0, -1, 1\}$ be non-zero zero-divisors. If there is a non-zero nilpotent, then there is a non-zero element z such that $z^2 = 0$, and then we have the 3-cycle $1 \leftrightarrow z \leftrightarrow 2z \leftrightarrow 1$. If there is no non-zero nilpotent, then for any non-zero zero-divisor z, there is a non-zero t $(\neq z)$ such that $zt = 0$. In this case, we have the cycle $z \leftrightarrow 2t \leftrightarrow 2z \leftrightarrow t \leftrightarrow z$ (note that $2t \neq z$ and $2z \neq t$ as otherwise we would have $z^2 = 0$).

Next, let $Char(R) = 4$. If R has a unit u distinct from $1, -1$, then we have the cycle $1 \leftrightarrow 2 \leftrightarrow 2u \leftrightarrow 1$ (as $2, 2u$ are non-zero nilpotents). Suppose R has no such units. Then all elements of $R - R_1$ are non-zero zero-divisors, where $R_1 = \{m \cdot 1 \mid m = 1, 2, 3, 4\}$. We show that there exists a non-zero zero-divisor $z \in R - R_1$ such that $2z \neq 0$. If possible, let $2x = 0$ for some $x \in R - R_1$. Clearly, $|R - R_1| > 1$, as $|R|$ is a multiple of 4. Let $z_1 \in R - R_1$. Now $z_1 + 1 \notin R_1$. So $2(z_1 + 1) = 0$, which gives $2 = -2z_1 = 0$, which is not possible. Hence, we have some $z \in R - R_1$ such that $2z \neq 0$. Noting that $2z$ is a non-zero nilpotent, we have the 3-cycle $1 \leftrightarrow 2 \leftrightarrow 2z \leftrightarrow 1$.

Lastly, let $Char(R) = 6$. Let $R_1 = \{m \cdot 1 \mid m = 1, 2, \ldots, 6\}$. If $R - R_1$ has a unit u, then we have the cycle $2 \leftrightarrow 3u \leftrightarrow 4 \leftrightarrow 3 \leftrightarrow 2$. Suppose $R - R_1$ contains non-zero zero-divisors only. If $R - R_1$ has a non-zero zero-divisor z such that $z^2 = 0$ then we have the 3-cycle $1 \leftrightarrow 5z \leftrightarrow z \leftrightarrow 1$. Now suppose there exists no such non-zero zero-divisor in $R - R_1$. We know that the subgraph $\Gamma(R)$ of $\Gamma_1(R)$ is a connected graph. So we have some $z \in R - R_1$ such that z is adjacent to at least one of $2, 3, 4$, i.e., $tz = 0$ for some $t \in \{2, 3, 4\}$. If $2z = 0$, then $3z \neq 0$. So we have the cycle $z \leftrightarrow z + 1 \leftrightarrow z + 4 \leftrightarrow z + 3 \leftrightarrow 2 \leftrightarrow z$. Again, if $3z = 0$, then $2z, 4z \neq 0$, and we have the cycle $2z \leftrightarrow z+1 \leftrightarrow 2z+4 \leftrightarrow z+3 \leftrightarrow 2z+2 \leftrightarrow 3 \leftrightarrow 2z$ (note that $2z \neq 3$ as otherwise $z = -3$). Finally, if $4z = 0$, we have the cycle $2z \leftrightarrow 2z + 1 \leftrightarrow 2z + 4 \leftrightarrow 2z + 3 \leftrightarrow 2 \leftrightarrow 2z$ (note that $2z \neq \pm 2$ as otherwise $4 = 0$).

So in all three cases, $\Gamma_1(R)$ contains a cycle. ■

With the help of the previous two lemmas, we obtain the following.

Proposition 3.4. *Let R be a finite commutative ring with unity such that $Char(R) \neq 2$ and $R \ncong \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. Then $\Gamma_1(R)$ contains a cycle.*

Proof. If $R \cong \mathbb{Z}_n$ for some $n \notin \{2, 3, 4, 6\}$, then by Lemma 3.2, $\Gamma_1(R)$ contains a cycle. It is easy to see that if S is a ring which contains a subring isomorphic to $\mathbb{Z}_n$ for any $n \notin \{2,3,4,6\}$, then any cycle in $\Gamma_1(\mathbb{Z}_n)$ (for corresponding value of n) is present in $\Gamma_1(S)$ as well. Hence, if $Char(R) \notin \{2,3,4,6\}$, then $\Gamma_1(R)$ contains a cycle. Again, let $R \ncong \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and $Char(R) \in \{3,4,6\}$. Then by Lemma 3.3, we have that $\Gamma_1(R)$ contains a cycle. Hence the result. ■

Remark 3.5. Note that $\Gamma_1(R)$ can contain cycles even when $Char(R) = 2$. For example, if $R \cong \mathbb{F}_{2^n}$, then $\Gamma_1(R) \cong K_{2^n-1}$. Also, if R is a boolean ring distinct from $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, and hence $\Gamma_1(R)$ contains a 3-cycle formed by the vertices $\{(\bar{1},\bar{0},\ldots,\bar{0}),(\bar{0},\bar{1},\ldots,\bar{0}),(\bar{0},\bar{0},\bar{1},\ldots,\bar{0})\}$. However, $\Gamma_1(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is acyclic.

Next, we completely characterize the finite commutative rings R for which $\Gamma_1(R)$ is a star graph. A star graph is a simple undirected graph where there is one vertex (called the central vertex) which is adjacent to all other vertices and the remaining vertices are not adjacent to each other.

Theorem 3.6. *Let R be a finite commutative ring with unity. Then $\Gamma_1(R)$ is a star graph having at least three vertices if and only if $R \cong \mathbb{Z}_4$ or $R \cong \frac{\mathbb{Z}_2[x]}{<x^2>}$. In both the cases, $\Gamma_1(R) \cong P_3$.*

Proof. It is easy to see that if $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$, then $\Gamma_1(R) \cong P_3$. Conversely, let $\Gamma_1(R)$ be a star graph having at least 3 vertices. So we have that $|R| > 3$. Since $\Gamma_1(R)$ is acyclic, by Proposition 3.4 we have that either R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_6$ or $Char(R) = 2$. Now $\Gamma_1(\mathbb{Z}_4)$ is a star graph, but $\Gamma_1(\mathbb{Z}_6)$ is not. Now let $Char(R) = 2$. As we have seen in Remark 3.5, $\Gamma_1(R)$ is not acyclic if R is a field with $|R| > 2$ and $Char(R) = 2$. So R contains non-zero zero-divisors. If possible, let the central vertex be a unit element. Then if we have more than one non-zero zero-divisor, the subgraph $\Gamma(R)$ is disconnected, which is not possible. So we can have only one non-zero zero-divisor, say z_1. But in this case, $z_1^2 = 0$, and hence all units will be adjacent to z_1, which is a contradiction since $\Gamma_1(R)$ is a star graph having at least 3 vertices. So the central vertex has to be a non-zero zero-divisor, say z. First, let z be the only non-zero zero-divisor in R. Then by [10, Theorem I], we have that $|R| \leq 4$. Since we also have that $|R| > 3$, this gives $|R| = 4$. Hence, keeping in mind that $Char(R) = 2$, we have that R is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2, \frac{\mathbb{Z}_2[x]}{<x^2>}$, or $\mathbb{F}_4$. Among them, $\Gamma_1(R)$ is a star graph only for $R \cong \frac{\mathbb{Z}_2[x]}{<x^2>}$. Again, let R have non-zero zero-divisors other than z. Then the subgraph $\Gamma(R)$ is also a star graph (note that $\Gamma(R)$ cannot have isolated vertices). If $\Gamma(R) \cong K_2$, then by [3, Theorem 2.10] we have that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (since $Char(R) = 2$). But we have seen that $\Gamma_1(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not a star graph. So $\Gamma(R)$ is a star graph with at least 3 vertices. If there is no non-zero t such that $t^2 = 0$, then by [7, Corollary 4.6] we have that R is a direct product of fields. Clearly, here the fields are of characteristic 2. If R is a product of more than 2 fields then $\Gamma_1(R)$ has a 3-cycle formed by the vertices $(\bar{1},\bar{0},\ldots,\bar{0}),(\bar{0},\bar{1},\bar{0},\ldots,\bar{0}),(\bar{0},\bar{0},\bar{1},\ldots,\bar{0})$. So R is of the form $F_1 \times F_2$. Now as we have already considered $\mathbb{Z}_2 \times \mathbb{Z}_2$, we may assume that (without loss of generality) F_1 has a unit distinct from 1. Then we have the 4-cycle $(1,0) \leftrightarrow (0,1) \leftrightarrow (u,0) \leftrightarrow (1+u,1) \leftrightarrow (1,0)$. So now we assume that there is a non-zero t such that $t^2 = 0$. It is easy to see that t is the central vertex as any unit is adjacent to t. So by [7, Lemma 2.6], we have that R is isomorphic to $\frac{\mathbb{Z}_2[x]}{<x^2>}$ or $\frac{\mathbb{Z}_2[x]}{<x^3>}$. Now in the latter case, we have a 3-cycle formed by the vertices $1+ < x^3 >, x+ < x^3 >, x^2+ < x^3 >$. So having considered all these, $\Gamma_1(R)$ is a star graph if and only if $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. ∎

We now observe an interesting property of acyclic $\Gamma_1(R)$ graphs when $Char(R) = 2$.

Proposition 3.7. *Let R be a finite commutative ring with unity such that $Char(R) = 2$ and $\Gamma_1(R)$ is acyclic. Then no two units of R are adjacent to each other in $\Gamma_1(R)$.*

Proof. Let u, v be two unit vertices adjacent to each other in $\Gamma_1(R)$. Then $u+v$ is a unit which is distinct from both u and v. Hence we have a 3-cycle formed by the vertices $u, v, u+v$, which is a contradiction. Thus, no two units are adjacent to each other. ∎

Corollary 3.8. *If $\Gamma_1(R)$ is a tree, then no two units are adjacent to each other.*

Proof. Let $\Gamma_1(R)$ be a tree. Then $\Gamma_1(R)$ is acyclic and connected. Thus, by Proposition 3.4, $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_6$ or we have $Char(R) = 2$ (note that $\Gamma_1(\mathbb{Z}_3)$ is disconnected). Now in $\mathbb{Z}_4$ and $\mathbb{Z}_6$, the only units are 1 and -1, so they cannot be adjacent. Again, if $Char(R) = 2$, then two units cannot be adjacent to each other by Proposition 3.7. ∎

Proposition 3.4 and Corollary 3.8 help us to characterize the rings for which $\Gamma_1(R)$ is a path graph or a cycle, as we show next.

Theorem 3.9. *Let R be a finite commutative ring with unity. Then $\Gamma_1(R) \cong P_n$ (for some $n \in \mathbb{N} - \{1, 2\}$) if and only if R is isomorphic to $\mathbb{Z}_4, \mathbb{Z}_6$, or $\frac{\mathbb{Z}_2[x]}{<x^2>}$.*

Proof. It is easy to see that $\Gamma_1(\mathbb{Z}_6) \cong P_5$ and $\Gamma_1(\mathbb{Z}_4) \cong \Gamma_1(\frac{\mathbb{Z}_2[x]}{<x^2>}) \cong P_3$. Conversely, let $\Gamma_1(R)$ be a path with at least 3 vertices. So by Proposition 3.4, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_6$ or $Char(R) = 2$. Now having already considered $\mathbb{Z}_6$ and $\mathbb{Z}_4$, we may assume that $Char(R) = 2$. By Corollary 3.8, units cannot be adjacent to each other. Suppose there exists a unit vertex which is not a pendant (i.e., having degree 1) vertex. If R has more than one non-zero zero-divisor, then $\Gamma(R)$ becomes disconnected as there is a non-pendant unit vertex and two units cannot be adjacent to each other. So either there is only one non-zero zero-divisor or units are only of degree one. In the first case, $|R| \leq 4$ (by [10, Theorem I]. This gives $|R| = 4$ as $\Gamma_1(R)$ has at least three vertices. It is easily seen that among such rings, $\Gamma_1(R)$ is a path (with at least 3 vertices) only for $R \cong \frac{\mathbb{Z}_2[x]}{<x^2>}$. Now consider the case when all units are of degree 1. Clearly, 1 is a terminal vertex. So 1 is adjacent to some zero-divisor z, i.e., $1+z$ is a unit. Hence, $1+z$ is the other terminal vertex and we also note that $z \leftrightarrow 1+z$. This implies $\Gamma_1(R) \cong P_3$. So by Theorem 3.6, R is isomorphic to $\frac{\mathbb{Z}_2[x]}{<x^2>}$. ■

Theorem 3.10. *Let R be a finite commutative ring with unity. Then $\Gamma_1(R) \cong C_n$ (for some $n \in \mathbb{N} - \{1, 2\}$) if and only if $R \cong \mathbb{F}_4$ or $R \cong \mathbb{Z}_5$.*

Proof. It is easy to see that $\Gamma_1(\mathbb{F}_4) \cong C_3$ and $\Gamma_1(\mathbb{Z}_5) \cong C_4$. Conversely, let $\Gamma_1(R)$ be a cycle. We know that $\Gamma_1(R)$ is acyclic for $R \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. From the proof of Lemma 3.2 and Proposition 3.4, we can see that for a ring R distinct from $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and with $Char(R) \neq 2, 6$, the graph $\Gamma_1(R)$ contains a 3-cycle or 4-cycle. So if $\Gamma_1(R)$ is a cycle for a ring R distinct from $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and with $Char(R) \neq 2, 6$, then $4 \leq |R| \leq 5$. Among the finite commutative rings of said orders, $\Gamma_1(R)$ is a cycle only for $R \cong \mathbb{F}_4$ and $R \cong \mathbb{Z}_5$. Now for $Char(R) = 6$ (where $R \ncong \mathbb{Z}_6$), $\Gamma_1(R)$ contains a cycle of length at most 6 (from the proof of Lemma 3.3). Since we have already considered 3-cycles and 4-cycles, we assume $6 \leq |R| \leq 7$. Now the only finite commutative ring of order 7 with unity is $\mathbb{Z}_7$, and $Char(\mathbb{Z}_7) \neq 6$. So $|R| = 6$. Since R is a commutative ring with unity, this gives $R \cong \mathbb{Z}_6$, which is a contradiction as $\Gamma_1(\mathbb{Z}_6)$ is acyclic. Finally, let $Char(R) = 2$. If $\Gamma_1(R) \cong C_3$, then $|R| = 4$. We have studied the cases $\mathbb{F}_4, \frac{\mathbb{Z}_2[x]}{<x^2>}, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$ before. So we may assume that $\Gamma_1(R) \cong C_n$ for some $n > 3$. In this case no two units can be adjacent to each other (as their sum would then be another unit, and we would have a 3-cycle). So if there exists at least two units in R, then the subgraph $\Gamma(R)$ is disconnected (note that then R has at least two non-zero zero-divisors as otherwise we would have an adjacent pair of unit vertices since $\Gamma_1(R)$ is assumed to be cycle), which is not possible. Consequently, R has only one unit 1. Thus 1 is adjacent to some non-zero zero-divisor z, and hence $1+z$ is unit. This contradicts that 1 is the only unit. Thus, if $\Gamma_1(R)$ is a cycle, then R is isomorphic to $\mathbb{F}_4$ or $\mathbb{Z}_5$. ■

Remark 3.11. It is interesting to note that $\Gamma_1(R) \cong \Gamma_1(S)$ does not imply $R \cong S$. For example, $\Gamma_1(\mathbb{Z}_4) \cong \Gamma_1(\frac{\mathbb{Z}_2[x]}{<x^2>})$.

4. $\Gamma_1(F)$ WHERE F IS A FINITE FIELD

Now we consider the graph $\Gamma_1(F)$ for a finite field F. We know that a finite field is of order p^n, where p is a prime and $n \in \mathbb{N}$, and also that the characteristic of a field of order p^n is p. We start with the following result:

Theorem 4.1. *Let F be a finite field. Then $\Gamma_1(F)$ is a regular graph.*

Proof. Let $|F| = p^n$ for some prime p and $n \in \mathbb{N}$. Now in a field, there are no non-zero zero-divisors. Hence, the product of two non-zero elements cannot be 0, and the sum of two elements is a unit unless they are additive inverses of each other.

Case I. Let $p = 2$. Then $Char(F) = 2$, and hence each element is its own additive inverse. Thus each vertex is adjacent to all other vertices. So $deg(v) = 2^n - 2$ for every vertex v.

Case II. Let p be an odd prime. The additive inverse of a is $p - a$, and an element cannot be its own inverse since p is odd. Thus each vertex is adjacent to all other vertices except its additive inverse. So $deg(v) = p^n - 3$ for all vertex v. Hence the result. ∎

Corollary 4.2. *Let F be a finite field with $|F| = p^n$. If $p = 2$, then $\Gamma_1(F) \cong K_{2^{n-1}}$, and if p is odd, then $\overline{\Gamma_1(F)} \cong \frac{(p^n-1)}{2}$ copies of K_2, where $\overline{\Gamma_1(F)}$ is the complement of the graph $\Gamma_1(F)$.*

Proof. For $p = 2$, $\Gamma_1(F)$ is a complete graph by Theorem 3.1. Let p be odd. Since each vertex v is adjacent to all other vertices except its unique additive inverse $-v$, the complement of the graph consists of $(p^n - 1)/2$ copies of K_2. ∎

Corollary 4.3. *Let F be a finite field with $|F| > 3$. Then $\Gamma_1(F)$ is connected and Eulerian.*

Proof. This readily follows from Theorem 4.1 or Corollary 4.2. ∎

We are interested if the converse of Theorem 4.1 holds to some extent, i.e., if for a finite commutative ring R, $\Gamma_1(R)$ is k-regular for some particular value of k, then whether R will be a field or not. As we show in the following result, it does hold true for certain values of k.

Theorem 4.4. *Let R be a commutative ring of order n with unity. If n is odd and $\Gamma_1(R)$ is $(n-3)$-regular, then R is a field of characteristic p, where p is an odd prime. If n is even and $\Gamma_1(R)$ is $(n-2)$-regular, then R is a field of characteristic 2.*

Proof. If R has no non-zero zero-divisors, then R, being a finite integral domain, is a field. If possible, let R have non-zero zero-divisors.

(i) Let n be odd and the graph $\Gamma_1(R)$ be $(n-3)$-regular. Since the characteristic of a finite ring divides the order of the ring, the characteristic must be odd in this case. Suppose $Char(R) = 2k+1$. Now we write $2k \cdot 1$ as $2k$. Clearly, $2k$ is not adjacent to 1. Let z be a non-zero zero-divisor. If $2k+z$ is a zero-divisor, then $2k \not\leftrightarrow z$, because $2kz = 0$ would imply $z = z + 2kz = (2k+1)z = 0$. Note that z is neither 1 nor $2k$ (we know that there is a $z_1 \neq 0$ such that $zz_1 = 0$; so $z = 2k$ would imply $z_1 = 0$ as before). Hence $deg(2k) \leq n-4$, which is a contradiction. So suppose $2k + z = u$ for some unit u. Then $2k - u = -z$, and also $2ku \neq 0$ since u is a unit. So $2k \not\leftrightarrow -u$. Now $-u \neq 1$ (as $-u = 1$ would imply $z = 0$). If possible, let $-u = 2k$. Then $2k - u = -z$ implies $4k = -z$. Hence $4kz_2 = 0$ for some zero-divisor z_2 ($\neq 0$). This gives $2z_2 = 4kz_2 + 2z_2 = 2(2k+1)z_2 = 0$, and consequently $2kz_2 = 0$. Hence, $z_2 = 2kz_2 + z_2 = (2k+1)z_2 = 0$. This is a contradiction; so $-u \neq 2k$. Hence, $2k$ is not adjacent to 1, $2k$,$-u$. This implies $deg(2k) \leq n-4$. Hence our initial assumption that there are non-zero zero-divisors in R is wrong. So R is a finite integral domain, and hence a field. The order of R being odd, $Char(R)$ must be an odd prime p.

(ii) Let n be even and $\Gamma_1(R)$ be $(n-2)$-regular. Suppose that $Char(R) = 2k$. Let $l = -1$. Now l is not adjacent to 1. Let z be a non-zero zero-divisor. If $l+z$ is a zero-divisor, then l is not adjacent to z. Consequently, $deg(l)$ is at most $n-3$, which is a contradiction. Again, let $l + z = u$ for some unit u. Then $l - u = -z$, and hence $l \not\leftrightarrow -u$. Note that if $l = -u$, then $z = 2u$, which is impossssible since u is a unit. So $l \neq -u$. Since $-u \neq 1$ (as $-u = 1$ would imply $l - u = 0$, i.e., $z = 0$), we again have that $deg(l)$ is at most $n-3$. So our initial assumption, that there are non-zero zero-divisors in R is wrong. Hence R is a finite integral domain, i.e., a field. Moreover, n being even, the characteristic of R must be 2.

■

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APPENDIX: ALL POSSIBLE $\Gamma_1(R)$ GRAPHS WITH AT MOST 6 VERTICES

We here give all possible $\Gamma_1(R)$ graphs with at most 6 vertices, where R is a finite commutative ring with unity. We also mention the rings R for which the respective graphs are realized as $\Gamma_1(R)$.

Figure 3. $\Gamma_1(\mathbb{Z}_2)$

Figure 4. $\Gamma_1(\mathbb{Z}_3)$

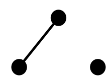

Figure 5. $\Gamma_1(\mathbb{Z}_2 \times \mathbb{Z}_2)$

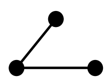

Figure 6. $\Gamma_1(\mathbb{Z}_4) \cong \Gamma_1(\frac{\mathbb{Z}_2[x]}{<x^2>})$

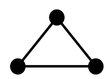

Figure 7. $\Gamma_1(\mathbb{F}_4)$

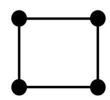

Figure 8. $\Gamma_1(\mathbb{Z}_5)$

Figure 9. $\Gamma_1(\mathbb{Z}_6)$

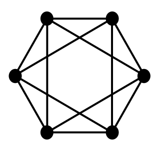

Figure 10. $\Gamma_1(\mathbb{Z}_7)$

The single vertex graph will correspond to a ring of two elements with unity; so it is realized as $\Gamma_1(\mathbb{Z}_2)$. In fact, if p is a prime, then any $\Gamma_1(R)$ graph with exactly $p-1$ vertices corresponds to a ring of order p. Noting that $\mathbb{Z}_p$ is the only finite commutative ring of order p with unity, we have that the only $\Gamma_1(R)$ graphs of (exactly) 2 vertices, 4 vertices, and 6 vertices are $\Gamma_1(\mathbb{Z}_3)$, $\Gamma_1(\mathbb{Z}_5)$, and $\Gamma_1(\mathbb{Z}_7)$, respectively (shown in Figure 4, Figure 8, and Figure 10, respectively). Now for graphs with exactly 3 vertices, we need to consider finite commutative rings of order 4 with unity. We have 4 such rings viz., $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{<x^2>}$, and $\mathbb{F}_4$. So the only graphs of 4 vertices realizable as $\Gamma_1(R)$ in this case are those shown in figures 5, 6, 7. Note that $\Gamma_1(\mathbb{Z}_4) \cong \Gamma_1(\frac{\mathbb{Z}_2[x]}{<x^2>})$. Finally, for $\Gamma_1(R)$ graphs of exactly 5 vertices, we need to consider rings of order 6. The only finite commutative ring of order 6 with unity $1\ (\neq 0)$ is $\mathbb{Z}_6$. Hence, the graph shown in Figure 9 is the only graph of 6 vertices realizable as $\Gamma_1(R)$. So figures 3-10 give all possible $\Gamma_1(R)$ graphs with at most 6 vertices.