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HOLOMORPH OF GENERALIZED BOL LOOPS II

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Abstract

The notion of the holomorph of a generalized Bol loop (GBL) is characterized afresh. The holomorph of a right inverse property loop (RIPL) is shown to be a GBL if and only if the loop is a GBL and some bijections of the loop are right (middle) regular. The holomorph of a RIPL is shown to be a GBL if and only if the loop is a GBL and some elements of the loop are right (middle) nuclear. Necessary and sufficient conditions for the holomorph of a RIPL to be a Bol loop are deduced. Some algebraic properties and commutative diagrams are established for a RIPL whose holomorph is a GBL.

Keywords: generalized Bol loop, holomorph of a loop.

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1. Introduction

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y, respectively, for each $a, b \in L$, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho \in L$ and $x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right and left inverse elements of x respectively. Here, $e^\rho \in L$ and $e^\lambda \in L$ satisfy the relations $xe^\rho = x$ and $e^\lambda x = x$ for all $x \in L$ if they exist in a quasigroup $(L, \cdot)$ and are respectively called the right and left identity elements. Now, if $e^\rho = e^\lambda = e \in L$, then e is called the identity element and $(L, \cdot)$ is called a loop. In case $x^\lambda = x^\rho$, then, we simply write $x^\lambda = x^\rho = x^{-1} = xJ$ and refer to x^{-1} as the inverse of x. If $x, y, z \in L$ such that $(x \cdot yz) = (xy \cdot z)(x, y, z)$, then (x, y, z) is called the associator of x, y, z.

Let x be an arbitrarily fixed element in a loop $(G, \cdot)$. For any $y \in G$, the left and right translation maps of $x \in G$, L_x and R_x are respectively defined by

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

A loop $(L, \cdot)$ is called a (right) Bol loop if it satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y). \tag{1}$$

A loop $(L, \cdot)$ is called a left Bol loop if it satisfies the identity

$$y(z \cdot yx) = (y \cdot zy)x. \tag{2}$$

A loop $(L, \cdot)$ is called a Moufang loop if it satisfies the identity

$$(xy) \cdot (zx) = (x \cdot yz)x. \tag{3}$$

A loop $(L, \cdot)$ is called a right inverse property loop (RIPL) if it satisfies right inverse property (RIP)

$$(yx)x^\rho = y. \tag{4}$$

A loop $(L, \cdot)$ is called a left inverse property loop (LIPL) if it satisfies left inverse property (LIP)

$$x^\lambda(xy) = y. \tag{5}$$

A loop $(L, \cdot)$ is called an automorphic inverse property loop (AIPL) if it satisfies automorphic inverse property (AIP)

$$(xy)^{-1} = x^{-1}y^{-1}. \tag{6}$$

A loop $(L, \cdot)$ in which the mapping $x \mapsto x^2$ is a permutation, is called a Bruck loop if it is both a Bol loop and either AIPL or obeys the identity $xy^2 \cdot x = (yx)^2$ (Robinson [33]).

Let $(L, \cdot)$ be a loop with a single valued self-map $\sigma : x \longrightarrow \sigma(x)$:

$(L, \cdot)$ is called a σ-generalized (right) Bol loop or right B-loop if it satisfies the identity

$$(xy \cdot z)\sigma(y) = x(yz \cdot \sigma(y)) \tag{7}$$

$(L, \cdot)$ is called a σ-generalized left Bol loop or left B-loop if it satisfies the identity

$$\sigma(y)(z \cdot yx) = (\sigma(y) \cdot zy)x \tag{8}$$

$(L, \cdot)$ is called a σ-M-loop if it satisfies the identity

$$(xy) \cdot (z\sigma(x)) = (x \cdot yz)\sigma(x). \tag{9}$$

Let $(G, \cdot)$ be a groupoid (quasigroup, loop) and let A, B and C be three bijective mappings, that map G onto G. The identity map on G shall be denoted by I. The triple $\alpha = (A, B, C)$ is called an autotopism of $(G, \cdot)$ if and only if

$$xA \cdot yB = (x \cdot y)C \;\forall\; x, y \in G.$$

Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$.

If $A = B = C$, then A is called an automorphism of the groupoid (quasigroup, loop) $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the automorphism group of $(G, \cdot)$. Let G and H be groups such that $\varphi : G \to H$ is an isomorphism. If $\varphi(g) = h$, then this would be expressed as $g \overset{\varphi}{\cong} h$.

Given any two sets X and Y. The statement '$f : X \to Y$ is defined as $f(x) = y,\ x \in X,\ y \in Y$' will be at times be expressed as '$f : X \to Y \ \uparrow\ f(x) = y$'.

The right nucleus of $(L, \cdot)$ is defined by $N_\rho(L, \cdot) = \{x \in L \mid zy \cdot x = z \cdot yx \;\forall\; y, z \in L\}$. The middle nucleus of $(L, \cdot)$ is defined by $N_\mu(L, \cdot) = \{x \in L \mid zx \cdot y = z \cdot xy \;\forall\; y, z \in L\}$.

Definition. Let $(G, \cdot)$ be a quasigroup. Then

1. a bijection U is called autotopic if there exists $(U, V, W) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\Sigma(G, \cdot)$.

2. a bijection U is called ρ-regular if there exists $(I, U, U) \in AUT(G, \cdot)$; the set of all such mappings forms a group $\mathcal{P}(G, \cdot)$.

3. a bijection U is called μ-regular if there exists a bijection U' such that $(U, U'^{-1}, I) \in AUT(G, \cdot)$. U' is called the adjoint of U. The set of all μ-regular mappings forms a group $\Phi(G, \cdot) \leq \Sigma(G, \cdot)$. The set of all adjoint mapping forms a group $\Psi(G, \cdot)$.

Definition. Let $(Q,\cdot)$ be a loop and $A(Q) \leq AUM(Q,\cdot)$ be a group of automorphisms of the loop $(Q,\cdot)$. Let $H = A(Q) \times Q$. Define $\circ$ on H as

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y) \text{ for all } (\alpha, x), (\beta, y) \in H.$$

$(H,\circ)$ is a loop and is called the A-holomorph of $(Q,\cdot)$.

The left and right translations maps of an element $(\alpha, x) \in H$ are respectively denoted by $\mathbb{L}_{(\alpha,x)}$ and $\mathbb{R}_{(\alpha,x)}$.

Remark 1. $(H,\circ)$ has a subloop $\{I\}\times Q$ that is isomorphic to $(Q,\cdot)$. As observed in Lemma 6.1 of Robinson [33], given a loop $(Q,\cdot)$ with an A-holomorph $(H,\circ)$, $(H,\circ)$ is a Bol loop if and only if $(Q,\cdot)$ is a θ-generalized Bol loop for all $\theta \in A(Q)$. Also in Theorem 6.1 of Robinson [33], it was shown that $(H,\circ)$ is a Bol loop if and only if $(Q,\cdot)$ is a Bol loop and $x^{-1} \cdot x\theta \in N_\rho(Q,\cdot)$ for all $\theta \in A(Q)$.

The birth of Bol loops can be traced back to Gerrit Bol [11] in 1937 when he established the relationship between Bol loops and Moufang loops, the latter which was discovered by Ruth Moufang [29]. Thereafter, a theory of Bol loops was evolved through the Ph.D. thesis of Robinson [33] in 1964 where he studied the algebraic properties of Bol loops, Moufang loops and Bruck loops, isotopy of Bol loop and some other notions on Bol loops. Some later results on Bol loops and Bruck loops can be found in Bruck [12], Solarin [44], Adeniran and Akinleye [4], Bruck [13], Burn [15], Gerrit Bol [11], Blaschke and Bol [10], Sharma [36, 37], Adeniran and Solarin [6]. In the 1980s, the study and construction of finite Bol loops caught the attention of many researchers among whom are Burn [15, 16, 17], Solarin and Sharma [41, 40, 42] and others like Chein and Goodaire [21, 19, 20], Foguel *et al.* [24], Kinyon and Phillips [27, 28] in the present millennium. One of the most important results in the theory of Bol loops is the solution of the open problem on the existence of a simple Bol loop which was finally laid to rest by Nagy [30, 31, 32].

In 1978, Sharma and Sabinin [38, 39] introduced and studied the algebraic properties of the notion of half-Bol loops(left B-loops). Thereafter, Adeniran [2], Adeniran and Akinleye [4], Adeniran and Solarin [7] studied the algebraic properties of generalized Bol loops. Also, Ajmal [8] introduced and studied the algebraic properties of generalized Bol loops and their relationship with M-loops.

Some of their results are highlighted below.

Theorem 2 (Adeniran and Akinleye [4])**.** *If $(L,\cdot)$ is a generalized Bol loop, then:*

1. *$(L,\cdot)$ is an RIPL.*

2. *$x^\lambda = x^\rho$ for all $x \in L$.*

3. *$R_{y\cdot\sigma(y)} = R_y R_{\sigma(y)}$ for all $y \in L$.*

4. $[xy \cdot \sigma(x)]^{-1} = (\sigma(x))^{-1}y^{-1} \cdot x^{-1}$ *for all* $x, y \in L$.
5. $(R_{y^{-1}}, L_yR_{\sigma(y)}, R_{\sigma(y)}), (R_y^{-1}, L_yR_{\sigma(y)}, R_{\sigma(y)}) \in AUT(L, \cdot)$ *for all* $y \in L$.

Theorem 3 (Sharma and Sabinin [38]). *If* $(L, \cdot)$ *is a half Bol loop, then:*

1. $(L, \cdot)$ *is an LIPL.*
2. $x^\lambda = x^\rho$ *for all* $x \in L$.
3. $L_{(x)}L_{(\sigma(x))} = L_{(\sigma(x)x)}$ *for all* $x \in L$.
4. $(\sigma(x) \cdot yx)^{-1} = x^{-1} \cdot y^{-1}(\sigma(x))^{-1}$ *for all* $x, y \in L$.
5. $(R_{(x)}L_{(\sigma(x))}, L_{(x)^{-1}}, L_{(\sigma(x))}), (R_{(\sigma(x))}L_{(x)^{-1}}, L_{\sigma(x)}, L_{(x)^{-1}}) \in AUT(L, \cdot)$ *for all* $x \in L$.

Theorem 4 (Ajmal [8]). *Let* $(L, \cdot)$ *be a loop. The following statements are equivalent:*

1. $(L, \cdot)$ *is an M-loop;*
2. $(L, \cdot)$ *is both a left B-loop and a right B-loop;*
3. $(L, \cdot)$ *is a right B-loop and satisfies the LIP;*
4. $(L, \cdot)$ *is a left B-loop and satisfies the RIP.*

Theorem 5 (Ajmal [8]). *Every isotope of a right B-loop with the LIP is a right B-loop.*

Example 6. Let R be a ring of all 2×2 matrices taken over the field of three elements and let $G = R \times R$. For all $(u, f), (v, g) \in G$, define $(u, f) \cdot (v, g) = (u + v, f + g + uv^3)$. Then $(G, \cdot)$ is a loop which is not a right Bol loop but which is a σ-generalized Bol loop with $\sigma : x \mapsto x^2$.

We shall need the following result.

Theorem 7 (Belousov [9]). *Let* $(G, \cdot)$ *be a loop with an identity element* e. *Let*

$$\psi : \mathcal{P}(G, \cdot) \to N_\rho(G, \cdot) \uparrow \psi(U) = eU, \ \phi : \Phi(G, \cdot) \to \Psi(G, \cdot) \ \uparrow \phi(U) = U',$$

$$\varpi : \Phi(G, \cdot) \to N_\mu(G, \cdot) \uparrow \varpi(U) = eU \ \text{and} \ \beta : \Psi(G, \cdot) \to N_\mu(G, \cdot) \uparrow \beta(U') = eU'.$$

Then $\mathcal{P}(G, \cdot) \overset{\psi}{\cong} N_\rho(G, \cdot)$, $\Phi(G, \cdot) \overset{\phi}{\cong} \Psi(G, \cdot)$, $\Phi(G, \cdot) \overset{\varpi}{\cong} N_\mu(G, \cdot)$, $\Psi(G, \cdot) \overset{\beta}{\cong} N_\mu(G, \cdot)$.

Interestingly, Adeniran [3] and Robinson [33], Chiboka and Solarin [23], Bruck [12], Bruck and Paige [14], Robinson [34], Huthnance [25] and Adeniran [3] have respectively studied the holomorphs of Bol loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and Bruck loops. A set of results on the holomorph of some varieties of loops can be found in Jaiyeola [26]. The latest study on the holomorph of generalized Bol loops can be found in Adeniran *et al.* [5].

In this present work, the notion of the holomorph of a generalized Bol loop (GBL) is characterized afresh. The holomorph of a right inverse property loop (RIPL) is shown to be a GBL if and only if the loop is a GBL and some bijections of the loop are right (middle) regular. The holomorph of a RIPL is shown to be a GBL if and only if the loop is a GBL and some elements of the loop are right (middle) nuclear. Necessary and sufficient conditions for the holomorph of a RIPL to be a Bol loop are deduced. Some algebraic properties and commutative diagrams are established for a RIPL whose holomorph is a GBL.

2. Main Results

Theorem 8. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x)\mapsto(\alpha,\sigma(x))$ for all $(\alpha,x)\in H$. The A-holomorph $(H,\circ)$ of $(Q,\cdot)$ is a σ'-generalised Bol loop if and only if $C=\big(R_x^{-1}, L_xR_{[\sigma(x\gamma^{-1})]\alpha^{-1}}, R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\big)\in AUT(Q,\cdot)$ for all $x\in Q$ and all $\alpha,\gamma\in A(Q)$.*

Proof. Note that

- $(H,\circ)$ is a RIPL if and only if $(Q,\cdot)$ is a RIPL.
- $(Q,\cdot)$ is a σ-generalised Bol loop if and only if $B=(R_x^{-1}, L_xR_{\sigma(x)}, R_{\sigma(x)})\in AUT(Q,\cdot)$ for all $x\in Q$.

Define $\sigma' : H\to H$ as $\sigma'(\alpha,x)=(\alpha,\sigma(x))$. Let $(\alpha,x),(\beta,y),(\gamma,z)\in H$, then $(H,\circ)$ is a σ'-generalised Bol loop if and only if $(\mathbb{R}_{(\alpha,x)^{-1}},\mathbb{L}_{(\alpha,x)}\mathbb{R}_{\sigma'(\alpha,x)},\mathbb{R}_{\sigma'(\alpha,x)})\in AUT(H,\circ)$ for all $(\alpha,x)\in H$, i.e., $(\mathbb{R}_{(\alpha,x)^{-1}},\mathbb{L}_{(\alpha,x)}\mathbb{R}_{(\alpha,\sigma(x))},\mathbb{R}_{(\alpha,\sigma(x))})\in AUT(H,\circ)$

(10)
$$\Longleftrightarrow (\beta,y)\mathbb{R}_{(\alpha,x)^{-1}}\circ(\gamma,z)\mathbb{L}_{(\alpha,x)}\mathbb{R}_{(\alpha,\sigma(x))}=[(\beta,y)\circ(\gamma,z)]\mathbb{R}_{(\alpha,\sigma(x))}$$
$$\Leftrightarrow [(\beta,y)\circ(\alpha,x)^{-1}]\circ[((\alpha,x)\circ(\gamma,z))\circ(\alpha,\sigma(x))]=[(\beta,y)\circ(\gamma,z)]\circ(\alpha,\sigma(x)).$$

Let $(\beta,y)\circ(\alpha,x)^{-1}=(\tau,t)$. Since $(\alpha,x)^{-1}=(\alpha^{-1},(x\alpha^{-1})^{-1})$, then

$$(\tau,t)=(\beta\alpha^{-1},(yx^{-1})\alpha^{-1}). \tag{11}$$

From (10) and (11),

$$(12)\quad \begin{aligned} &(\tau,t)\circ[(\alpha\gamma,x\gamma\cdot z)\circ(\alpha,\sigma(x))]=(\beta\gamma,y\gamma\cdot z)\circ(\alpha,\sigma(x))\\ &\Leftrightarrow\Big(\tau\alpha\gamma\alpha,(t\alpha\gamma\alpha)\big((x\gamma\cdot z)\alpha\cdot\sigma(x)\big)\Big)=\big(\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot\sigma(x)\big).\end{aligned}$$

Putting (11) in (12), we have

$$(13)\quad \begin{aligned} &\Big(\beta\alpha^{-1}\alpha\gamma\alpha,(yx^{-1})\alpha^{-1}(\alpha\gamma\alpha)\big((x\gamma\cdot z)\alpha\cdot\sigma(x)\big)\Big)=\big(\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot\sigma(x)\big)\\ &\Leftrightarrow\Big(\beta\gamma\alpha,(yx^{-1})\gamma\alpha\big[(x\gamma\cdot z)\alpha\cdot\sigma(x)\big]\Big)=\big(\beta\gamma\alpha,(y\gamma\cdot z)\alpha\cdot\sigma(x)\big)\\ &\Leftrightarrow(yx^{-1})\gamma\alpha\cdot[(x\gamma\cdot z)\alpha\cdot\sigma(x)]=(y\gamma\cdot z)\alpha\cdot\sigma(x)\\ &\Leftrightarrow\big[(yx^{-1})\gamma\cdot[(x\gamma\cdot z)\cdot(\sigma(x)\alpha^{-1})]\big]\alpha=[(y\gamma\cdot z)\cdot(\sigma(x)\alpha^{-1})]\alpha\\ &\Leftrightarrow(y\gamma x^{-1}\gamma)[(x\gamma\cdot z)\cdot(\sigma(x)\alpha^{-1})]=(y\gamma\cdot z)(\sigma(x)\alpha^{-1}).\end{aligned}$$

Let $\bar{y}=y\gamma$, then (13) becomes

$$\begin{aligned} &(\bar{y}\cdot x^{-1}\gamma)[(x\gamma\cdot z)(\sigma(x)\alpha^{-1})]=(\bar{y}\cdot z)(\sigma(x)\alpha^{-1})\\ &\Leftrightarrow\Big(R_{x\gamma}^{-1},L_{x\gamma}R_{[\sigma(x)\alpha^{-1}]},R_{[\sigma(x)\alpha^{-1}]}\Big)\in AUT(Q,\cdot)\end{aligned}$$

and replacing $x\gamma$ by x, $\big(R_x^{-1},L_xR_{[\sigma(x\gamma^{-1})]\alpha^{-1}},R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\big)\in AUT(Q,\cdot)$. ■

Theorem 9. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma':(\alpha,x)\mapsto(\alpha,\sigma(x))$ for all $(\alpha,x)\in H$. $(H,\circ)$ is a σ'-generalised Bol loop if and only if*

1. *$(Q,\cdot)$ is a σ-GBL;*
2. $\Big(I,R_{\sigma(x)}^{-1}R_{\sigma(x\gamma^{-1})},R_{\sigma(x)}^{-1}R_{\sigma(x\gamma^{-1})}\Big)\in AUT(Q,\cdot)$*; and*
3. $\Big(I,R_{\sigma(x)}^{-1}R_{[\sigma(x)]\alpha^{-1}},R_{\sigma(x)}^{-1}R_{[\sigma(x)]\alpha^{-1}}\Big)\in AUT(Q,\cdot)$

for all $x\in Q$ and $\alpha,\gamma\in A(Q)$.

Proof. From Theorem 8, $(H,\circ)$ is a σ'-generalised Bol loop if and only if

$$C=\Big(R_x^{-1},L_xR_{[\sigma(x\gamma^{-1})]\alpha^{-1}},R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\Big)\in AUT(Q,\cdot)\Leftrightarrow$$

$\Big(R_x^{-1}, L_xR_{\sigma''(x)}, R_{\sigma''(x)}\Big) \in AUT(Q,\cdot)$ where $\sigma''(x) = [\sigma(x\gamma^{-1})]\alpha^{-1}$. Taking $\alpha = \gamma = I$ in C, then $\sigma'' = \sigma$ which implies that $(Q,\cdot)$ is a σ-GBL and thus $B = (R_x^{-1}, L_xR_{\sigma(x)}, R_{\sigma(x)}) \in AUT(Q,\cdot)$ for all $x \in Q$. So,

$$B^{-1}C = \Big(I, R^{-1}_{\sigma(x)}R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}, R^{-1}_{\sigma(x)}R_{[\sigma(x\gamma^{-1})]\alpha^{-1}}\Big) \in AUT(Q,\cdot) \tag{14}$$

Substitute $\alpha = I$ in (14) to get

$$D(x) = \Big(I, R^{-1}_{\sigma(x)}R_{[\sigma(x\gamma^{-1})]}, R^{-1}_{\sigma(x)}R_{[\sigma(x\gamma^{-1})]}\Big) \in AUT(Q,\cdot)$$

and also substitute $\gamma = I$ in (14) to get

$$E(x) = \Big(I, R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}, R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\Big) \in AUT(Q,\cdot)$$

This proves the forward. The converse is achieved by computing and showing that $BD(x)E(x\gamma^{-1}) = C$. ■

Theorem 10. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \sigma(x))$ for all $(\alpha, x) \in H$. $(H,\circ)$ is a σ'-generalised Bol loop if and only if*

1. *$(Q,\cdot)$ is a σ-GBL; and*
2. *$\sigma(x)^{-1}\sigma\big(x\gamma^{-1}\big), \sigma(x)^{-1}(\sigma(x))\alpha^{-1} \in N_\rho(Q,\cdot)$;*

for all $x, y \in Q$ and $\alpha, \gamma \in A(Q)$.

Proof. This is achieved by Theorem 9 by using the autotopisms $D(x)$ and $E(x)$. ■

Lemma 11. *Let $(Q,\cdot)$ be a RIPL with a bijective self map σ and let $(H,\circ)$ be the holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \sigma(x))$ for all $(\alpha, x) \in H$. If $(H,\circ)$ is a σ'-GBL, then $A(Q) = \{\sigma R_{n_1}\sigma^{-1}, R^{-1}_{n_2} | n_1, n_2 \in N_\rho(Q,\cdot)\}$.*

Proof. Using Theorem 10:

$$\sigma(x)\cdot\sigma(x)^{-1}\sigma\big(x\gamma^{-1}\big), \sigma(x)\cdot\sigma(x)^{-1}(\sigma(x))\alpha^{-1} \in \sigma(x)N_\rho(Q,\cdot)$$

$$\Rightarrow \sigma\big(x\gamma^{-1}\big) = \sigma(x)n_1 \text{ and } (\sigma(x))\alpha^{-1} = \sigma(x)n_2 \text{ for some } n_1, n_2 \in N_\rho(Q,\cdot)$$

$$\Rightarrow \gamma = \sigma R_{n_1}\sigma^{-1} \text{ and } \alpha = R^{-1}_{n_2} \text{ for some } n_1, n_2 \in N_\rho(Q,\cdot).$$

■

Theorem 12. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ for all $(\alpha,x) \in H$. $(H,\circ)$ is a σ'-generalised Bol loop if and only if*

1. *$(Q,\cdot)$ is a σ-GBL;*
2. *$\Big(R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, \big(JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J\big)^{-1}, I\Big) \in AUT(Q,\cdot)$; and*
3. *$\Big(R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}, \big(JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J\big)^{-1}, I\Big) \in AUT(Q,\cdot)$*

for all $x \in Q$ and $\alpha,\gamma \in A(Q)$.

Proof. This is achieved with Theorem 9 by using the fact that in a RIPL, $(U,V,W) \in AUT(Q,\cdot) \Rightarrow (W,JVJ,U) \in AUT(Q,\cdot)$. ∎

Theorem 13. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ for all $(\alpha,x) \in H$. The following are equivalent*

1. *$(H,\circ)$ is a σ'-GBL.*
2. (a) *$(Q,\cdot)$ is a σ-GBL;*
   (b) *$R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}$ and $R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}$ are ρ-regular for all $x \in Q$ and $\alpha,\gamma \in A(Q)$.*
3. (a) *$(Q,\cdot)$ is a σ-GBL;*
   (b) *$R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}$ and $R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}$ are μ-regular with adjoints $JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J$ and $JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J$ respectively, for all $x \in Q$ and $\alpha,\gamma \in A(Q)$.*

Proof. Use Theorem 9 and Theorem 12. ∎

Corollary 14. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ for all $(\alpha,x) \in H$. The following are equivalent*

1. *$(H,\circ)$ is a σ'-GBL.*
2. (a) *$(Q,\cdot)$ is a σ-GBL;*
   (b) *$R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} \in \mathcal{P}(Q,\cdot)$ for all $x \in Q$ and $\alpha,\gamma \in A(Q)$.*
3. (a) *$(Q,\cdot)$ is a σ-GBL;*

(b) $R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} \in \Phi(Q,\cdot)$ *and* $JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J$, $JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J \in \Psi(Q,\cdot)$ *for all* $x \in Q$ *and* $\alpha,\gamma \in A(Q)$.

Proof. Use Theorem 13. ■

Corollary 15. *Let* $(Q,\cdot)$ *be a RIPL with a self map* σ *and let* $(H,\circ)$ *be the A-holomorph of* $(Q,\cdot)$ *with a self map* σ' *such that* $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ *for all* $(\alpha,x) \in H$. *The following are equivalent*

1. $(H,\circ)$ *is a* σ'*-GBL.*
2. (a) $(Q,\cdot)$ *is a* σ*-GBL;*

   (b) $R_{\sigma(x\gamma^{-1})}, R_{[\sigma(x)]\alpha^{-1}} \in R_{\sigma(x)}\mathcal{P}(Q,\cdot)$ *for all* $x \in Q$ *and* $\alpha,\gamma \in A(Q)$.
3. (a) $(Q,\cdot)$ *is a* σ*-GBL;*

   (b) $R_{\sigma(x\gamma^{-1})}, R_{[\sigma(x)]\alpha^{-1}} \in R_{\sigma(x)}\Phi(Q,\cdot)$ *and* $R_{\sigma(x)}J \in R_{\sigma(x\gamma^{-1})}J\Psi(Q,\cdot)$, $R_{\sigma(x)}J \in R_{[\sigma(x)]\alpha^{-1}}J\Psi(Q,\cdot)$ *for all* $x \in Q$ *and* $\alpha,\gamma \in A(Q)$.

Proof. Use Corollary 14. ■

Lemma 16. *Let* $(L,\cdot)$ *be a loop. Then*

1. $\delta\mathcal{P}(L,\cdot)\delta^{-1} = \mathcal{P}(L,\cdot)$ *for all* $\delta \in AUM(L,\cdot)$.
2. $\delta\Phi(L,\cdot)\delta^{-1} = \Phi(L,\cdot)$ *and* $\delta\Psi(L,\cdot)\delta^{-1} = \Psi(L,\cdot)$ *for all* $\delta \in AUM(L,\cdot)$.

Proof.

1. Let $\delta \in AUM(L,\cdot)$ and $U \in \mathcal{P}(L,\cdot)$.

   Then $(\delta,\delta,\delta)(I,U,U)(\delta^{-1},\delta^{-1},\delta^{-1}) = (I,\delta U\delta^{-1},\delta U\delta^{-1}) \in AUT(L,\cdot) \Rightarrow \delta U\delta^{-1} \in \mathcal{P}(L,\cdot)$. Hence the conclusion.
2. These are similar to the proof of 1. ■

Corollary 17. *Let* $(Q,\cdot)$ *be a RIPL with a self map* σ *and let* $(H,\circ)$ *be the A-holomorph of* $(Q,\cdot)$ *with a self map* σ' *such that* $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ *for all* $(\alpha,x) \in H$. *If* $(H,\circ)$ *is a* σ'*-GBL, then*

1. $\delta R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\delta^{-1}, \delta R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\delta^{-1} \in \mathcal{P}(L,\cdot)$ *for all* $\delta \in AUM(L,\cdot)$.

   In particular, $\alpha R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\alpha^{-1}, \gamma R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\gamma^{-1} \in \mathcal{P}(L,\cdot)$ *for all* $x \in L$.

2. $\delta R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\delta^{-1}, \delta R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\delta^{-1} \in \Phi(L,\cdot)$ *and*

$$\delta JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}(\delta J)^{-1}, \delta R^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}(\delta J)^{-1} \in \Psi(L,\cdot)$$

for all $\delta \in AUM(L,\cdot)$. *In particular,*

$$\alpha R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\alpha^{-1}, \gamma R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\gamma^{-1} \in \Phi(L,\cdot)$$

and $\alpha JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}(\alpha J)^{-1}, \gamma JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}(\gamma J)^{-1} \in \Psi(L,\cdot)$ *for all* $x \in L$.

Proof. Use Corollary 14 and Lemma 16. ∎

Corollary 18. *Let* $(Q,\cdot)$ *be a RIPL with a self map* σ *and let* $(H,\circ)$ *be the A-holomorph of* $(Q,\cdot)$ *with a self map* σ' *such that* $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ *for all* $(\alpha,x) \in H$. *The following are equivalent*

1. $(H,\circ)$ *is a* σ'*-GBL.*
2. (a) $(Q,\cdot)$ *is a* σ*-GBL;*
   (b) $\sigma(x)^{-1}\sigma(x\gamma^{-1}), \sigma(x)^{-1}[\sigma(x)]\alpha^{-1} \in N_\rho(Q,\cdot)$ *for all* $x \in Q$ *and* $\alpha,\gamma \in A(Q)$.
3. (a) $(Q,\cdot)$ *is a* σ*-GBL;*
   (b) $\sigma(x)^{-1}\sigma(x\gamma^{-1}), \sigma(x)^{-1}[\sigma(x)]\alpha^{-1} \in N_\mu(Q,\cdot), \big(\sigma(x\gamma^{-1})\big)^{-1}\sigma(x)$, $\big([\sigma(x)]\alpha^{-1}\big)^{-1}\sigma(x) \in N_\mu(Q,\cdot)$ *for all* $x \in Q$ *and* $\alpha,\gamma \in A(Q)$.

Proof. We shall use Corollary 14 and Theorem 7.

1. Since $\mathcal{P}(G,\cdot) \overset{\psi}{\cong} N_\rho(G,\cdot)$, then $R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} \in \mathcal{P}(Q,\cdot) \Leftrightarrow eR^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, eR^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} \in N_\rho(G,\cdot) \Leftrightarrow \sigma(x)^{-1}\sigma(x\gamma^{-1})$, $\sigma(x)^{-1}[\sigma(x)]\alpha^{-1} \in N_\rho(Q,\cdot)$ for all $x \in Q$ and $\alpha,\gamma \in A(Q)$.
2. This is similar to 1. ∎

Theorem 19. *Let* $(Q,\cdot)$ *be a RIPL with a self map* σ *and let* $(H,\circ)$ *be the A-holomorph of* $(Q,\cdot)$ *with a self map* σ' *such that* $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ *for all* $(\alpha,x) \in H$. *If* $(H,\circ)$ *is a* σ'*-GBL, then:*

1. $JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J = L_{\sigma(x)^{-1}\sigma(x\gamma^{-1})}$;

(a) $\Big[y^{-1}[\sigma(x\gamma^{-1})]^{-1}\cdot\sigma(x)]^{-1} = \big[\sigma(x)^{-1}\sigma(x\gamma^{-1})\big]y$,

(b) $\big[\sigma(x\gamma^{-1})^{-1}\sigma(x)\big]^{-1} = \sigma(x)^{-1}\sigma(x\gamma^{-1})$.

2. $R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})} = R_{\big\{[\sigma(x\gamma^{-1})]^{-1}\sigma(x)\big\}^{-1}} = R_{\sigma(x)^{-1}\sigma(x\gamma^{-1})}$;

(a) $y\sigma(x)^{-1}\cdot\sigma(x\gamma^{-1}) = y\Big\{[\sigma(x\gamma^{-1})]^{-1}\sigma(x)^{-1}\Big\}^{-1} = y\big[\sigma(x)^{-1}\cdot\sigma(x\gamma^{-1})\big]$,

(b) $\sigma(x)\Big\{[\sigma(x\gamma^{-1})]^{-1}\sigma(x)^{-1}\Big\}^{-1} = \sigma(x\gamma^{-1})$.

3. $JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J = L_{\sigma(x)^{-1}[\sigma(x)]\alpha^{-1}}$;

(a) $y^{-1}\big(([\sigma(x)]\alpha^{-1})^{-1}\cdot\sigma(x)\big)^{-1} = \big[\sigma(x)^{-1}[(\sigma(x))\alpha^{-1}]\big]y$;

(b) $\big(([\sigma(x)]\alpha^{-1})^{-1}\sigma(x)\big)^{-1} = \sigma(x)^{-1}\big[(\sigma(x))\alpha^{-1}\big]$.

4. $R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} = R_{\big\{\big[(\sigma(x))\alpha^{-1}\big]^{-1}\sigma(x)\big\}^{-1}} = R_{\sigma(x)^{-1}}R_{[\sigma(x)]\alpha^{-1}}$;

(a) $y(\sigma(x))^{-1}\cdot[\sigma(x)]\alpha^{-1} = y\Big\{\big[(\sigma(x))\alpha^{-1}\big]^{-1}\sigma(x)\Big\}^{-1} =$
$y\big[\sigma(x)^{-1}\cdot[(\sigma(x))\alpha^{-1}]\big]$,

(b) $\sigma(x)\Big\{\big[(\sigma(x))\alpha^{-1}\big]^{-1}\sigma(x)\Big\}^{-1} = (\sigma(x))\alpha^{-1}$.

Proof. 1. From Theorem 12,
$\Big(R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, \big(JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J\big)^{-1}, I\Big) \in AUT(Q,\cdot)$ implies

$$yR^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\cdot z = y\cdot zJR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J.$$

Put $y=e$ to get $JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J = L_{\sigma(x)^{-1}\sigma(x\gamma^{-1})}$. (a) and (b) follow from this.

2. From Theorem 9, $\Big(I, R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}, R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\Big) \in AUT(Q,\cdot)$ implies

$$y\cdot zR^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})} = (yz)R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}.$$

Put $z=e$ and subsequently $y=e$ to get
$R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})} = R_{\big\{[\sigma(x\gamma^{-1})]^{-1}\sigma(x)\big\}^{-1}} = R_{\sigma(x)^{-1}\sigma(x\gamma^{-1})}$. (a) and (b) follow from this.

3. This is similar to 1.

4. This is similar to 2. ∎

Theorem 20. *Let $(Q,\cdot)$ be a RIPL with a bijective self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ for all $(\alpha,x) \in H$. The following are equivalent*

1. $(H,\circ)$ *is a* σ'*-GBL.*

2. (a) $(Q,\cdot)$ *is a* σ*-GBL;*
   (b) $\sigma(x)^{-1}\sigma^2\big(\sigma^{-1}(x)\cdot n\big) \in N_\rho(Q,\cdot) \ \ni\ \gamma = \sigma R_n \sigma^{-1}\ \forall\ \gamma \in A(Q),\ x \in Q$ *and some* $n \in N_\rho(Q,\cdot)$.

3. (a) $(Q,\cdot)$ *is a* σ*-GBL;*
   (b) $\sigma(x)^{-1}\sigma^2\big(\sigma^{-1}(x)\cdot n\big) \in N_\mu(Q,\cdot) \ \ni\ \gamma = \sigma R_n \sigma^{-1}\ \forall\ \gamma \in A(Q)\ , x \in Q$ *and some* $n \in N_\mu(Q,\cdot)$.
   (c) $\big[\sigma^2\big(\sigma^{-1}(x)\cdot n\big)\big]^{-1}\sigma(x) \in N_\mu(Q,\cdot) \ \ni\ \gamma = \sigma R_n \sigma^{-1}\ \forall\ \gamma \in A(Q)\ , x \in Q$ *and some* $n \in N_\mu(Q,\cdot)$.
   (d) $(\sigma(x)\cdot n')^{-1}\sigma(x) \in N_\mu(Q,\cdot) \ \ni\ \alpha = R_{n'}^{-1}\ \forall\ \alpha \in A(Q)\ , x \in Q$ *and some* $n' \in N_\mu(Q,\cdot)$.

Hence, $\sigma(xn^{-1}) = \sigma(x)n'^{-1}$ *for all* $x \in Q$ *and some* $n, n' \in N_\mu(Q,\cdot)$.

Proof. This is achieved by Corollary 18 and Lemma 11. ∎

Corollary 21. *Let $(Q,\cdot)$ be a RIPL with a bijective self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ for all $(\alpha,x) \in H$. $(H,\circ)$ is a σ'-GBL implies*

1. $(Q,\cdot)$ *is a* σ*-GBL.*

2. $\sigma(x)^{-1}\sigma^2\big(\sigma^{-1}(x)\cdot n\big) \in N_\rho(Q,\cdot)\ \forall\ x \in Q$ *and some* $n \in N_\rho(Q,\cdot)$.

3. $\big[\sigma^2\big(\sigma^{-1}(x)\cdot n\big)\big]^{-1}\sigma(x) \in N_\mu(Q,\cdot)\ \forall\ x \in Q$ *and some* $n \in N_\mu(Q,\cdot)$.

4. $(\sigma(x)\cdot n)^{-1}\sigma(x) \in N_\mu(Q,\cdot)\ \forall\ x \in Q$ *and some* $n \in N_\mu(Q,\cdot)$.

5. $\sigma(xn^{-1}) = \sigma(x)n'^{-1}$ *for all* $x \in Q$ *and some* $n, n' \in N_\mu(Q,\cdot)$.

Proof. This follows from Theorem 20. ∎

Corollary 22. *Let $(Q,\cdot)$ be a RIPL and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$. The following are equivalent*

1. $(H, \circ)$ *is a Bol loop.*
2. (a) $(Q, \cdot)$ *is a Bol loop;*
   (b) $\gamma = R_n^{-1}\ \forall\ \gamma \in A(Q)$ *and some* $n \in N_\rho(Q, \cdot)$.
3. (a) $(Q, \cdot)$ *is a Bol loop;*
   (b) $\gamma = R_n^{-1}\ \forall\ \gamma \in A(Q)$ $, x \in Q$ *and some* $n \in N_\mu(Q, \cdot)$;
   (c) $(x \cdot n)^{-1}x \in N_\mu(Q, \cdot)\ \ni\ \gamma = R_n^{-1}\ \forall\ \gamma \in A(Q)$ $, x \in Q$ *and some* $n \in N_\mu(Q, \cdot)$.

Proof. This is achieved by Corollary 21 with $\sigma = I$. ∎

Theorem 23. *Let $(Q, \cdot)$ be a RIPL with a bijective self map σ and let $(H, \circ)$ be the A-holomorph of $(Q, \cdot)$ with a self map σ' such that $\sigma' : (\alpha, x) \mapsto (\alpha, \sigma(x))$ for all $(\alpha, x) \in H$. The following are equivalent*

1. $(H, \circ)$ *is a* σ'*-GBL.*
2. (a) $(Q, \cdot)$ *is a* σ*-GBL;*
   (b) $\gamma = \sigma\rho\sigma^{-1}$ *for some* $\rho \in \mathcal{P}(Q, \cdot)$ *for all* $\gamma \in A(Q)$;
   (c) $\alpha \in \mathcal{P}(Q, \cdot)$ *for all* $\alpha \in A(Q)$.
3. (a) $(Q, \cdot)$ *is a* σ*-GBL;*
   (b) $\gamma = \sigma J\varphi(\sigma J)^{-1}$ *and* $\alpha = J\varphi J$ *for some* $\varphi \in \Psi(Q, \cdot)$ *and for all* $\gamma, \alpha \in A(Q)$;
   (c) $\alpha = J\varphi J$ *for some* $\varphi \in \Psi(Q, \cdot)$ *and for all* $\alpha \in A(Q, \cdot)$.

Proof. We need Corollary 15.

$$R_{\sigma(x\gamma^{-1})} \in R_{\sigma(x)}\mathcal{P}(Q, \cdot) \Leftrightarrow R_{\sigma(x\gamma^{-1})} = R_{\sigma(x)}\rho \text{ for some } \rho \in \mathcal{P}(Q, \cdot) \Leftrightarrow$$
$$y \cdot \sigma(x\gamma^{-1}) = (y\sigma(x))\rho \Leftrightarrow (I, \sigma^{-1}\gamma^{-1}\sigma, \rho) \in AUT(Q, \cdot) \Leftrightarrow \sigma^{-1}\gamma^{-1}\sigma = \rho \Leftrightarrow$$
$$\gamma = \sigma\rho^{-1}\sigma^{-1} \Leftrightarrow \gamma = \sigma\rho_1\sigma^{-1} \text{ for some } \rho_1 \in \mathcal{P}(Q, \cdot).$$

$$R_{[\sigma(x)]\alpha^{-1}} \in R_{\sigma(x)}\mathcal{P}(Q, \cdot) \Leftrightarrow R_{[\sigma(x)]\alpha^{-1}} = R_{\sigma(x)}\rho \text{ for some } \rho \in \mathcal{P}(Q, \cdot) \Leftrightarrow$$
$$y \cdot [\sigma(x)]\alpha^{-1} = (y\sigma(x))\rho \Leftrightarrow (I, \alpha^{-1}, \rho) \in AUT(Q, \cdot) \Leftrightarrow \alpha = \rho^{-1} \Leftrightarrow$$
$$\alpha = \rho_1 \text{ for some } \rho_1 \in \mathcal{P}(Q, \cdot).$$

$$R_{\sigma(x\gamma^{-1})} \in R_{\sigma(x)}\Phi(Q, \cdot) \Leftrightarrow R_{\sigma(x\gamma^{-1})} = R_{\sigma(x)}\varrho \text{ for some } \varrho \in \Phi(Q, \cdot) \Leftrightarrow$$
$$y \cdot \sigma(x\gamma^{-1}) = (y \cdot \sigma(x))\varrho \Leftrightarrow (I, \sigma^{-1}\gamma^{-1}\sigma, \varrho) \in AUT(Q, \cdot) \Leftrightarrow$$
$$(\varrho, J\sigma^{-1}\gamma^{-1}\sigma J, I) \in AUT(Q, \cdot) \Leftrightarrow (\varrho, (J\sigma^{-1}\gamma\sigma J)^{-1}, I) \in AUT(Q, \cdot) \Leftrightarrow$$
$$\varrho' = J\sigma^{-1}\gamma\sigma J \Leftrightarrow \gamma = \sigma J\varrho'\gamma(\sigma J)^{-1} \Leftrightarrow$$
$$\gamma = \sigma J\varphi\gamma(\sigma J)^{-1} \text{ for some } \varphi \in \Psi(Q, \cdot).$$

$$R_{\sigma(x)}J \in R_{\sigma(x\gamma^{-1})}J\Psi(Q,\cdot) \Leftrightarrow R_{\sigma(x)}J = R_{\sigma(x\gamma^{-1})}J\varphi \text{ for some } \varphi \in \Psi(Q,\cdot) \Leftrightarrow$$
$$(y\cdot\sigma(x))^{-1} = \big([y\cdot\sigma(x\gamma^{-1})]^{-1}\big)\varphi \Leftrightarrow (I,\sigma^{-1}\gamma\sigma, J\varphi J) \in AUT(Q,\cdot) \Leftrightarrow$$
$$(J\varphi J, J\sigma^{-1}\gamma\sigma J, I) \in AUT(Q,\cdot) \Leftrightarrow \big(J\varphi J, (J\sigma^{-1}\gamma^{-1}\sigma J)^{-1}, I\big) \in AUT(Q,\cdot) \Leftrightarrow$$
$$J\varphi J \in \Phi(Q,\cdot) \text{ and } J\sigma^{-1}\gamma^{-1}\sigma J \in \Psi(Q,\cdot) \Leftrightarrow \varrho = J\varphi J \in \Phi(Q,\cdot)$$
$$\text{for some } \varrho \in \Phi(Q,\cdot) \text{ and } \gamma = \sigma J\varphi(\sigma J)^{-1} \text{ for some } \varphi \in \Psi(Q,\cdot)$$

$$R_{[\sigma(x)]\alpha^{-1}} \in R_{\sigma(x)}\Phi(Q,\cdot) \Leftrightarrow R_{[\sigma(x)]\alpha^{-1}} = R_{\sigma(x)}\varrho \text{ for some } \varrho \in \Phi(Q,\cdot) \Leftrightarrow$$
$$y\cdot[\sigma(x)]\alpha^{-1} = (y\cdot\sigma(x))\varrho \Leftrightarrow (I,\alpha^{-1},\varrho) \in AUT(Q,\cdot) \Leftrightarrow (\varrho, J\alpha^{-1}J, I) \in AUT(Q,\cdot)$$
$$\Leftrightarrow (\varrho, (J\alpha J)^{-1}, I) \in AUT(Q,\cdot) \Leftrightarrow \varrho' = J\alpha J \Leftrightarrow \alpha = J\varphi J \text{ for some } \varphi \in \Psi(Q,\cdot).$$

$$R_{\sigma(x)}J \in R_{[\sigma(x)]\alpha^{-1}}J\Psi(Q,\cdot) \Leftrightarrow R_{\sigma(x)}J = R_{[\sigma(x)]\alpha^{-1}}J\varphi \text{ for some } \varphi \in \Psi(Q,\cdot) \Leftrightarrow$$
$$(y\cdot\sigma(x))^{-1} = \big(\big[y\cdot[\sigma(x)]\alpha^{-1}\big]^{-1}\big)\varphi \Leftrightarrow (I,\alpha,J\varphi J,) \in AUT(Q,\cdot) \Leftrightarrow$$
$$(J\varphi J, J\alpha J, I) \in AUT(Q,\cdot) \Leftrightarrow \big(J\varphi J, (J\alpha^{-1}J)^{-1}, I\big) \in AUT(Q,\cdot) \Leftrightarrow$$
$$J\varphi J \in \Phi(Q,\cdot) \text{ and } J\alpha^{-1}J \in \Psi(Q,\cdot) \Leftrightarrow \varrho = J\varphi J \text{ for some } \varrho \in \Phi(Q,\cdot)$$
$$\text{and } \alpha = J\varphi J \text{ for some } \varphi \in \Psi(Q,\cdot).$$

■

Corollary 24. *Let $(Q,\cdot)$ be a RIPL with a bijective self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma' : (\alpha,x) \mapsto (\alpha,\sigma(x))$ for all $(\alpha,x) \in H$. If $(H,\circ)$ is a σ'-GBL, then*

$$A(Q) = \big\{\sigma\rho\sigma^{-1}, \rho, \sigma J\varphi(\sigma J)^{-1}, J\varphi J \,|\, \textit{for some } \rho \in \mathcal{P}(Q,\cdot) \textit{ and some } \varphi \in \Psi(Q,\cdot)\big\}.$$

Proof. Use Theorem 23. ■

Corollary 25. *Let $(Q,\cdot)$ be a RIPL and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$. The following are equivalent*

1. *$(H,\circ)$ is a Bol loop.*
2. (a) *$(Q,\cdot)$ is a Bol loop;*
   (b) *$\alpha,\gamma \in \mathcal{P}(Q,\cdot)$ for all $\alpha,\gamma \in A(Q)$;*
3. (a) *$(Q,\cdot)$ is a Bol loop;*
   (b) *$\alpha,\gamma \in J\Psi(Q,\cdot)J$ for all $\alpha,\gamma \in A(Q)$;*
   (c) *$\varrho = J\varphi J$ for some $\varphi \in \Psi(Q,\cdot)$ and some $\varrho \in \Phi(Q,\cdot)$.*

Proof. Apply Theorem 23 with $\sigma = I$. ■

Corollary 26. *Let $(Q,\cdot)$ be a RIPL and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$. If $(H,\circ)$ is a Bol loop, then*

$$A(Q)=\big\{\rho, J\varphi J|\ \textit{for some}\ \rho\in\mathcal{P}(Q,\cdot)\ \textit{and some}\ \varphi\in\Psi(Q,\cdot)\big\}.$$

Proof. Use Corollary 24 with $\sigma=I$. ■

Theorem 27. *Let $(Q,\cdot)$ be a RIPL with identity element e and a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma':(\alpha,x)\mapsto(\alpha,\sigma(x))$ for all $(\alpha,x)\in H$. Let*

$$\psi:\mathcal{P}(Q,\cdot)\to N_\rho(Q,\cdot)\uparrow\psi(U)=eU,\ \ \phi:\Phi(Q,\cdot)\to\Psi(Q,\cdot)\ \uparrow\phi(U)=U',$$

$$\varpi:\Phi(Q,\cdot)\to N_\mu(Q,\cdot)\uparrow\varpi(U)=eU\ \textit{and}\ \beta:\Psi(Q,\cdot)\to N_\mu(Q,\cdot)\uparrow\beta(U')=eU'.$$

If $(H,\circ)$ is a σ'-GBL, then

1. $R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\overset{\psi,\varpi}{\cong}\sigma(x)^{-1}\sigma(x\gamma^{-1})\ \forall\ \gamma\in A(Q)\ ,x\in Q.$
2. $R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\overset{\psi,\varpi}{\cong}\sigma(x)^{-1}[\sigma(x)]\alpha^{-1}\ \forall\ \alpha\in A(Q)\ ,x\in Q.$
3. $JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J\overset{\beta}{\cong}\sigma(x)^{-1}\sigma(x\gamma^{-1})\ \forall\ \gamma\in A(Q)\ ,x\in Q.$
4. $JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J\overset{\beta}{\cong}\sigma(x)^{-1}[\sigma(x)]\alpha^{-1}\ \forall\ \alpha\in A(Q)\ ,x\in Q.$
5. $R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}\overset{\phi}{\cong}R^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J\ \forall\ \gamma\in A(Q)\ ,x\in Q.$
6. $R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}\overset{\phi}{\cong}JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J\ \forall\ \alpha\in A(Q)\ ,x\in Q.$

Proof. This is achieved by using Theorem 19, Corollary 14 and Theorem 7. ■

Theorem 28. *Let $(Q,\cdot)$ be a RIPL with a self map σ and let $(H,\circ)$ be the A-holomorph of $(Q,\cdot)$ with a self map σ' such that $\sigma':(\alpha,x)\mapsto(\alpha,\sigma(x))$ for all $(\alpha,x)\in H$. If $(H,\circ)$ is a σ'-GBL, then*

1. *the correspondence*

$$\begin{tikzcd}
 & \sigma(x)^{-1}\sigma(x\gamma^{-1}) \\
R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})} \arrow[ur, leftrightarrow, "\psi,\varpi"] \arrow[r, mapsto, "\phi", "\textit{isomorphism}"'] & JR^{-1}_{\sigma(x\gamma^{-1})}R_{\sigma(x)}J \arrow[u, "\beta"', "\textit{isomorphism}"]
\end{tikzcd}$$

is true for all $\gamma\in A(Q)$ and $x\in Q$, $\psi=\phi\beta$ and $\varpi=\phi\beta$.

2. *the correspondence*

$$\begin{tikzcd}
 & \sigma(x)^{-1}[\sigma(x)]\alpha^{-1} \\
R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} \arrow[r, mapsto, "\phi", "isomorphism"'] \arrow[ur, leftrightarrow, "{\psi,\varpi}"] & JR^{-1}_{[\sigma(x)]\alpha^{-1}}R_{\sigma(x)}J \arrow[u, "\beta", "isomorphism"']
\end{tikzcd}$$

is true for all $\alpha \in A(Q)$ *and* $x \in Q$, $\psi = \phi\beta$ *and* $\varpi = \phi\beta$.

3. *the commutative diagram*

$$\begin{tikzcd}
\mathcal{P}(L,\cdot) \arrow[r, "\psi"] \arrow[d, dotted, "\delta_1"'] & N_\mu(L,\cdot) \\
\Phi(L,\cdot) \arrow[r, "\phi", "isomorphism"'] \arrow[ur, "\varpi"] & \Psi(L,\cdot) \arrow[u, "\beta", "isomorphism"']
\end{tikzcd}$$

is true, $\delta_1 = \psi\beta^{-1}\phi^{-1} = \psi\varpi^{-1}$ *and* $R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})} \overset{\delta_1}{\cong} R^{-1}_{\sigma(x)}R_{\sigma(x\gamma^{-1})}$ *for all* $\gamma \in A(Q)$ *and* $x \in Q$.

4. *the commutative diagram*

$$\begin{tikzcd}
\mathcal{P}(L,\cdot) \arrow[r, "\psi"] & N_\mu(L,\cdot) \\
\Phi(L,\cdot) \arrow[r, "\phi", "isomorphism"'] \arrow[ur, "\varpi"] \arrow[u, dotted, "\delta_2"] & \Psi(L,\cdot) \arrow[u, "\beta", "isomorphism"']
\end{tikzcd}$$

is true, $\varpi = \phi\beta = \delta_2\psi$ *and* $R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}} \overset{\delta_2}{\cong} R^{-1}_{\sigma(x)}R_{[\sigma(x)]\alpha^{-1}}$ *for all* $\alpha \in A(Q)$ *and* $x \in Q$.

Proof. The proof follows from Theorem 27 and Theorem 7. ■

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