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ON THE LENGTH OF RATIONAL CONTINUED FRACTIONS OVER $\mathbb{F}_q(X)$

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Abstract

Let $\mathbb{F}_q$ be a finite field and $A(Y) \in \mathbb{F}_q(X, Y)$. The aim of this paper is to prove that the length of the continued fraction expansion of $A(P); P \in \mathbb{F}_q[X]$, is bounded.

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1. Introduction

Let x be a rational number. It is known that we can write the continued fraction expansion of x in a unique finite way as follows:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}},$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}^*$, $\forall 1 \leq i \leq n-1$ and $a_n \geq 1$.

Denote $n = \Psi(x)$ the length of the continued fraction of x such that Ψ takes on values in $\mathbb{N}$. This function appears in several papers [6, 9, 10].

In 1971, Mendès France [8] asked in the real case, whether it was indeed true that

$$\lim_{n \to +\infty} \Psi\left(\left(\frac{a}{b}\right)^n\right) = +\infty?$$

without any other assumptions as $1 < b < a$, $\gcd(a, b) = 1$. This problem was solved by Pourchet in private letter to Mendès France in 1972 and by Choquet in a series of Comptes Rendus [3].

In 1997, this problem was solved by Grisel [2] on rational fractions with coefficients in a finite field and he investigates the behavior of the $\Psi((\frac{A}{B})^n)$ as n goes to infinity.

In 1972, Mendès France [7] studied the length Ψ of continued fractions of rational functions with rational coefficients and he proved the following theorem which is based on the Euclidean algorithm.

Theorem 1.1. *Let F be a rational function with rational coefficients ($F(x) \in \mathbb{Q}(x)$). The sequence $(\Psi(F(n)))$ is ultimately periodic.*

In this work, we are interested to generalize the last result for a special field with characteristic $p > 0$ which is the field of formal power series over the finite field $\mathbb{F}_q$ ($q = p^s; s \geq 1$). The present paper is organized as follows: In Section 2, we start by introducing the field of formal power series and the continued fraction expansion over this field. In Section 3, we state the main theorem where we prove that the length of the continued fraction expansion of $A(P); P \in \mathbb{F}_q[X]$ is bounded and we present some definitions and lemmas that we will use to prove our result and we close this section with the proof of our main theorem (see Theorem 3.1).

2. Field of formal power series $\mathbb{F}_q((X^{-1}))$

Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q[X]$ the ring of polynomials with coefficient in $\mathbb{F}_q$ and $\mathbb{F}_q(X)$ the field of rational functions. Let $\mathbb{F}_q((X^{-1}))$ be the field of formal power series

$$\mathbb{F}_q((X^{-1})) = \Big\{ f = \sum_{n \geq n_0} b_n X^{-n}; b_n \in \mathbb{F}_q; n_0 \in \mathbb{Z} \Big\}.$$

Define the absolute value

$$|f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases}$$

Where, $\deg f = \sup\{-n : b_n \neq 0\}$, for $f = \sum_{n \geq n_0} b_n X^{-n}$. Thus, $|\ .\ |$ is a not an archimedean absolute value over $\mathbb{F}_q((X^{-1}))$, that is:

$$\begin{aligned} |f+g| &\leq \max\,(|f|, |g|) \qquad \textit{and} \\ |f+g| &= \max\,(|f|, |g|) \qquad \textit{if } |f| \neq |g|. \end{aligned}$$

By analogy with the real case, we have a continued fraction algorithm in $\mathbb{F}_q((X^{-1}))$. A formal power series $f = \sum_{n\geq n_0} b_n X^{-n}$ has a unique decomposition as $f = [f] + \{f\}$ with $[f] \in \mathbb{F}_q[X]$ and $|\{f\}| < 1$. The polynomial $[f]$ is called the polynomial part of f and $\{f\}$ is called the fractional part of f. We can write for any $f \in \mathbb{F}_q((X^{-1}))$

$$f = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}} = [a_0, a_1, \ldots, a_n, \ldots],$$

where $a_0 = [f]$ and $a_i = [f_i] \in \mathbb{F}_q[X]$ with $\deg(a_i) \geq 1$ for any $i \geq 1$ and $f_i = \frac{1}{\{f_{i-1}\}}$. The sequence $(a_i)_{i\geq 0}$ is called the sequence of partial quotients of f and we denote by $f_n = [a_n, a_{n+1}, \ldots]$ the nth complete quotient of f.

Now, we define two sequences of polynomials $(P_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ as follows:

$$P_0 = a_0, \quad Q_0 = 1, \quad P_1 = a_0 a_1 + 1, \quad Q_1 = a_1$$

and

$$P_n = a_n P_{n-1} + P_{n-2}, \quad Q_n = a_n Q_{n-1} + Q_{n-2}, \text{ for any } n \geq 2.$$

We easily check that

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}, \text{ for any } n \geq 1$$

and

$$\frac{P_n}{Q_n} = [a_0, a_1, a_2....., a_n], \text{ for any } n \geq 0$$

$\dfrac{P_n}{Q_n}$ is called the n^{th}convergent of f and it satisfies the following:

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = f = [a_0, a_1, \ldots, a_n, \ldots].$$

Similar to the real case, we have the following two theorems:

Theorem 2.1. *Let $f \in \mathbb{F}_q((X^{-1}))$. The sequence of partial quotients of the continued fraction of f is finite if and only if $f \in \mathbb{F}_q(X)$.*

Theorem 2.2. *Let $f \in \mathbb{F}_q((X^{-1}))$. The sequence of partial quotients of the continued fraction of f is ultimately periodic (periodic from a certain rank) if and only if f is quadratic over $\mathbb{F}_q(X)$.*

For more information on the theory of continued fractions in the field of formal power series over a finite field, see [1, 4, 5, 11] et [12].

3. Results

We will start by giving the main theorem:

Theorem 3.1. *Let $F(Y) \in \mathbb{F}_q(X,Y)$ and*

$$\mathcal{A} = \{\Psi\left(F(P)\right);\ P \in \mathbb{F}_q[X]\}.$$

Then $\mathcal{A}$ is finite.

Before giving the proof of this theorem, we will introduce some definitions and lemmas that we will need.

Definition 3.2. Let $P(Y) = A_m Y^m + A_{m-1}Y^{m-1} + \cdots + A_0$ where $A_i \in \mathbb{F}_q[X]$ and A_m is not equal to zero. Then $\sigma(P) = A_m$.

Definition 3.3. Let $A(Y)$ and $B(Y)$ in $\mathbb{F}_q[X][Y]$.

$$A(Y) \prec B(Y) \text{ if and only if } \begin{cases} \deg A < \deg B, \\ \quad \text{or} \\ \deg A = \deg B \text{ and } |\sigma(A)| < |\sigma(B)|. \end{cases}$$

Lemma 3.4. *Let $A(Y)$ and $B(Y)$ in $\mathbb{F}_q[X][Y]$. If $A(Y) \prec B(Y)$ and $P \in \mathbb{F}_q[X]$ such that $|P|$ is sufficiently large, then $\left|\frac{A(P)}{B(P)}\right| < 1$.*

Proof. We denote by d (resp. s) the degree of $A(P)$ (resp. $B(P)$) such that $d \leq s$. We have:

$$A(P) = \sum_{i=0}^{d} a_i P^i;\ a_i \in \mathbb{F}_q[X] \text{ and } B(P) = \sum_{i=0}^{s} b_i P^i;\ b_i \in \mathbb{F}_q[X].$$

For $|P| > \sup_{0\leq i\leq d}(|a_i|)$, we have $|A(P)| = |a_d P^d|$, similarly, for $|P| > \sup_{0\leq i\leq s}(|b_i|)$, we have $|B(P)| = |b_s P^s|$.

Then, for $|P| > \sup\left(\sup_{0\leq i\leq d}(|a_i|), \sup_{0\leq i\leq s}(|b_i|)\right)$,

$$\left|\frac{A(P)}{B(P)}\right| = \frac{|a_d P^d|}{|b_s P^s|} = \left|\frac{a_d}{b_s}\right| P^{d-s} < 1.$$

■

Lemma 3.5. *Let $A(Y)$ and $B(Y)$ be two polynomials with coefficient in $\mathbb{F}_q[X]$. Then there exists a polynomial G in $\mathbb{F}_q[X]$ such that, for all $H \in \mathbb{F}_q[X]$; $|H| < |G|$, there exist $Q_1, R_1 \in \mathbb{F}_q[X,Y]$ with*

$$\begin{cases} A(GY+H) = B(GY+H)Q_1(Y) + R_1(Y), \\ R_1 \prec B. \end{cases}$$

Proof. By the Euclidean division of $A(Y)$ by $B(Y)$ in $\mathbb{F}_q(X,Y)$, we obtain

$$\begin{cases} A(Y) = B(Y)Q(Y) + R(Y), \\ Q \in \mathbb{F}_q(X)[Y], \\ R \in \mathbb{F}_q(X)[Y], \\ \deg R < \deg B. \end{cases}$$

Let $s = (\deg A - \deg B)$ then $Q(Y) = \sum_{i=0}^{s} \frac{a_i(X)}{b_i(X)} Y^i$. We denote by $G = \operatorname{lcm}(b_i)_{1\leq i\leq s}$. Thus, we obtain

$$\begin{aligned} A(GY+H) &= B(GY+H)Q(GY+H) + R(GY+H) \\ &= B(GY+H)\left(Q(GY+H) - \{Q(H)\}\right) + R(GY+H) \\ &\quad + B(GY+H)\{Q(H)\}. \end{aligned}$$

Hence

$$A(GY+H) = B(GY+H)Q_1(Y) + R_1(Y)$$

with

$$\begin{cases} Q_1(Y) = Q(GY+H) - \{Q(H)\} \in \mathbb{F}_q[X][Y], \\ R_1(Y) = R(GY+H) + B(GY+H)\{Q(H)\} \in \mathbb{F}_q[X][Y], \\ R_1 \prec B. \end{cases}$$

■

***Proof of Theorem* 3.1.** Put $F(Y) = \frac{A(Y)}{B(Y)}$ with $A(Y)$ and $B(Y)$ be two polynomials with coefficient in $\mathbb{F}_q[X]$ such that $B(Y) \prec A(Y)$. Then, from by Lemma 3.5, there exists a polynomial $G_1 \in \mathbb{F}_q[X]$ such that $|G_1| > 1$ and for all $H_1 \in \mathbb{F}_q[X]$; $|H_1| < |G_1|$, we have

$$\begin{cases} A(G_1Y+H_1) = B(G_1Y+H_1)Q_1(Y) + R_1(Y), \\ Q_1 \in \mathbb{F}_q[X][Y], \\ R_1 \in \mathbb{F}_q[X][Y], \\ R_1 \prec B. \end{cases}$$

We set $B_1(Y) = B(G_1Y+H_1)$, the same lemma shows that there exists a polynomial $G_2 \in \mathbb{F}_q[X]$ such that $|G_2| > 1$ and for all $|H_2| < |G_2|$, we obtain

$$\begin{cases} B_1(G_2Y+H_2) = R_1(G_2Y+H_2)Q_2(Y) + R_2(Y), \\ Q_2 \in \mathbb{F}_q[X][Y], \\ R_2 \in \mathbb{F}_q[X][Y], \\ R_2 \prec R_1. \end{cases}$$

This procedure is repeated infinitely many and then we obtain a sequence of polynomials in $\mathbb{F}_q[X][Y]$, $R_1(Y) \prec R_2(Y) \prec \cdots \prec R_s(Y) \prec \cdots$. It is clear that the sequence of the degree of R_n in Y is decreasing.

Let $G = \prod_{k=1}^{s} G_k$. For $|H| < |G|$, we have

$$\begin{aligned}
\frac{A(GY+H)}{B(GY+H)} &= \frac{A(G_1G_2\dots G_sY+H)}{B(G_1G_2\dots G_sY+H)}; \quad H = K_1G_1 + H_1 \text{ with } |H_1| < |G_1| \\
&= \frac{A(G_1(G_2\dots G_sY+K_1)+H_1)}{B(G_1(G_2\dots G_sY+K_1)+H_1)} \\
&= Q_1(Y) + \frac{R_1(G_2\dots G_sY+K_1)}{B_1(G_2\dots G_sY+K_1)} \\
&= Q_1(Y) + \frac{1}{\dfrac{B_1(G_2G_3\dots G_sY+K_2G_2+H_2)}{R_1(G_2G_3\dots G_sY+K_2G_2+H_2)}}; \quad |H_2| < |G_2| \\
&= Q_1(Y) + \frac{1}{\dfrac{B_1(G_2(G_3\dots G_sY+K_2)+H_2)}{R_1(G_2(G_3\dots G_sY+K_2)+H_2)}} \\
&= Q_1(Y) + \frac{1}{Q_2(Y) + \dfrac{R_2(G_3\dots G_sY+K_2)}{R_1(G_3\dots G_sY+K_2)}} \\
&\ \ \vdots \\
&= Q_1(Y) + \frac{1}{Q_2(Y) + \dfrac{1}{\ddots + \dfrac{1}{Q_s(Y)}}}.
\end{aligned}$$

with s depends on H and with $(Q_i(Y))_{1\le i\le s}$ are polynomials with coefficients in $\mathbb{F}_q[X]$ of degree not equal to 0.

Then

$$\frac{A(GY+H)}{B(GY+H)} = [\underbrace{Q_1(Y), Q_2(Y), \dots, Q_s(Y)}_{s}].$$

This implies that $\Psi\left(\frac{A(GY+H)}{B(GY+H)}\right)$ depends only on H.

Now, let $P \in F_q[X]$, then $P(X) = G(X)Y(X) + H(X)$ for a suitable polynomial $H(X)$ with $H \prec G$. Since there are only finitely many possibilities for H one concludes. ∎

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