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ON A PERIODIC PART OF PSEUDO-BCI-ALGEBRAS

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Abstract

In the paper the connections between the set of some maximal elements of a pseudo-BCI-algebra and deductive systems are established. Using these facts, a periodic part of a pseudo-BCI-algebra is studied.

Keywords: pseudo-BCI-algebra, deductive system, periodic part.

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1. Introduction

Among many algebras of logic, BCI-algebras, introduced in [8], form an important and interesting class of algebras. They have connections with BCI-logic being the BCI-system in combinatory logic, which has application in the language of functional programming.

The notion of pseudo-BCI-algebras has been introduced in [1] as an extension of BCI-algebras. Pseudo-BCI-algebras are algebraic models of some extension of a non-commutative version of the BCI-logic. These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras. More about those algebras a reader can find in [7].

The paper is devoted to pseudo-BCI-algebras. In Section 2, we give some necessary material needed in the sequel. In Section 3, first we investigate the p-semisimple part $M(X)$ of a pseudo-BCI-algebra X and give conditions for $M(X)$ to be a deductive system of X. For $D \subseteq X$, the set $M(D) = \{(x \to 1) \to 1 : x \in D\}$ is also investigated. We end this section by giving some facts about deductive systems of a pseudo-BCI-algebra. Finally, using the results of Section 3, we study a periodic part of a pseudo-BCI-algebra in Section 4.

2. Preliminaries

A *pseudo-BCI-algebra* is a structure $(X; \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is a binary relation on a set X, $\rightarrow$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that for all $x, y, z \in X$, we have

(a1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,

(a2) $x \leq (x \rightarrow y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \rightarrow y$,

(a3) $x \leq x$,

(a4) if $x \leq y$ and $y \leq x$, then $x = y$,

(a5) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI-algebra $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ can be regarded as an algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that every pseudo-BCI-algebra satisfying $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI-algebra satisfying $x \leq 1$ for all $x \in X$ is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Troughout this paper, we will often use X to denote a pseudo-BCI-algebra. Any pseudo-BCI-algebra X satisfies the following, for all $x, y, z \in X$:

(b1) if $1 \leq x$, then $x = 1$,

(b2) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,

(b4) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,

(b5) $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,

(b6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \quad x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,

(b7) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(b8) $1 \rightarrow x = 1 \rightsquigarrow x = x$,

(b9) $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y, \quad ((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y$,

(b10) $x \rightarrow y \leq (y \rightarrow x) \rightsquigarrow 1, \quad x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightarrow 1$,

(b11) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightsquigarrow 1), \quad (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightarrow 1)$,

(b12) $x \rightarrow 1 = x \rightsquigarrow 1$.

If $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1), $(X; \leq)$ is a poset with 1 as a maximal element. Note that a pseudo-BCI-algebra has also other maximal elements.

Example 2.1 [3]**.** Let $X = \{a, b, c, d, 1\}$ and define binary operations $\rightarrow$ and $\rightsquigarrow$ on X by the following tables:

$\rightarrow$	a	b	c	d	1
a	1	1	1	d	1
b	b	1	1	d	1
c	b	b	1	d	1
d	d	d	d	1	d
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	1	1	d	1
b	c	1	1	d	1
c	a	b	1	d	1
d	d	d	d	1	d
1	a	b	c	d	1

Then $(X; \rightarrow, \rightsquigarrow, 1)$ is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because $d \nleq 1$.

Example 2.2 [9]**.** Let $Y_1 = (-\infty, 0]$ and let $\leq$ be the usual order on Y_1. Define binary operations $\rightarrow$ and $\rightsquigarrow$ on Y_1 by

$$x \rightarrow y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ \frac{2y}{\pi} \arctan(\ln(\frac{y}{x})) & \text{if} \quad y < x, \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} 0 & \text{if} \quad x \leq y, \\ y e^{-\tan(\frac{\pi x}{2y})} & \text{if} \quad y < x \end{cases}$$

for all $x, y \in Y_1$. Then $(Y_1; \leq, \rightarrow, \rightsquigarrow, 0)$ is a pseudo-BCK-algebra, and hence it is a nonproper pseudo-BCI-algebra.

Example 2.3 [6]**.** Let $Y_2 = \mathbb{R}^2$ and define binary operations $\rightarrow$ and $\rightsquigarrow$ and a binary relation $\leq$ on Y_2 by

$$(x_1, y_1) \rightarrow (x_2, y_2) = (x_2 - x_1, (y_2 - y_1)e^{-x_1}),$$
$$(x_1, y_1) \rightsquigarrow (x_2, y_2) = (x_2 - x_1, y_2 - y_1 e^{x_2 - x_1}),$$

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow (x_1, y_1) \rightarrow (x_2, y_2) = (0, 0) = (x_1, y_1) \rightsquigarrow (x_2, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in Y_2$. Then $(Y_2; \leq, \rightarrow, \rightsquigarrow, (0, 0))$ is a proper pseudo-BCI-algebra. Notice that Y_2 is not a pseudo-BCK-algebra because there exists $(x, y) = (1, 1) \in Y_2$ such that $(x, y) \nleq (0, 0)$.

Example 2.4 [6]**.** Let Y be the direct product of pseudo-BCI-algebras Y_1 and Y_2 from Examples 2.2 and 2.3, respectively. Then Y is a proper pseudo-BCI-algebra, where $Y = (-\infty, 0] \times \mathbb{R}^2$ and binary operations $\to$ and $\rightsquigarrow$ and binary relation $\leq$ are defined on Y by

$$(x_1, y_1, z_1) \to (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_1 \leq x_2, \\ (\frac{2x_2}{\pi} \arctan(\ln(\frac{x_2}{x_1})), y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \rightsquigarrow (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_1 \leq x_2, \\ (x_2 e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 = y_2 \text{ and } z_1 = z_2.$$

Notice that Y is not a pseudo-BCK-algebra because there exists $(x, y, z) = (0, 1, 1) \in Y$ such that $(x, y, z) \not\leq (0, 0, 0)$.

For any pseudo-BCI-algebra $(X; \to, \rightsquigarrow, 1)$, the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of X (called pseudo-BCK-part of X). Then $(K(X); \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra. Note that a pseudo-BCI-algebra X is a pseudo-BCK-algebra if and only if $X = K(X)$.

It is easily seen that for the pseudo-BCI-algebras X, Y_1, Y_2 and Y from Examples 2.1, 2.2, 2.3 and 2.4 we have $K(X) = \{a, b, c, 1\}$, $K(Y_1) = Y_1$, $K(Y_2) = \{(0, 0)\}$ and $K(Y) = \{(x, 0, 0) : x \leq 0\}$, respectively.

We will denote by $M(X)$ the set of all maximal elements of X and call it the p-semisimple part of X. Obviously, $1 \in M(X)$. Notice that $M(X) \cap K(X) = \{1\}$. Indeed, if $a \in M(X) \cap K(X)$, then $a \leq 1$ and a is a maximal element of X, which means that $a = 1$. Moreover, observe that 1 is the only maximal element of a pseudo-BCK-algebra. Therefore, for a pseudo-BCK-algebra X, $M(X) = \{1\}$. In [2] and [6] there is shown that $M(X) = \{x \in X : x = (x \to 1) \to 1\}$ and it is a subalgebra of X.

Observe that for the pseudo-BCI-algebras X, Y_1, Y_2 and Y from Examples 2.1, 2.2, 2.3 and 2.4 we have $M(X) = \{d, 1\}$, $M(Y_1) = \{0\}$, $M(Y_2) = Y_2$ and $M(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

Let X be a pseudo-BCI-algebra. For any $a \in X$, we define a subset $V(a)$ of X as follows

$$V(a) = \{x \in X : x \leq a\}.$$

Note that $V(a)$ is non-empty, because $a \leq a$ gives $a \in V(a)$. Notice also that $V(a) \subseteq V(b)$ for any $a, b \in X$ such that $a \leq b$.

If $a \in M(X)$, then the set $V(a)$ is called a *branch* of X determined by element a. The following facts are proved in [6]: (1) branches determined by different elements are disjoint, (2) a pseudo-BCI-algebra is a set-theoretic union of branches, (3) comparable elements are in the same branch.

The pseudo-BCI-algebra Y_1 from Example 2.2 has only one branch (as the pseudo-BCK-algebra) and the pseudo-BCI-algebra X from Example 2.1 has two branches: $V(d) = \{d\}$ and $V(1) = \{a, b, c, 1\}$. Every $\{(x, y)\}$ is a branch of the pseudo-BCI-algebra Y_2 from Example 2.3, where $(x, y) \in Y_2$. For the pseudo-BCI-algebra Y from Example 2.4, the sets $V((0, a_1, a_2)) = \{(x, a_1, a_2) \in Y : x \leq 0\}$, where $(0, a_1, a_2) \in M(X)$, are branches of Y.

Proposition 2.5 [2]. *Let X be a pseudo-BCI-algebra. For all $a, x, y \in X$, the following are equivalent:*

(i) $a \in M(X)$,

(ii) $(a \to x) \rightsquigarrow x = a = (a \rightsquigarrow x) \to x$.

Proposition 2.6 [2]. *Let X be a pseudo-BCI-algebra and let $x \in X$ and $a, b \in M(X)$. If $x \in V(a)$, then $x \to b = a \to b$ and $x \rightsquigarrow b = a \rightsquigarrow b$.*

Proposition 2.7 [2]. *Let X be a pseudo-BCI-algebra and let $x, y \in X$ and $a, b \in M(X)$. If $x \in V(a)$ and $y \in V(b)$, then $x \to y \in V(a \to b)$ and $x \rightsquigarrow y \in V(a \rightsquigarrow b)$.*

Proposition 2.8 [5]. *Let X be a pseudo-BCI-algebra and let $x, y \in X$. The following are equivalent:*

(i) *x and y belong to the same branch of X,*

(ii) $x \to y \in K(X)$,

(iii) $x \rightsquigarrow y \in K(X)$,

(iv) $x \to 1 = x \rightsquigarrow 1 = y \to 1 = y \rightsquigarrow 1$.

Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then X is *p-semisimple* if it satisfies for all $x \in X$,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if X is a p-semisimple pseudo-BCI-algebra, then $K(X) = \{1\}$. Hence, if X is a p-semisimple pseudo-BCK-algebra, then $X = \{1\}$. Moreover, as it is proved in [6], $M(X)$ is a p-semisimple pseudo-BCI-subalgebra of X and X is p-semisimple if and only if $X = M(X)$.

It is not difficult to see that the pseudo-BCI-algebras X, Y_1 and Y from Examples 2.1, 2.2 and 2.4, respectively, are not p-semisimple, and the pseudo-BCI-algebra Y_2 from Example 2.3 is a p-semisimple algebra.

Proposition 2.9 [6]. *Let X be a pseudo-BCI-algebra. Then, for all $a, b, x, y \in X$, the following are equivalent:*

(i) *X is p-semisimple,*

(ii) $(x \to y) \rightsquigarrow y = x = (x \rightsquigarrow y) \to y$,

(iii) $(x \to 1) \rightsquigarrow 1 = x = (x \rightsquigarrow 1) \to 1$,

(iv) $(x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$.

Proposition 2.10 [6]. *A pseudo-BCI-algebra $(X; \to, \rightsquigarrow, 1)$ is p-semisimple if and only if $(X; \cdot, ^{-1}, 1)$ is a group, where, for any $x, y \in X$, $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$, $x^{-1} = x \to 1 = x \rightsquigarrow 1$, $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.*

3. Deductive systems

We say that a subset D of a pseudo-BCI-algebra X is a *deductive system* of X if it satisfies: (i) $1 \in D$, (ii) for all $x, y \in X$, if $x \in D$ and $x \to y \in D$, then $y \in D$. Under this definition, $\{1\}$ and X are the simplest examples of deductive systems. Note that the condition (ii) can be replaced by (ii') for all $x, y \in X$, if $x \in D$ and $x \rightsquigarrow y \in D$, then $y \in D$. It can be easily proved that for any $x, y \in X$, if $x \in D$ and $x \leq y$, then $y \in D$.

A deductive system D of a pseudo-BCI-algebra X is called: (1) *proper* if $D \neq X$ and (2) *closed* if D is closed under operations $\to$ and $\rightsquigarrow$, that is, if D is a subalgebra of X. It is not difficult to show (see [2]) that a deductive system D of a pseudo-BCI-algebra X is closed if and only if for any $x \in D$, $x \to 1 = x \rightsquigarrow 1 \in D$. Obviously, the pseudo-BCK-part $K(X)$ is a closed deductive system of X.

A deductive system D of a pseudo-BCI-algebra X is called *compatible* if for all $x, y \in X$,

$$x \to y \in D \text{ iff } x \rightsquigarrow y \in D.$$

Further, if D is a compatible deductive system of X, then the relation θ_D defined by

$$(x, y) \in \theta_D \quad \text{iff} \quad x \to y \in D \text{ and } y \to x \in D \tag{1}$$

is a congruence, where $[1]_{\theta_D} \subseteq D$ is a closed compatible deductive system of X. Moreover, $[1]_{\theta_D} = D$ if and only if D is closed.

We say that $\theta \in \mathrm{Con}(X)$ is a *relative congruence* of X if the quotient algebra $(X/\theta; \to, \rightsquigarrow, [1]_\theta)$ is a pseudo-BCI-algebra. It is proved in [3] that relative congruences of X correspond one-to-one to closed compatible deductive systems of X, that is, every relative congruence of X is given by (1) for some closed compatible deductive system D. For every relative congruence θ_D, the quotient algebra

$(X/\theta_D; \to, \rightsquigarrow, [1]_{\theta_D})$ will be usually denoted by $(X/D; \to, \rightsquigarrow, [1]_D)$ and then we will write $[x]_D$ instead of $[x]_{\theta_D}$.

Remark. Although the set $M(X)$ is a subalgebra of a pseudo-BCI-algebra X, it does not have to be a deductive system of X. Let X be the pseudo-BCI-algebra from Example 2.1. It is easy to see that $M(X) = \{d, 1\}$ is not a deductive system of X.

From [2] we have the following.

Proposition 3.1. *Let X be a pseudo-BCI-algebra. The following are equivalent:*

(i) *$M(X)$ is a deductive system of X,*

(ii) *for all $x, y \in X$ and $a \in M(X)$, $a \to x = a \to y$ implies $x = y$,*

(iii) *for all $x, y \in X$ and $a \in M(X)$, $a \rightsquigarrow x = a \rightsquigarrow y$ implies $x = y$.*

Here we have the following theorem.

Theorem 3.2. *Let X be a pseudo-BCI-algebra. The following are equivalent:*

(i) *$M(X)$ is a deductive system of X,*

(ii) *$x = (a \to 1) \to (a \to x)$ for all $x \in X$ and $a \in M(X)$,*

(iii) *$x = (a \rightsquigarrow 1) \rightsquigarrow (a \rightsquigarrow x)$ for all $x \in X$ and $a \in M(X)$.*

Proof. (i) $\Rightarrow$ (ii): Assume that $M(X)$ is a deductive system of X. Let $x \in X$ and $a \in M(X)$. Then, by (b4) and Proposition 2.5, we have

$$a \to (((a \to 1) \to (a \to x)) \rightsquigarrow x) = ((a \to 1) \to (a \to x)) \rightsquigarrow (a \to x) = a \to 1.$$

Hence, by Proposition 3.1, $((a \to 1) \to (a \to x)) \rightsquigarrow x = 1$, that is,

$$(a \to 1) \to (a \to x) \leq x.$$

On the other hand, again by (b4),

$$\begin{aligned} x \rightsquigarrow ((a \to 1) \to (a \to x)) &= (a \to 1) \to (a \to (x \rightsquigarrow x)) \\ &= (a \to 1) \to (a \to 1) \\ &= 1, \end{aligned}$$

that is,

$$x \leq (a \to 1) \to (a \to x).$$

Hence, $x = (a \to 1) \to (a \to x)$ and (ii) is satisfied.

(ii) $\Rightarrow$ (i): Assume that (ii) is satisfied. We use Proposition 3.1. Let $x, y \in X$ and $a \in M(X)$. Suppose that $a \to x = a \to y$. Then, by (ii), we get

$$x = (a \to 1) \to (a \to x) = (a \to 1) \to (a \to y) = y.$$

Therefore, by Proposition 3.1, $M(X)$ is a deductive system of X.

(i) $\Leftrightarrow$ (iii): Analogous. ∎

Remark. From [3] we know that $K(X)$ is a closed compatible deductive system of a pseudo-BCI-algebra X and $X/K(X) \cong M(X)$. We also have the following proposition.

Proposition 3.3. *Let X be a pseudo-BCI-algebra. If $M(X)$ is a compatible deductive system of X, then $X/M(X) \cong K(X)$. Moreover, $[x]_{M(X)} \neq [y]_{M(X)}$ for all $x, y \in V(a)$ such that $x \neq y$, where $a \in M(X)$.*

Proof. Since $M(X)$ is a (closed) compatible deductive system of X, we have $X/M(X)$ is a pseudo-BCI-algebra. Define a function $f : K(X) \to X/M(X)$ as follows:

$$f(x) = [x]_{M(X)} \text{ for all } x \in K(X).$$

Since

$$f(x \to y) = [x \to y]_{M(X)} = [x]_{M(X)} \to [y]_{M(X)} = f(x) \to f(y)$$

and

$$f(x \rightsquigarrow y) = [x \rightsquigarrow y]_{M(X)} = [x]_{M(X)} \rightsquigarrow [y]_{M(X)} = f(x) \rightsquigarrow f(y),$$

so f is a homomorphism. Take $x, y \in K(X)$ such that $x \neq y$. Then $x \to y \neq 1$ or $y \to x \neq 1$. Hence, $x \to y \notin M(X)$ or $y \to x \notin M(X)$, that is, $[x]_{M(X)} \neq [y]_{M(X)}$. Thus, f is injective. Now, take $x \in X$ and $a = (x \to 1) \to 1$. Then $a \in M(X)$ and $x \in V(a)$. Hence, by Proposition 2.8, $a \to x \in K(X)$. Thus, since $[a]_{M(X)} = [1]_{M(X)}$, we have

$$\begin{aligned} f(a \to x) &= [a \to x]_{M(X)} \\ &= [a]_{M(X)} \to [x]_{M(X)} \\ &= [1]_{M(X)} \to [x]_{M(X)} \\ &= [x]_{M(X)}. \end{aligned}$$

Hence f is also surjective. Therefore f is an isomorphism, that is, $X/M(X) \cong K(X)$.

Finally, let $a \in M(X)$, $x, y \in V(a)$ and $x \neq y$. Then, $x \to y, y \to x \in K(X)$. If $[x]_{M(X)} = [y]_{M(X)}$, then $x \to y \in M(X)$ and $y \to x \in M(X)$. Hence, $x \to y = 1$ and $y \to x = 1$, that is, $x = y$ and we get a contradiction. Thus, $[x]_{M(X)} \neq [y]_{M(X)}$. ∎

Theorem 3.4. *Let X be a pseudo-BCI-algebra. Then $X \cong K(X) \times M(X)$ if and only if $M(X)$ is a compatible deductive system of X.*

Proof. Assume that $X \cong K(X) \times M(X)$. Let $f : X \to K(X) \times M(X)$ be an isomorphism. Let π_K and π_M be projection maps onto $K(X)$ and $M(X)$, respectively. Denote

$$f_K = \pi_K \circ f : X \to K(X)$$

and

$$f_M = \pi_M \circ f : X \to M(X).$$

Obviously, f_K and f_M are both homomorphisms. The following are easy to show:

(i) $f(x) = (f_K(x), f_M(x))$ for all $x \in X$,

(ii) $f_K(x) = 1$ for all $x \in M(X)$,

(iii) $f_M(x) = 1$ for all $x \in K(X)$,

(iv) if x and y are in the same branch $V(a)$, then $f_M(x) = f_M(y) = a$,

(v) if x and y are in the same branch and $x \neq y$, then $f_K(x) \neq f_K(y)$.

Now, by (ii) and (v), it follows that $M(X) = \ker(f_K)$, that is, it is a (closed) compatible deductive system of X.

Conversely, assume that $M(X)$ is a compatible deductive system of X. Obviously, it is closed. Hence $X/M(X)$ is a pseudo-BCI-algebra. From Proposition 3.3, we know that $X/M(X) \cong K(X)$. Since also $X/K(X) \cong M(X)$, it suffices to show that $X \cong X/M(X) \times X/K(X)$. Define a function $g : X \to X/M(X) \times X/K(X)$ as follows:

$$g(x) = ([x]_{M(X)}, [x]_{K(X)}) \text{ for all } x \in X.$$

Obviously, g is a homomorphism. First, we prove that it is injective. Let $x, y \in X$ and $g(x) = g(y)$. Then, $([x]_{M(X)}, [x]_{K(X)}) = ([y]_{M(X)}, [y]_{K(X)})$, whence $[x]_{M(X)} = [y]_{M(X)}$ and $[x]_{K(X)} = [y]_{K(X)}$. Hence, $x \to y, y \to x \in M(X)$ and $x \to y, y \to x \in K(X)$. These are possible only in case $x \to y = y \to x = 1$. Thus, $x = y$ and g is injective.

Next, we show that g is surjective. Let $([x]_{M(X)}, [y]_{K(X)}) \in X/M(X) \times X/K(X)$. Denote $a = (x \to 1) \to 1$ and $b = (y \to 1) \to 1$. Then, $a, b \in M(X)$. Since $(a \to x) \rightsquigarrow x = a \in M(X)$ and $x \rightsquigarrow (a \to x) = a \to 1 \in M(X)$, we have $[x]_{M(X)} = [a \to x]_{M(X)}$. Moreover, since $y \in V(b)$, by Proposition 2.8, $b \to y, y \to b \in K(X)$. Hence, $[y]_{K(X)} = [b]_{K(X)}$. Thus,

$$([x]_{M(X)}, [y]_{K(X)}) = ([a \to x]_{M(X)}, [b]_{K(X)}).$$

Let $z = (b \to 1) \to (a \to x)$. We have $a \to x \in K(X)$ by Proposition 2.8, and $z \in V((b \to 1) \to 1) = V(b)$ by Proposition 2.7, whence $[z]_{K(X)} = [b]_{K(X)}$. Moreover, by (b4) and Proposition 2.5, we have

$$(a \to x) \rightsquigarrow z = (a \to x) \rightsquigarrow ((b \to 1) \to (a \to x)) = (b \to 1) \to 1 = b \in M(X)$$

and

$$z \rightsquigarrow (a \to x) = ((b \to 1) \to (a \to x)) \rightsquigarrow (a \to x) = b \to 1 \in M(X).$$

Hence, $[z]_{M(X)} = [a \to x]_{M(X)}$. Thus,

$$g(z) = ([z]_{M(X)}, [z]_{K(X)}) = ([a \to x]_{M(X)}, [b]_{K(X)}) = ([x]_{M(X)}, [y]_{K(X)}).$$

Hence, g is surjective.

Therefore, g is an isomorphism, that is, $X \cong X/M(X) \times X/K(X) \cong K(X) \times M(X)$. ∎

Remark. It is easy to see that for the pseudo-BCI-algebra X from Example 2.1, $M(X)$ is not a (compatible) deductive system of X and $X \ncong K(X) \times M(X)$, and for the pseudo-BCI-algebra Y from Example 2.4, $M(Y)$ is a compatible deductive system of Y and $Y \cong K(Y) \times M(Y)$.

For any non-empty subset A of a pseudo-BCI-algebra X, denote

$$M(A) = \{(x \to 1) \to 1 : x \in A\}.$$

Obviously, $M(A) \subseteq M(X)$ and $A \cap M(X) \subseteq M(A)$.

Proposition 3.5. *Let X be a pseudo-BCI-algebra and D be a deductive system of X. Then*

(i) *$M(D) = D \cap M(X)$,*

(ii) *$M(D)$ is a deductive system of $M(X)$.*

Proof. (i) It suffices to prove that $M(D) \subseteq D \cap M(X)$. Let $x \in D$. Then, by (a2), $(x \to 1) \to 1 \in D$. Thus, $(x \to 1) \to 1 \in D \cap M(X)$, that is, $M(D) = D \cap M(X)$.

(ii) Obviously, $1 \in M(D)$. Let $x, y \in M(X)$ be such that $x \in M(D)$ and $x \to y \in M(D)$. Then, we have $x, x \to y \in D$ and $x, x \to y \in M(X)$. Hence, since D is a deductive system of X, $y \in D \cap M(X) = M(D)$. Therefore, $M(D)$ is a deductive system of X. ∎

Remark. If $M(D)$ is a deductive system of $M(X)$, then D does not have to be a deductive system of X. Let X be the pseudo-BCI-algebra from Example 2.1. Then, for $D = \{a, 1\}$, $M(D) = \{1\}$ is a deductive system of $M(X)$, but D is not a deductive system of X.

From [2] we have the following fact.

Proposition 3.6. *Let X be a pseudo-BCI-algebra. If D is a subalgebra of X, then $M(D)$ is a closed deductive system of $M(X)$.*

Remark. If $M(D)$ is a closed deductive system of $M(X)$, then D does not have to be a subalgebra of X. Let X be the pseudo-BCI-algebra from Example 2.1. Then for $D = \{a, b, 1\}$ we have $M(D) = \{1\}$ is a closed deductive system of $M(X)$, but D is not a subalgebra of X.

Proposition 3.7. *Let X be a pseudo-BCI-algebra. A deductive system D of X is closed if and only if a deductive system $M(D)$ is closed in $M(X)$.*

Proof. By Proposition 3.6, it suffices to prove that if $M(D)$ is closed in $M(X)$, then D is closed in X. Assume that a deductive system $M(D)$ is closed in $M(X)$. Let $x \in D$. Then, $(x \to 1) \to 1 \in M(D)$. Hence, using (b9) and (b12), $x \to 1 = ((x \to 1) \to 1) \to 1 \in M(D)$. By Proposition 3.5(i), $x \to 1 \in D$, that is, D is closed. ∎

Moreover, it is not difficult to show the following.

Proposition 3.8. *Let X be a pseudo-BCI-algebra. If D is a compatible deductive system of X, then $M(D)$ is a compatible deductive system of $M(X)$.*

Remark. The converse of Proposition 3.8 does not hold. Let X be the pseudo-BCI-algebra from Example 2.1. Then, for $D = \{c, 1\}$, $M(D) = \{1\}$ is a compatible deductive system of $M(X)$, but D is a deductive system of X which is not compatible.

Theorem 3.9. *Let X be a pseudo-BCI-algebra and $A \subseteq M(X)$. Then, $D = \bigcup_{a\in A} V(a)$ is a subalgebra of X if and only if A is a subalgebra of $M(X)$.*

Proof. Assume that D is a subalgebra of X. Let $a, b \in A$. Then, $V(a), V(b) \subseteq D$, that is, $a \to b, a \rightsquigarrow b \in D$. Since $a \to b$ and $a \rightsquigarrow b$ are maximal elements of X, $V(a \to b), V(a \rightsquigarrow b) \subseteq D$, that is, $a \to b, a \rightsquigarrow b \in A$. Therefore, A is a subalgebra of $M(X)$.

Conversely, assume that A is a subalgebra of $M(X)$. Let $x, y \in D$. Then, there are $a, b \in A$ such that $x \in V(a)$, $y \in V(b)$ and $a \to b, a \rightsquigarrow b \in A$. By Proposition 2.7, $x \to y \in V(a \to b) \subseteq D$ and $x \rightsquigarrow y \in V(a \rightsquigarrow b) \subseteq D$. Therefore, D is a subalgebra of X. ∎

Theorem 3.10. *Let X be a pseudo-BCI-algebra and $A \subseteq M(X)$. Then, $D = \bigcup_{a\in A} V(a)$ is a deductive system of X if and only if A is a deductive system of $M(X)$.*

Proof. If D is a deductive system of X, then by Proposition 3.5, $M(D) = D \cap M(X) = A$ is a deductive system of $M(X)$.

Conversely, assume that A is a deductive system of $M(X)$. Obviously, $1 \in D$. Let $x, y \in X$ be such that $x, x \to y \in D$. Denote $a = (x \to 1) \to 1$, $b = ((x \to y) \to 1) \to 1$ and $c = (y \to 1) \to 1$. It is clear that $a, b, c \in M(X)$, $x \in V(a)$, $x \to y \in V(b)$ and $y \in V(c)$. Moreover, by Proposition 2.7, $x \to y \in V(a \to c)$. Hence, $a \to c = b \in A$. Since $a \in A$ and A is a deductive system of $M(X)$, we get $c \in A$. Thus, $y \in V(c) \subseteq D$. Therefore, D is a deductive system of X. ■

The following fact is proved in [2].

Proposition 3.11. *Let X be a p-semisimple pseudo-BCI-algebra and $D \subseteq X$. Then, D is a closed deductive system of X if and only if it is a subalgebra of X.*

Theorem 3.12. *Let $(X; \to, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra. Then the following are equivalent:*

(i) *D is a closed deductive system of $(X; \to, \rightsquigarrow, 1)$,*

(ii) *D is a subalgebra of $(X; \to, \rightsquigarrow, 1)$,*

(iii) *D is a subgroup of $(X; \cdot, ^{-1}, 1)$.*

Proof. (i)⇔(ii): Follows by Proposition 3.11.

(i)⇒(iii): Assume that D is a closed deductive system. Let $x, y \in D$. Then, $x \rightsquigarrow (x \cdot y) = x^{-1} \cdot (x \cdot y) = y \in D$. Hence, $x \cdot y \in D$. Moreover, since D is closed, $x^{-1} = x \to 1 \in D$ for any $x \in D$. Thus, D is a subgroup.

(iii)⇒(i): Assume that D is a subgroup. Obviously, $1 \in D$. Let $x, x \to y \in D$. Then, $y \cdot x^{-1} \in D$ and so $y = (y \cdot x^{-1}) \cdot x \in D$. Thus, D is a deductive system. Moreover, $x \to 1 = x^{-1} \in D$ for any $x \in X$, that is, D is closed. ■

Moreover, we have the following simple proposition.

Proposition 3.13. *Let $(X; \to, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra. The following are equivalent:*

(i) *D is a closed compatible deductive system of $(X; \to, \rightsquigarrow, 1)$,*

(ii) *D is a normal subgroup of $(X; \cdot, ^{-1}, 1)$.*

Combining Theorems 3.9, 3.10 and 3.12 we have the following theorem.

Theorem 3.14. *Let $(X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra, $A \subseteq M(X)$ and $D = \bigcup_{a \in A} V(a)$. Then the following are equivalent:*

(i) *D is a subalgebra of $(X; \to, \rightsquigarrow, 1)$,*

(ii) *A is a subalgebra of $(M(X); \to, \rightsquigarrow, 1)$,*

(iii) *A is a subgroup of* $(M(X); \cdot, ^{-1}, 1)$

(iv) *A is a closed deductive system of* $(M(X); \rightarrow, \rightsquigarrow, 1)$,

(v) *D is a closed deductive system of* $(X; \rightarrow, \rightsquigarrow, 1)$.

4. Periodic part

Troughout this section, we recall some facts from [4] needed in the sequel. Let X be a pseudo-BCI-algebra. Define

$$x \rightarrow^0 y = y,$$
$$x \rightarrow^n y = x \rightarrow (x \rightarrow^{n-1} y),$$

where $x, y \in X$ and $n = 1, 2, \ldots$. Similarly, we define $x \rightsquigarrow^n y$ for any $n = 0, 1, 2, \ldots$.

Proposition 4.1 [4]**.** *Let X be a pseudo-BCI-algebra. The following hold for any $x, y, z \in X$ and $m, n = 0, 1, 2, \ldots$:*

(i) $x \rightarrow^n 1 = x \rightsquigarrow^n 1$,

(ii) $x \rightarrow^n x = x \rightarrow^{n-1} 1, \quad x \rightsquigarrow^n x = x \rightsquigarrow^{n-1} 1$,

(iii) $(x \rightarrow 1) \rightarrow^n 1 = (x \rightarrow^n 1) \rightarrow 1, \quad (x \rightsquigarrow 1) \rightsquigarrow^n 1 = (x \rightsquigarrow^n 1) \rightsquigarrow 1$,

(iv) $x \rightarrow (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \rightarrow z), \quad x \rightsquigarrow (y \rightarrow^n z) = y \rightarrow^n (x \rightsquigarrow z)$,

(v) $x \rightarrow^m (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \rightarrow^m z)$,

(vi) $x \rightarrow^n 1 = ((x \rightarrow 1) \rightarrow 1) \rightarrow^n 1, \quad x \rightsquigarrow^n 1 = ((x \rightsquigarrow 1) \rightsquigarrow 1) \rightsquigarrow^n 1$.

Lemma 4.2. *Let X be a pseudo-BCI-algebra. The following hold for any $x, y \in X$:*

(i) $x \rightarrow^{m+n} y = x \rightarrow^m (x \rightarrow^n y)$ *for any* $m, n = 0, 1, 2, \ldots$,

(ii) $x \rightarrow^{mn} y = x \rightarrow^m (\ldots \rightarrow^m (x \rightarrow^m y) \ldots)$ (n times) *for any* $m = 0, 1, 2, \ldots$ *and* $n = 1, 2, \ldots$.

Proof. Routine. ■

Lemma 4.3. *Let X be a pseudo-BCI-algebra. The following hold for any $x \in X$ and $m, n = 0, 1, 2, \ldots$:*

(i) $(x \to^m 1) \to^n 1 = (x \to 1) \to^{mn} 1$,

(ii) $(x \rightsquigarrow^m 1) \rightsquigarrow^n 1 = (x \rightsquigarrow 1) \rightsquigarrow^{mn} 1$.

Proof. (i) We prove it by induction under n. For $n = 0$ it is obvious. Assume it for $n = k$:

$$(x \to^m 1) \to^k 1 = (x \to 1) \to^{mk} 1.$$

We have, by assumption and Proposition 4.1(i,iii,v),

$$\begin{aligned}
(x \to^m 1) \to^{k+1} 1 &= (x \to^m 1) \to ((x \to^m 1) \to^k 1) \\
&= (x \to^m 1) \to ((x \to 1) \to^{mk} 1) \\
&= (x \to^m 1) \to ((x \to 1) \rightsquigarrow^{mk} 1) \\
&= (x \to 1) \rightsquigarrow^{mk} ((x \to^m 1) \to 1) \\
&= (x \to 1) \rightsquigarrow^{mk} ((x \to 1) \to^m 1) \\
&= (x \to 1) \to^m ((x \to 1) \rightsquigarrow^{mk} 1) \\
&= (x \to 1) \to^m ((x \to 1) \to^{mk} 1) \\
&= (x \to 1) \to^{m(k+1)} 1.
\end{aligned}$$

So, the equation (i) holds for any $n = 0, 1, 2, \ldots$.

(ii) Follows from (i) and Proposition 4.1(i). ■

Let X be a pseudo-BCI-algebra. For any $x \in X$, if there exists the least natural number n such that $x \to^n 1 = 1$, then n is called a *period* of x denoted $p(x)$. If, for any natural number n, $x \to^n 1 \neq 1$, then a period of x is called to be infinite and denoted $p(x) = \infty$. Obviously, $p(1) = 1$.

Proposition 4.4 [4]**.** *Let X be a pseudo-BCI-algebra. Then the following hold for any $x, y \in X$,*

(i) $p(x) = p(x \to 1) = p(x \rightsquigarrow 1)$,

(ii) *if* $x \leq y$, *then* $p(x) = p(y)$,

(iii) $p(x \to y) = p(y \to x)$, $\quad p(x \rightsquigarrow y) = p(y \rightsquigarrow x)$,

(iv) $p(x \to y) = p(x \rightsquigarrow y)$.

Proposition 4.5 [4]**.** *Let $(X; \to, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra and $(X; \cdot, ^{-1}, 1)$ be a group related with X. Then $p(x) = o(x)$ for any $x \in X$, where $o(x)$ means an order of an element x in a group $(X; \cdot, ^{-1}, 1)$.*

Proposition 4.6 [4]**.** *Let X be a pseudo-BCI-algebra. Then*

(i) *X is a pseudo-BCK-algebra if and only if $p(x) = 1$ for any $x \in X$,*

(ii) *X is p-semisimple if and only if $p(x) > 1$ for any $x \in X \backslash \{1\}$.*

Proposition 4.7. *Let X be a pseudo-BCI-algebra and $a \in M(X)$. If $x \in V(a)$, then $p(x) = p(a)$.*

Proof. Assume that $a \in M(X)$ and $x \in V(a)$. By Proposition 2.6, $x \to 1 = a \to 1$. Hence, by Proposition 4.4(i), we have $p(x) = p(x \to 1) = p(a \to 1) = p(a)$. ■

Corollary 4.8. *In any pseudo-BCI-algebra, all elements in the same branch have the same period.*

Remark. By Proposition 4.7 we can reduce the study of periods of elements of a pseudo-BCI-algebra to the study of periods of maximal elements.

Proposition 4.9. *Let X be a pseudo-BCI-algebra. If x and y are in the same branch, then $p(x \to y) = p(x \rightsquigarrow y) = 1$.*

Proof. Follows from Propositions 4.4(iv) and 2.8. ■

Proposition 4.10. *Let X be a pseudo-BCI-algebra, $x \in X$, $m, n \in \mathbb{N}$ and $p(x) = m$. Then, $x \to^n 1 = 1$ if and only if $m|n$.*

Proof. Let $x \in X$ and $p(x) = m$ for some $m \in \mathbb{N}$. Assume that $x \to^n 1 = 1$ for some $n \in \mathbb{N}$. Suppose that $n = mp + r$, for some $p, r \in \mathbb{N}$ and $1 \leq r < m$. Then, by Lemma 4.2,

$$\begin{aligned} 1 &= x \to^n 1 = x \to^{mp+r} 1 = x \to^r (x \to^{mp} 1) \\ &= x \to^r (x \to^m (\ldots \to^m (x \to^m 1) \ldots)) \ (p \text{ times}) \\ &= x \to^r 1. \end{aligned}$$

But, $r < m = p(x)$ which is impossible. Therefore, $m|n$.

Conversely, assume that $m|n$, that is, $n = mp$ for some $p \in \mathbb{N}$. Then, by Lemma 4.2(ii), we get $x \to^n 1 = x \to^{mp} 1 = x \to^m (\ldots \to^m (x \to^m 1) \ldots)$ (p times) $= 1$. ■

Let X be a pseudo-BCI-algebra. The set

$$P(X) = \{x \in X : p(x) < \infty\}$$

is called a *periodic part* of X. Moreover, denote

$$P_M(X) = \{x \in M(X) : p(x) < \infty\}.$$

Obviously, $P_M(X) \subseteq P(X)$.

Proposition 4.11. *Let X be a pseudo-BCI-algebra. Then the following hold:*

(i) $K(X) \subseteq P(X)$,

(ii) $P(X) = \bigcup_{a \in P_M(X)} V(a)$.

Proof. (i) Obvious.

(ii) Follows from Proposition 4.7. ■

Remark. Note that for the pseudo-BCI-algebra X from Example 2.1, $P_M(X) = M(X)$ and $P(X) = X$, and for the pseudo-BCI-algebra Y from Example 2.4, $P_M(Y) = \{(0,0,0)\}$ and $P(Y) = K(Y)$.

Remark. It is well known that a torsion part of a non-abelian group does not have to be a subgroup. Hence, by Theorem 3.12 and Proposition 4.5, $P_M(X)$ does not have to be a closed deductive system of a p-semisimple pseudo-BCI-algebra X. Thus, by Theorem 3.14 and Proposition 4.11(ii), $P(X)$ does not have to be a closed deductive system of a pseudo-BCI-algebra X.

The following facts follow from Theorem 3.14 and Propositions 4.4(iv) and 4.11.

Proposition 4.12. *Let X be a pseudo-BCI-algebra and let $P_M(X)$ be a subalgebra of $M(X)$. Then*

(i) *$P_M(X)$ is a closed compatible deductive system of $M(X)$,*

(ii) *$P(X)$ is a closed compatible deductive system of X.*

Let X be a pseudo-BCI-algebra. Denote by $D(a)$ a deductive system of X generated by $\{a\}$, where $a \in X$. From [3] we know that, for any $a \in X$,

$$\begin{aligned} D(a) &= \{1\} \cup \{x \in X : a \to^n x = 1 \text{ for some } n \in \mathbb{N}\} \\ &= \{1\} \cup \{x \in X : a \rightsquigarrow^n x = 1 \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$

Proposition 4.13. *Let X be a pseudo-BCI-algebra. Then a deductive system $D(a)$ is closed for any $a \in P(X)$.*

Proof. If $a \in P(X)$, then there exists $k \in \mathbb{N}$ such that $p(a) = k$. Hence, $a \to^k 1 = 1 \in D(a)$. Moreover, it is not difficult to show that also $a \to 1, a \to^2 1, \ldots, a \to^{k-1} 1 \in D(a)$. Now, let $x \in D(a)$. We show that $x \to 1 \in D(a)$, that is, $D(a)$ is closed. If $x = 1$, then the thesis is obvious. Assume that $x \neq 1$. Then there exists $n \in \mathbb{N}$ such that $a \to^n x = 1$. Thus, by (b12) and Proposition 4.1(iv),

$$x \to 1 = x \rightsquigarrow 1 = x \rightsquigarrow (a \to^n x) = a \to^n (x \rightsquigarrow x) = a \to^n 1.$$

Further, remark that there is $l \in \mathbb{N}$ such that $0 \leq l \leq k-1$ and $n = kp + l$ for some $p \in \mathbb{N}$. Hence, by Lemma 4.2 and the equation $a \to^k 1 = 1$ we get

$$\begin{aligned} a \to^n 1 &= a \to^{kp+l} 1 = a \to^l (a \to^{kp} 1) \\ &= a \to^l (a \to^k (\ldots \to^k (a \to^k 1)\ldots)) \ (p \text{ times}) \\ &= a \to^l 1. \end{aligned}$$

Thus,

$$x \to 1 = a \to^n 1 = a \to^l 1 \in D(a).$$

Therefore, a deductive system $D(a)$ is closed. ■

A pseudo-BCI-algebra X is called: (1) *periodic* if $P(X) = X$, and (2) *aperiodic* if $p(x) = \infty$ for any $x \notin K(X)$. Obviously, every pseudo-BCK-algebra is periodic as well as aperiodic.

Remark. It is not difficult to see that the pseudo-BCI-algebra X from Example 2.1 is periodic, and the pseudo-BCI-algebra Y from Example 2.4 is aperiodic.

Theorem 4.14. *Let X be a pseudo-BCI-algebra and let $P_M(X)$ be a subalgebra of $M(X)$. Then $X/P(X)$ is an aperiodic p-semisimple pseudo-BCI-algebra and $X/P(X) \cong M(X)/P_M(X)$.*

Proof. Note that by Proposition 4.12, $P_M(X)$ is a closed compatible deductive system of $M(X)$ and $P(X)$ is a closed compatible deductive system of X. Hence, $M(X)/P_M(X)$ and $X/P(X)$ are both pseudo-BCI-algebras.

First, we show that $M(X)/P_M(X)$ is p-semisimple. We will denote by $[x]^M_{P_M(X)}$ for $x \in M(X)$ elements of $M(X)/P_M(X)$. Assume that $[x]^M_{P_M(X)} \to [1]^M_{P_M(X)} = [1]^M_{P_M(X)}$ for some $x \in M(X)$. Then, $[x \to 1]^M_{P_M(X)} = [1]^M_{P_M(X)}$, that is, $x \to 1 \in P_M(X)$. Hence, by Proposition 4.4(i), $p(x) = p(x \to 1) < \infty$, that is, $x \in P_M(X)$. Thus, $[x]^M_{P_M(X)} = [1]^M_{P_M(X)}$. Therefore, a pseudo-BCI-algebra $M(X)/P_M(X)$ is p-semisimple.

Next, we show that $X/P(X)$ and $M(X)/P_M(X)$ are isomorphic. Define a function $f : X/P(X) \to M(X)/P_M(X)$ as follows:

$$f([x]_{P(X)}) = [(x \to 1) \to 1]^M_{P_M(X)}$$

for any $x \in X$. Obviously, f is well-defined. We show that it is an isomorphism. Let $x, y \in X$. By (b11) and (b12), we have

$$\begin{aligned}f([x]_{P(X)} \to [y]_{P(X)}) &= f([x \to y]_{P(X)})\\ &= [((x \to y) \to 1) \to 1]^M_{P_M(X)}\\ &= [((x \to 1) \rightsquigarrow (y \to 1)) \to 1]^M_{P_M(X)}\\ &= [((x \to 1) \to 1) \to ((y \to 1) \to 1)]^M_{P_M(X)}\\ &= [(x \to 1) \to 1]^M_{P_M(X)} \to [(y \to 1) \to 1]^M_{P_M(X)}\\ &= f([x]_{P(X)}) \to f([y]_{P(X)}).\end{aligned}$$

Similarly, $f([x]_{P(X)} \rightsquigarrow [y]_{P(X)}) = f([x]_{P(X)}) \rightsquigarrow f([y]_{P(X)})$. Hence, f is a homomorphism. Moreover, since $M(X)/P_M(X)$ is p-semisimple, it is easy to see that f is surjective. Now, let $x, y \in X$ be such that $[x]_{P(X)} \neq [y]_{P(X)}$. Then, $x \to y \notin P(X)$ or $y \to x \notin P(X)$. Assume, for example, that $x \to y \notin P(X)$. Proof of the case $y \to x \notin P(X)$ is analogous. Since $x \to y \notin P(X)$, by (b11), (b12) and Proposition 4.4, we have

$$((x \to 1) \to 1) \to ((y \to 1) \to 1) = ((x \to y) \to 1) \to 1 \notin P_M(X),$$

that is,

$$f([x]_{P(X)}) = [(x \to 1) \to 1]^M_{P_M(X)} \neq [(y \to 1) \to 1]^M_{P_M(X)} = f([y]_{P(X)}).$$

Hence, f is injective and so an isomorphism. Thus, we immediately have that $X/P(X)$ is p-semisimple.

Finally, to prove that $X/P(X)$ is aperiodic, it is sufficient to prove that $M(X)/P_M(X)$ is aperiodic. Since $M(X)/P_M(X)$ is p-semisimple, we have to show that for any $x \in M(X)$, $[x]^M_{P_M(X)} \neq [1]^M_{P_M(X)}$ implies $p([x]^M_{P_M(X)}) = \infty$. Assume that there is $x \in M(X)$ such that $[x]^M_{P_M(X)} \neq [1]^M_{P_M(X)}$ and $p([x]^M_{P_M(X)}) = n$ for some $n \in \mathbb{N}$. Then,

$$[x \to^n 1]^M_{P_M(X)} = [x]^M_{P_M(X)} \to^n [1]^M_{P_M(X)} = [1]^M_{P_M(X)}.$$

Hence, $x \to^n 1 \in P_M(X)$, that is, there exists $m \in \mathbb{N}$ such that $p(x \to^n 1) = m$. Thus, $(x \to^n 1) \to^m 1 = 1$. Hence, by Lemma 4.3(i), $(x \to 1) \to^{mn} 1 = 1$, so $p(x \to 1) \leq mn$. By Proposition 4.4(i), $p(x) \leq mn$, whence $x \in P_M(X)$. Thus, $[x]^M_{P_M(X)} = [1]^M_{P_M(X)}$ and we get a contradiction. Therefore, $M(X)/P_M(X)$ is aperiodic, whence also $X/P(X)$ is aperiodic. ■

Example 4.15. Let Z be the set of all bijections $f : \mathbb{N} \to \mathbb{N}$. Define binary operations $\to$ and $\rightsquigarrow$ on Z by

$$\begin{aligned}f \to g &= g \circ f^{-1},\\ f \rightsquigarrow g &= f^{-1} \circ g\end{aligned}$$

for all $f, g \in Z$. Then the algebra $(Z; \rightarrow, \rightsquigarrow, id_{\mathbb{N}})$ is a p-semisimple pseudo-BCI-algebra which is neither periodic nor aperiodic. Moreover, it is not difficult to see that

$$P(Z) = P_M(Z) = \{f \in Z : \exists_{k \in \mathbb{N}} \ \forall_{n \geq k} \ f(n) = n\}$$

is a closed compatible deductive system of Z. Hence, by Theorem 4.14, $Z/P(Z)$ is an aperiodic p-semisimple pseudo-BCI-algebra.

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