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ON THE SUBSEMIGROUP GENERATED BY ORDERED IDEMPOTENTS OF A REGULAR SEMIGROUP

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Abstract

An element e of an ordered semigroup S is called an ordered idempotent if $e \leq e^2$. Here we characterize the subsemigroup $< E_{\leq}(S) >$ generated by the set of all ordered idempotents of a regular ordered semigroup S. If S is a regular ordered semigroup then $< E_{\leq}(S) >$ is also regular. If S is a regular ordered semigroup generated by its ordered idempotents then every ideal of S is generated as a subsemigroup by ordered idempotents.

Keywords: ordered regular, ordered inverse, ordered idempotent, downward closed, completely regular.

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1. Introduction

Idempotents play an important role in the theory of semigroups as well as in ring theory. Particularly, in case of different major subclasses of the regular semigroups S, the set $E(S)$ of all idempotents of S is like the nucleous of a cell, that possesses several properties of S in an encrypted form. Unfortunately, the set of all idempotents $E(S)$ of a semigroup (without order) does not form a subsemigroup in general. A regular semigroup S such that $E(S)$ is a subsemigroup is called orthodox. Hall [3] – [5], Meakin [10, 11], Yamada [13] and many others like Milles [12], McAlister [9] have studied orthodox semigroups and characterized such semigroup S by $E(S)$. Another approach, introduced by C. Eberhart, W. Williams and L. Kinch [2] is to study a semigroup S by the subsemigroup generated by $E(S)$. They considered the subsemigroup $< E >= \bigcup_{n=1}^{\infty} E^n$. (where $E = E(S)$ is not necessarily a subsemigroup of S) and established a connection between the regularity of S and $< E >$. T.E. Hall [5] studied subsemigroups of an idempotent generated regular semigroup. He showed that a regular semigroup is generated by its idempotents if and only if each principal factor is generated by its own idempotents.

In [1], we introduced the notion of ordered idempotents and characterized ordered semigroups S such that every element is an ordered idempotent. If T is a subsemigroup of S, then the set of ordered regular elements of T is denoted by $Reg_{\leq}(T)$ [7]. Thus, if $T =< E_{\leq}(S) >$ then $Reg_{\leq}(T) = T = Reg_{\leq}(S) \cap T$, in general. In [7], Hansda proved several equivalent conditions so that $Reg_{\leq}(T) = T = Reg_{\leq}(S) \cap T$ for $T = (Se], (eS]$ and $(eSf]$, where e, f are ordered idempotents. It is the purpose of this paper to characterize an ordered semigroup S by the subsemigroup generated by ordered idempotents of S. We show that in a regular ordered semigroup S the subsemigroup $< E_{\leq}(S) >$ generated by all ordered idempotents of S is also regular. Similar result holds for completely regular ordered semigroups.

The article is organized as follows. The basic definitions and properties of ordered semigroups are presented in Section 2. Section 3 is devoted to characterize the regular ordered semigroups generated by their ordered idempotents.

2. Preliminaries

In this paper $\mathbb{N}$ denotes the set of all natural numbers. An ordered semigroup S is a partially ordered set, and at the same time a semigroup $(S, \cdot)$ such that $(\forall a,\ b,\ x \in S)\ a \leq b \Rightarrow\ xa \leq xb$ and $ax \leq bx$. It is denoted by $(S, \cdot, \leq)$. For an ordered semigroup S and $\emptyset \neq H \subseteq S$, denote

$$(H] = \{t \in S : t \leq h,\ \text{for some } h \in H\}.$$

H is called downward closed if $H = (H]$.

Let I be a nonempty subset of an ordered semigroup S. I is called a left (right) ideal of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. I is an ideal of S if it is both a left and a right ideal of S. S is left (right) simple if it has no non-trivial proper left (right) ideal. Similarly we define simple ordered semigroups. S is called a t-simple ordered semigroup if it is both left and right simple.

An element $a \in S$ is called ordered regular if $a \leq axa$ for some $x \in S$. If every element of an ordered semigroup is ordered regular then S is called regular. Thus S is a regular ordered semigroup if and only if $a \in (aSa]$ for all $a \in S$. An element $b \in S$ is said to be an ordered inverse of a if $a \leq aba$ and $b \leq bab$. In a regular ordered semigroup every element has an ordered inverse. For, if $a \leq axa$ then $a \leq a(xax)a$ and $xax \leq (xax)a(xax)$ shows that xax is an ordered inverse of a. We denote the set of all ordered inverse of a by $V_{\leq}(a)$. If A is a nonempty subset of S, then we denote $\cup_{a \in A} V_{\leq}(a)$ by $V_{\leq}(A)$. An element $e \in S$ is defined to be an ordered idempotent if $e \leq e^2$ [1]. Ordered idempotents take a determining role in characterizing regular ordered semigroups [7], completely regular ordered semigroups, Clifford ordered semigroups [1], etc. If $a \leq axa$ then both ax and xa are ordered idempotents. The set of all ordered idempotents of S is denoted by $E_{\leq}(S)$. Kehayopulu [8] defined an ordered semigroups S to be completely regular if $a \in (a^2Sa^2]$ for all $a \in S$. Thus an ordered semigroup S is a completely regular ordered semigroup if and only if for every $a \in S$, $a \leq a^2xa^2$ for some $x \in S$.

3. Regular ordered semigroups generated by ordered idempotents

In this section we show that the subsemigroup generated by all ordered idempotents in a regular ordered semigroup S is always a regular ordered subsemigroup.

We denote the subsemigroup generated by the set $E_{\leq}(S)$ of all ordered idempotents of S by $< E_{\leq}(S) >$ or simply by $< E_{\leq} >$. Let $E^n_{\leq}$ be the set of all elements of S which can be written as the product of n (not necessarily distinct) ordered idempotents of S. Then $< E_{\leq}(S) >= \bigcup_{n=1}^{\infty} E^n_{\leq}$.

Lemma 3.1. *Let S be a regular ordered semigroup and $n \in \mathbb{N}$ be such that $n > 1$*

(1) *Then $x \in E^n_{\leq}$ implies that $V_{\leq}(x) \cap E^{n-1}_{\leq} \neq \phi$.*
(2) *Let $E^n_{\leq}$ be downward closed in S for every $n \in \mathbb{N}$. Then $x \in E^n_{\leq}$ if and only if $V_{\leq}(x) \cap E^{n-1}_{\leq} \neq \phi$.*

Proof. (1) We prove this results by induction on n. Consider $x \in E^2_{\leq}$. Then for some $e_1, e_2 \in E_{\leq}$, $x = e_1e_2$. Let $y \in V_{\leq}(x)$. Then $x \leq xyx$ and $y \leq yxy$. Now take $f = e_2ye_1$. Then it follows that $f = e_2ye_1 \leq e_2yxye_1 \leq e_2ye_1e_2ye_1 = f^2$. Thus $f \in E_{\leq}(S)$.

Now $x \leq xyx$ implies that $x \leq e_1e_2e_2ye_1e_1e_2 = xfx$. Furthermore $f = e_2ye_1 \leq e_2yxye_1 \leq e_2ye_1(e_1e_2)e_2ye_1 = fxf$. Hence $f \in V_{\leq}(x) \cap E_{\leq}$.

Suppose that the result holds for all $k < n$. Let $x \in E^n_{\leq}$. Then $x = e_1e_2 \ldots e_n$ where $e_1, e_2, \ldots, e_n \in E_{\leq}(S)$. Let $y = e_2 \ldots e_n$. Then $y \in E^{n-1}_{\leq}$. So by the induction hypothesis we have $z \in V_{\leq}(y) \cap E^{n-2}_{\leq}$. Consider $w \in V_{\leq}(x)$. Let $f = z(e_2 \ldots e_nwe_1)$. Now $e_2 \ldots e_nwe_1 \leq e_2 \ldots e_n(wxw)e_1 \leq (e_2 \ldots e_nwe_1)(e_2 \ldots e_nwe_1)$, since $x = e_1e_2 \ldots e_n$. Thus $e_2 \ldots e_nwe_1 \in E_{\leq}(S)$. Since $z \in E^{n-2}_{\leq}$, so $f = z(e_2 \ldots e_nwe_1) \in E^{n-1}_{\leq}$.

Now we have

$$\begin{aligned}
f &= z(e_2 \dots e_n w e_1) \\
&\leq z[e_2 \dots e_n (wxw) e_1)] && [w \in V_{\leq}(x)] \\
&\leq z[(e_2 \dots e_n w)(e_1 e_2 \dots e_n) w e_1] && [x = e_1 e_2 \cdots e_n] \\
&\leq z(e_2 \dots e_n w e_1)(e_2 \dots e_n z e_2 \dots e_n) w e_1 && [z \in V_{\leq}(e_2 e_3 \cdots e_n)] \\
&\leq z(e_2 \dots e_n w e_1) e_1 e_2 \dots e_n (z e_2 \dots e_n) w e_1 && [e_1 \leq e_1^2] \\
&\leq f(e_1 e_2 \dots e_n) f && [f = z(e_2 \dots e_n w e_1)] \\
&\leq fxf.
\end{aligned}$$

Since $w \in V_{\leq}(x)$, so $x \leq xwx$. This implies that $x \leq e_1e_2 \dots e_n wx \leq e_1 ywe_1(e_1e_2 \dots e_n) \leq e_1(ywe_1)x \leq e_1(yzywe_1)x \leq e_1e_2 \dots e_n ze_2 \dots e_n we_1 x \leq x(ze_2 \dots e_n we_1)x = xfx$. Thus $f \in V_{\leq}(x) \cap E_{\leq}^{n-1}$, that is, $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$. So the result follows by induction.

(2) The necessary part follows from (1).
Let $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$. Then there is $y \in V_{\leq}(x) \cap E_{\leq}^{n-1}$. Now $y \in E_{\leq}^{n-1}$ implies that there is $z \in V_{\leq}(y) \cap E_{\leq}^{n-2}$, by (1). Then $x \leq xyx \leq (xy)z(yx)$. Now $xy, yx \in E_{\leq}(S)$ and $z \in E_{\leq}^{n-2}$ implies that $xyzyx \in E_{\leq}^{n}$; and hence $x \in (E_{\leq}^{n}] = E_{\leq}^{n}$, since $E_{\leq}^{n}$ is downward closed. Thus the result follows. ■

Now we show that regularity of an ordered semigroup S ensures the same for $< E_{\leq}(S) >$.

Theorem 3.2. *Let S be a regular ordered semigroup and $E_{\leq}^{n}$ is downward closed in S for every $n > 1$. Then $V_{\leq}(E_{\leq}^{n-1}) = E_{\leq}^{n}$; and hence the subsemigroup $< E_{\leq} >$ of S generated by the ordered idempotents of S is also regular.*

Proof. Let $z \in V_{\leq}(E_{\leq}^{n-1})$. Then there is $w \in E_{\leq}^{n-1}$ such that $z \in V_{\leq}(w)$, which again implies that $w \in E_{\leq}^{n-1} \cap V_{\leq}(z)$. Then $z \in E_{\leq}^{n}$, by Lemma 3.1 and so $V_{\leq}(E_{\leq}^{n-1}) \subseteq E_{\leq}^{n}$.

Now let $x \in E_{\leq}^{n}$. Then $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$. Consider $y \in V_{\leq}(x) \cap E_{\leq}^{n-1}$. Then $x \in V_{\leq}(y)$ implies that $x \in V_{\leq}(E_{\leq}^{n-1})$. Thus $V_{\leq}(E_{\leq}^{n-1}) \subseteq E_{\leq}^{n}$. Hence $V_{\leq}(E_{\leq}^{n-1}) = E_{\leq}^{n}$. This completes the proof. ■

The following lemma has been given in [1]. For the sake of completeness, we would like to include a short proof here also.

Lemma 3.3. *Let S be a completely regular ordered semigroup. Then for every $a \in S$ there is $a' \in V_{\leq}(a)$ such that $a \leq aa'a$, $a \leq a^2a'$, and $a \leq a'a^2$.*

Proof. Let $a \in S$. Then there is $t \in S$ such that $a \leq a^2ta^2$. Then $a \leq a^3ta^2ta^2 \leq a^3ta^2ta^2ta^3 \leq aa'a$, where $a' = a^2ta^2ta^2ta^2$. Similarly we have $a' \leq a'aa'$. Thus $a' \in V_{\leq}(a)$. Also $a \leq a^2ta^2 \leq a^4ta^2ta^2ta^2 = a^2a'$. Similarly it can be shown that $a \leq a'a^2$. ■

Now we have the following results on completely regular ordered semigroups.

Theorem 3.4. *Let S be a completely regular ordered semigroup and $E_{\leq}^{n}$ is downward closed in S, for every $n \in \mathbb{N}$. Then $< E_{\leq}(S) >$ is a completely regular ordered subsemigroup.*

Proof. Consider $a \in < E_{\leq}(S) >$. Then $a = e_1 e_2 \dots e_m$ for some $m \in \mathbb{N}$, and $e_1, e_2, \dots, e_m \in E_{\leq}(S)$. Since a is completely regular, there is $h \in V_{\leq}(a)$ such that $a \leq aha$, $a \leq a^2 h$ and $a \leq ha^2$, by Lemma 3.3. Now $a \in V_{\leq}(h) \cap E_{\leq}^{m}$ implies that $h \in E_{\leq}^{m+1}$, by Lemma 3.1 and hence $h^3 \in < E_{\leq}(S) >$. Then $a \leq a^2 h^3 a^2$ implies that $< E_{\leq}(S) >$ is completely regular. ■

Following Eberhart, Williams and Kinch [2] let us define an equivalence relation γ on S in the following way: for $a, b \in S$,

> $a\gamma b$ if and only if there exists a sequence $x_1, x_2, \dots x_n$ of elements of S such that $x_1 \in V_{\leq}(a)$, $x_i \in V(x_{i-1})$; $i = 2, \dots, n$ and $b \in V(x_n)$.

In the following theorem we show that $< E_{\leq}(S) >$ is the union of all γ-equivalence classes of ordered idempotents.

Theorem 3.5. *Let S be a regular ordered semigroup and $E_{\leq}^{n}$ be downward closed in S for every $n \in \mathbb{N}$. Then for any $x \in S$, $x \in < E_{\leq} >$ if and only if there is an ordered idempotent $e \in E_{\leq}$ such that $x\gamma e$.*

Proof. Consider $x \in < E_{\leq} >$. Then $x \in E_{\leq}^{m}$; for some $m \in \mathbb{N}$. If $m = 1$, then the result follows trivially. Let $m > 1$. Then $V_{\leq}(x) \cap E_{\leq}^{m-1} \neq \phi$, by Lemma 3.1. Consider $x_1 \in V_{\leq}(x) \cap E_{\leq}^{m-1}$. Repeated application of this process yields a sequence of elements $x_1, x_2, \dots, x_{m-1}$ of S such that $x_1 \in V_{\leq}(x)$, $x_i \in V(x_{i-1}) \cap E_{\leq}^{m-1}$ for $i = 2, \dots, m-1$, whence $x_{m-1} \in E_{\leq}$. Say $e = x_{m-1}$. Thus $x\gamma e$.

Conversely assume that $x \in S$ and there is $e \in E_{\leq}(S)$ such that $x\gamma e$. Then there are elements $x_1, x_2, \dots, x_n \in S$ such that $x_1 \in V_{\leq}(x), x_i \in V_{\leq}(x_{i-1})$ and $e \in V(x_n)$ so that $x \leq xx_1x \leq (xx_1)(x_2x_1)x \leq \cdots \leq (xx_1)x_2 \dots x_n e x_n \dots x_2 x_1 x$ which can be rearranged as:

$$(3.1) \quad x \leq \begin{cases} (xx_1)(x_2x_3)\cdots(x_n e)(ex_n)\dots(x_3x_2)(x_1x) & \text{if } n \text{ is even} \\ (xx_1)(x_2x_3)\dots(x_{n-1}x_n)e(x_nx_{n-1})\dots(x_3x_2)(x_1x) & \text{if } n \text{ is odd.} \end{cases}$$

Since xx_1, x_1x, x_ne, ex_n and $x_ix_{i-1}, x_{j-1}x_j (i = 2, \dots, n; j = 2, \dots, n)$ are all ordered idempotents of S, so $x \in (E_{\leq}^{n+2}(S)]$ in both the cases. This implies that $x \in E_{\leq}^{n+2}(S)$ and hence $x \in < E_{\leq}(S) >$. ■

Definition 3.6. An ordered semigroup S is said to be generated by the ordered idempotents if $S =< E_{\leq}(S) >$.

Let T be a subsemigroup of S and $A \subseteq S$. We say that T is generated as an ordered subsemigroup by A if for every $t \in T$ there are $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A$ such that $t \leq a_1 a_2 \ldots a_n$. It is denoted by $T =< A >_{\leq}$. Now we show that every ideal I of an ordered idempotent generated ordered semigroup is also generated as an ordered subsemigroup by the ordered idempotents $E_{\leq}(I)$.

Theorem 3.7. *Let S be a regular ordered semigroup generated by its ordered idempotents and for every $n > 1$, let $E^n_{\leq}$ be downward closed in S. Then every ideal of S is also a regular ordered semigroup generated by ordered idempotents as an ordered subsemigroup.*

Proof. Suppose that $S =< E_{\leq}(S) >$. Consider an ideal I of S and $x \in I$. Since $x \in< E_{\leq}(S) >$, so $x\gamma e$ for some $e \in E_{\leq}(S)$, by Theorem 3.5; and hence there is a sequence of elements $x_1, x_2, \ldots, x_n \in S$ such that $x_1 \in V_{\leq}(x)$, $x_i \in V_{\leq}(x_{i-1}); i = 2, 3, \ldots, n$ and $e \in V(x_n)$. Then it follows that $x \leq xx_1x \leq xx_1x_2x_1x \leq x(x_1x_2 \ldots x_n)e(x_n \ldots x_2x_1)x$ which can be rearranged as:
(3.2)

$$x \leq \begin{cases} x((x_1x_2)(x_2x_3)\ldots(x_{n-1}x_n)e(x_nx_{n-1})\ldots(x_3x_2)(x_2x_1))x; & \text{if n is even} \\ x((x_1x_2)(x_2x_3)\ldots(x_ne)(ex_n)\ldots(x_3x_2)(x_2x_1))x; & \text{if n is odd.} \end{cases}$$

Since I is an ideal of S, $xx_1, x_1x \in I$. Now for $r = 1, 2, \ldots, n-1$; $x_{r+1}x_r \leq x_{r+1}x_rx_{r+1}x_r \leq x_{r+1}x_r(x_{r-1}x_r)x_{r+1}x_r \leq \cdots \leq x_{r+1}x_r \ldots x_2(x_1x)x_1x_2 \ldots x_{r+1}x_r$. Since $x_1x \in I$ the above inequality implies that $x_{r+1}x_r \in I$ for $r = 1, 2, \ldots n-1$. Similarly e, x_ne, ex_n and x_rx_{r+1} for $r = 1, 2, \ldots n-1$ all belong to I. Also for $r = 1, 2, \ldots, n-1$ the elements x_rx_{r+1}, $x_{r+1}x_r$, x_ne and ex_n are all ordered idempotents in S. Thus x_rx_{r+1}, $x_{r+1}x_r$, x_ne and $ex_n \in E_{\leq}(I)$ for all $r = 1, 2, \ldots, n-1$. So $(x_1x_2)(x_2x_3)\ldots(x_{n-1}x_n)e(x_nx_{n-1})\ldots(x_3x_2)(x_2x_1)$ and $(x_1x_2)(x_2x_3)\ldots(x_ne)(ex_n)\ldots(x_3x_2)(x_2x_1) \in< E_{\leq}(I) >$. Therefore from (3.2) it follows that $x \in< E_{\leq}(I) >_{\leq}$. Hence $< E_{\leq}(I) >_{\leq}= I$, in other words I is generated as an ordered subsemigroup by its ordered idempotents. Also every ideal of a regular ordered semigroup is also regular. This completes the proof. ■

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