

PSEUDO-BCH-ALGEBRAS

ANDRZEJ WALENDZIAK

Institute of Mathematics and Physics
Siedlce University
3 Maja 54, 08–110 Siedlce, Poland
e-mail: walent@interia.pl

Abstract

The notion of pseudo-BCH-algebras is introduced, and some of their properties are investigated. Conditions for a pseudo-BCH-algebra to be a pseudo-BCI-algebra are given. Ideals and minimal elements in pseudo-BCH-algebras are considered.

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1. INTRODUCTION

In 1966, Y. Imai and K. Iséki ([10, 11]) introduced BCK- and BCI-algebras. In 1983, Q.P. Hu and X. Li ([9]) introduced BCH-algebras. It is known that BCK- and BCI-algebras are contained in the class of BCH-algebras. J. Neggers and H.S. Kim ([16]) defined d-algebras which are a generalization of BCK-algebras.

In 2001, G. Georgescu and A. Iorgulescu ([8]) introduced the pseudo-BCK-algebras as an extension of BCK-algebras. In 2008, W.A. Dudek and Y.B. Jun ([3]) introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu in [6] and [7], respectively. Those algebras were investigated by several authors in [4, 5, 14] and [15]. As a generalization of d-algebras, Y.B. Jun, H.S. Kim and J. Neggers ([13]) introduced pseudo-d-algebras. Recently, R.A. Borzooei *et al.* ([1]) defined pseudo-BE-algebras.

In this paper we introduce pseudo-BCH-algebras as an extension of BCH-algebras. We give basic properties of pseudo-BCH-algebras and provide some

conditions for a pseudo-BCH-algebra to be a pseudo-BCI-algebra. Moreover we study the set $\text{Cen}\mathfrak{X}$ of all minimal elements of a pseudo-BCH-algebra \mathfrak{X} , the so-called centre of \mathfrak{X} . We also consider ideals in pseudo-BCH-algebras and establish a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre. Finally we show that the centre of a pseudo-BCH-algebra \mathfrak{X} defines a regular congruence on \mathfrak{X} .

2. DEFINITION AND EXAMPLES OF PSEUDO-BCH-ALGEBRAS

We recall that an algebra $\mathfrak{X} = (X; *, 0)$ of type $(2, 0)$ is called a *BCH-algebra* if it satisfies the following axioms:

- (BCH-1) $x * x = 0$;
- (BCH-2) $(x * y) * z = (x * z) * y$;
- (BCH-3) $x * y = y * x = 0 \implies x = y$.

A BCH-algebra \mathfrak{X} is said to be a *BCI-algebra* if it satisfies the identity

$$(\text{BCI}) \quad ((x * y) * (x * z)) * (z * y) = 0.$$

A *BCK-algebra* is a BCI-algebra \mathfrak{X} satisfying the law $0 * x = 0$.

Definition 2.1 ([3]). A pseudo-BCI-algebra is a structure $\mathfrak{X} = (X; \leq, *, \diamond, 0)$, where " \leq " is a binary relation on the set X , " $*$ " and " \diamond " are binary operations on X and " 0 " is an element of X , satisfying the axioms:

- (pBCI-1) $(x * y) \diamond (x * z) \leq z * y, \quad (x \diamond y) * (x \diamond z) \leq z \diamond y$;
- (pBCI-2) $x * (x \diamond y) \leq y, \quad x \diamond (x * y) \leq y$;
- (pBCI-3) $x \leq x$;
- (pBCI-4) $x \leq y, y \leq x \implies x = y$;
- (pBCI-5) $x \leq y \iff x * y = 0 \iff x \diamond y = 0$.

A pseudo-BCI-algebra \mathfrak{X} is called a *pseudo-BCK-algebra* if it satisfies the identities

$$(\text{pBCK}) \quad 0 * x = 0 \diamond x = 0.$$

Definition 2.2. A pseudo-BCH-algebra is an algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ satisfying the axioms:

- (pBCH-1) $x * x = x \diamond x = 0$;
- (pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;
- (pBCH-3) $x * y = y \diamond x = 0 \implies x = y$;
- (pBCH-4) $x * y = 0 \iff x \diamond y = 0$.

Remark 2.3. Observe that if $(X; *, 0)$ is a BCH-algebra, then letting $x \diamond y := x * y$, produces a pseudo-BCH-algebra $(X; *, \diamond, 0)$. Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if $(X; *, \diamond, 0)$ is a pseudo-BCH-algebra, then $(X; \diamond, *, 0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [3] we conclude that if $(X; \leq, *, \diamond, 0)$ is a pseudo-BCI-algebra, then $(X; *, \diamond, 0)$ is a pseudo-BCH-algebra.

We say that a pseudo-BCH-algebra \mathfrak{X} is *proper* if $* \neq \diamond$ and it is not a pseudo-BCI-algebra.

Remark 2.4. The class of all pseudo-BCH-algebras is a quasi-variety. Therefore, if \mathfrak{X}_1 and \mathfrak{X}_2 are two pseudo-BCH-algebras, then the direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is also a pseudo-BCH-algebra. In the case when at least one of \mathfrak{X}_1 and \mathfrak{X}_2 is proper, then \mathfrak{X} is proper.

Example 2.5. Let $X_1 = \{0, a, b, c\}$. We define the binary operations $*_1$ and \diamond_1 on X_1 as follows:

$*_1$	0	a	b	c		\diamond_1	0	a	b	c
0	0	0	0	0		0	0	0	0	0
a	a	0	a	0	and	a	a	0	a	0
b	b	b	0	0		b	b	b	0	0
c	c	b	c	0		c	c	c	a	0

It is easy to check that $\mathfrak{X}_1 = (X_1; *_1, \diamond_1, 0)$ is a pseudo-BCH-algebra. On the set $X_2 = \{0, 1, 2, 3\}$ consider the operation $*_2$ given by the following table:

$*_2$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

By simple calculation we can get that $\mathfrak{X}_2 = (X_2; *_2, *_2, 0)$ is a (pseudo)-BCH-algebra. The direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a pseudo-BCH-algebra. Observe that \mathfrak{X} is proper. Let $x = (a, 1)$, $y = (a, 3)$ and $z = (a, 2)$. Then $(x * y) \diamond (x * z) = (0, 1) \diamond (0, 0) = (0, 1)$ and $z * y = (0, 0)$. Since $(0, 1) \not\leq (0, 0)$, we conclude that \mathfrak{X} is not a pseudo-BCI-algebra, and therefore it is a proper pseudo-BCH-algebra.

Proposition 2.6. Any (proper) pseudo-BCH-algebra satisfying (pBCK) can be extended to a (proper) pseudo-BCH-algebra containing one element more.

Proof. Let $\mathfrak{X} = (X; *, \diamond, 0)$ be a pseudo-BCH-algebra satisfying (pBCK) and let $\delta \notin X$. On the set $Y = X \cup \{\delta\}$ consider the operations:

$$x *' y = \begin{cases} x * y & \text{if } x, y \in X, \\ \delta & \text{if } x = \delta \text{ and } y \in X, \\ 0 & \text{if } x \in Y \text{ and } y = \delta, \end{cases}$$

and

$$x \diamond' y = \begin{cases} x \diamond y & \text{if } x, y \in X, \\ \delta & \text{if } x = \delta \text{ and } y \in X, \\ 0 & \text{if } x \in Y \text{ and } y = \delta. \end{cases}$$

Obviously, $(Y; *', \diamond', 0)$ satisfies the axioms (pBCH-1), (pBCH-3), and (pBCH-4). Further, the axiom (pBCH-2) is easily satisfied for all $x, y, z \in X$. Moreover, by routine calculation we can verify it in the case when at least one of x, y, z is equal to δ . Thus, by definition, $(Y; *', \diamond', 0)$ is a pseudo-BCH-algebra. Clearly, if \mathfrak{X} is a proper pseudo-BCH-algebra, then $(Y; *', \diamond', 0)$ is also a proper pseudo-BCH-algebra. ■

From Example 2.5 and Proposition 2.6 we conclude that there are infinite many proper pseudo-BCH-algebras.

3. PROPERTIES OF PSEUDO-BCH-ALGEBRAS

Let $\mathfrak{X} = (X; *, \diamond, 0)$ be a pseudo-BCH-algebra. Define the relation \leq on X by $x \leq y$ if and only if $x * y = 0$ (or equivalently, $x \diamond y = 0$).

For any $x \in X$ and $n = 0, 1, 2, \dots$, we put

$$\begin{aligned} 0 *^0 x &= 0 & \text{and} & & 0 *^{n+1} x &= (0 *^n x) * x; \\ 0 \diamond^0 x &= 0 & \text{and} & & 0 \diamond^{n+1} x &= (0 \diamond^n x) \diamond x. \end{aligned}$$

Proposition 3.1. *In a pseudo-BCH-algebra \mathfrak{X} the following properties hold (for all $x, y, z \in X$):*

- (P1) $x \leq y, y \leq x \implies x = y$;
- (P2) $x \leq 0 \implies x = 0$;
- (P3) $x * (x \diamond y) \leq y, \quad x \diamond (x * y) \leq y$;
- (P4) $x * 0 = x = x \diamond 0$;
- (P5) $0 * x = 0 \diamond x$;
- (P6) $x \leq y \implies 0 * x = 0 \diamond y$;
- (P7) $0 \diamond (0 * (0 \diamond x)) = 0 \diamond x, \quad 0 * (0 \diamond (0 * x)) = 0 * x$;
- (P8) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$;
- (P9) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$.

Proof. (P1) follows from (pBCH-3).

(P2) Let $x \leq 0$. Then $x * 0 = 0$. Applying (pBCH-2) and (pBCH-1) we obtain

$$0 \diamond x = (x * 0) \diamond x = (x \diamond x) * 0 = 0 * 0 = 0,$$

that is, $0 \leq x$. Therefore $x = 0$ by (P1).

(P3) Using (pBCH-2) and (pBCH-1) we have $(x * (x \diamond y)) \diamond y = (x \diamond y) * (x \diamond y) = 0$. Hence $x * (x \diamond y) \leq y$. Similarly, $x \diamond (x * y) \leq y$.

(P4) Putting $y = 0$ in (P3), we have $x * (x \diamond 0) \leq 0$ and $x \diamond (x * 0) \leq 0$. From (P2) we obtain $x * (x \diamond 0) = 0$ and $x \diamond (x * 0) = 0$. Thus $x \leq x \diamond 0$ and $x \leq x * 0$.

On the other hand, $(x \diamond 0) * x = (x * x) \diamond 0 = 0 \diamond 0 = 0$ and $(x * 0) \diamond x = (x \diamond x) * 0 = 0 * 0 = 0$, and so $x \diamond 0 \leq x$ and $x * 0 \leq x$. By (P1), $x * 0 = x = x \diamond 0$.

(P5) Applying (pBCH-1) and (pBCH-2) we get $0 * x = (x \diamond x) * x = (x * x) \diamond x = 0 \diamond x$.

(P6) Let $x \leq y$. Then $x \diamond y = 0$ and therefore $0 * x = (x \diamond y) * x = (x * x) \diamond y = 0 \diamond y$.

(P7) From (P3) it follows that $0 * (0 \diamond x) \leq x$ and $0 \diamond (0 * x) \leq x$. Hence, using (P5) and (P6) we obtain (P7).

(P8) Applying (pBCH-1) and (pBCH-2) we have

$$\begin{aligned} (0 \diamond x) \diamond (0 * y) &= (((x * y) * (x * y)) \diamond x) \diamond (0 * y) \\ &= (((x * y) \diamond x) * (x * y)) \diamond (0 * y) \\ &= (((x \diamond x) * y) * (x * y)) \diamond (0 * y) \\ &= ((0 * y) * (x * y)) \diamond (0 * y) \\ &= ((0 * y) \diamond (0 * y)) * (x * y) \\ &= 0 * (x * y). \end{aligned}$$

(P9) The proof is similar to the proof of (P8). ■

From (P1) and (P3) we get

Corollary 3.2. *Every pseudo-BCH-algebra satisfies (pBCI-2)–(pBCI-5).*

Remark 3.3. In any pseudo-BCI-algebra the relation \leq is transitive (see [3], Proposition 3.2). However, in the pseudo-BCH-algebra \mathfrak{X} from Example 2.5 we have $(a, 1) \leq (a, 2)$ and $(a, 2) \leq (a, 3)$ but $(a, 1) \not\leq (a, 3)$.

Theorem 3.4. *Let \mathfrak{X} be a pseudo-BCH-algebra. Then \mathfrak{X} is a pseudo-BCI-algebra if and only if it satisfies the following implication:*

$$(3.1) \quad x \leq y \implies x * z \leq y * z, \quad x \diamond z \leq y \diamond z.$$

Proof. If \mathfrak{X} is a pseudo-BCI-algebra, then \mathfrak{X} satisfies (3.1) by Proposition 3.2 (b7) of [3]. Conversely, let (3.1) hold in \mathfrak{X} and let $x, y, z \in X$. By (P3), $x \diamond (x * z) \leq z$ and $x * (x \diamond z) \leq z$. Hence $(x \diamond (x * z)) * y \leq z * y$ and $(x * (x \diamond z)) \diamond y \leq z \diamond y$, and so $(x * y) \diamond (x * z) \leq z * y$ and $(x \diamond y) * (x \diamond z) \leq z \diamond y$. Therefore, \mathfrak{X} satisfies (pBCI-1). Consequently, \mathfrak{X} is a pseudo-BCI-algebra. ■

Theorem 3.5. *Let \mathfrak{X} be a pseudo-BCH-algebra. The following statements are equivalent:*

- (i) $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$;
- (ii) $0 * x = x = 0 \diamond x$ for every $x \in X$;
- (iii) $x * y = x \diamond y = y * x$ for all $x, y \in X$;
- (iv) $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ for all $x, y, z \in X$.

Proof. (i) \implies (ii). Let $x \in X$. We have $x = x * 0 = x * (x * x) = (x * x) * x = 0 * x$. By (P5), $0 \diamond x = x$.

(iv) \implies (ii). The proof is similar to the above proof.

(ii) \implies (iii). Let (ii) hold and $x, y \in X$. Applying (P8) and (pBCH-2) we obtain

$$\begin{aligned} x * y &= 0 * (x * y) = (0 \diamond x) \diamond (0 * y) \\ &= x \diamond y \\ &= (0 * x) \diamond y = (0 \diamond y) * x = y * x. \end{aligned}$$

(iii) \implies (i). Let $x, y, z \in X$. Using (iii) and (pBCH-2) we get

$$x * (y * z) = (y \diamond z) * x = (y * x) \diamond z = (x * y) * z.$$

(iii) \implies (iv) has a proof similar to the proof of implication (iii) \implies (i).

Hence all the conditions are equivalent. ■

Corollary 3.6. *If \mathfrak{X} is a pseudo-BCH-algebra satisfying the identity $0 * x = x$, then $(X; *, 0)$ is an Abelian group each element of which has order 2 (that is, a Boolean group).*

4. THE CENTRE OF A PSEUDO-BCH-ALGEBRA. IDEALS

An element a of a pseudo-BCH-algebra \mathfrak{X} is said to be *minimal* if for every $x \in X$ the following implication

$$x \leq a \implies x = a$$

holds.

Proposition 4.1. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $a \in X$. Then the following conditions are equivalent (for every $x \in X$):*

- (i) a is minimal;
- (ii) $x \diamond (x * a) = a$;
- (iii) $0 \diamond (0 * a) = a$;
- (iv) $a * x = (0 * x) \diamond (0 * a)$;
- (v) $a * x = 0 \diamond (x * a)$.

Proof. (i) \implies (ii). By (P2), $x \diamond (x * a) \leq a$ for all $x \in X$. Since a is minimal, we get (ii).

(ii) \implies (iii). Obvious.

(iii) \implies (iv). We have $a * x = (0 \diamond (0 * a)) * x = (0 * x) \diamond (0 * a)$.

(iv) \implies (v). Applying (P5) and (P8) we see that

$$0 \diamond (x * a) = 0 * (x * a) = (0 \diamond x) \diamond (0 * a) = (0 * x) \diamond (0 * a) = a * x.$$

(v) \implies (i). Let $x \leq a$. Then $x * a = 0$ and hence $a * x = 0 \diamond (x * a) = 0$. Thus $a \leq x$. Consequently, $x = a$. ■

Replacing $*$ by \diamond and \diamond by $*$ in Proposition 4.1 we obtain

Proposition 4.2. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $a \in X$. Then for every $x \in X$ the following conditions are equivalent:*

- (i) a is minimal;
- (ii) $x * (x \diamond a) = a$;
- (iii) $0 * (0 \diamond a) = a$;
- (iv) $a \diamond x = (0 \diamond x) * (0 \diamond a)$;
- (v) $a \diamond x = 0 * (x \diamond a)$.

Proposition 4.3. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $a \in X$. Then a is minimal if and only if there is an element $x \in X$ such that $a = 0 * x$.*

Proof. Let a be a minimal element of \mathfrak{X} . By Proposition 4.2, $a = 0 * (0 \diamond a)$. If we set $x = 0 \diamond a$, then $a = 0 * x$.

Conversely, suppose that $a = 0 * x$ for some $x \in X$. Using (P7) we get

$$0 * (0 \diamond a) = 0 * (0 \diamond (0 * x)) = 0 * x = a.$$

From Proposition 4.2 it follows that a is minimal. ■

For $x \in X$, set

$$\bar{x} = 0 \diamond (0 * x).$$

By (P5), $\bar{x} = 0 * (0 * x) = 0 \diamond (0 \diamond x) = 0 * (0 \diamond x)$.

Proposition 4.4. *Let \mathfrak{X} be a pseudo-BCH-algebra. For any $x, y \in X$ we have:*

- (a) $\overline{x * y} = \bar{x} * \bar{y}$;
- (b) $\overline{x \diamond y} = \bar{x} \diamond \bar{y}$;
- (c) $\bar{\bar{x}} = \bar{x}$.

Proof. (a) Applying (P8) and (P9) we get

$$\begin{aligned} \overline{x * y} &= 0 \diamond (0 * (x * y)) = 0 \diamond [(0 \diamond x) \diamond (0 * y)] \\ &= [0 * (0 \diamond x)] * [0 \diamond (0 * y)] = \bar{x} * \bar{y}. \end{aligned}$$

(b) has a proof similar to (a).

(c) By (P7), $0 * (0 \diamond (0 * x)) = 0 * x$, that is, $0 * \bar{x} = 0 * x$. Hence $\bar{\bar{x}} = 0 \diamond (0 * \bar{x}) = 0 \diamond (0 * x) = \bar{x}$. ■

Following the terminology from BCH-algebras (see [2], Definition 5) the set $\{x \in X : x = \bar{x}\}$ will be called the *centre* of \mathfrak{X} . We shall denote it by $\text{Cen}\mathfrak{X}$. By Proposition 4.1, $\text{Cen}\mathfrak{X}$ is the set of all minimal elements of \mathfrak{X} . We have

$$(4.1) \quad \text{Cen}\mathfrak{X} = \{\bar{x} : x \in X\}.$$

Define $\Phi : \mathfrak{X} \rightarrow \text{Cen}\mathfrak{X}$ by $\Phi(x) = \bar{x}$ for all $x \in X$. By Proposition 4.4, Φ is a homomorphism from \mathfrak{X} onto $\text{Cen}\mathfrak{X}$. We also obtain

Proposition 4.5. *Let \mathfrak{X} be a pseudo-BCH-algebra. Then $\text{Cen}\mathfrak{X}$ is a subalgebra of \mathfrak{X} .*

Proposition 4.6. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $x, y \in \text{Cen}\mathfrak{X}$. Then for every $z \in X$ we have*

$$(4.2) \quad x \diamond (z * y) = y * (z \diamond x).$$

Proof. Let $z \in X$. Using Propositions 4.2 and 4.1 we obtain

$$x \diamond (z * y) = [z * (z \diamond x)] \diamond (z * y) = [z \diamond (z * y)] * (z \diamond x) = y * (z \diamond x),$$

that is, (4.2) holds. ■

Following [5], a pseudo-BCI-algebra $(X; \leq, *, \diamond, 0)$ is said to be *p-semisimple* if it satisfies for all $x \in X$,

$$0 \leq x \implies x = 0.$$

From Theorem 3.1 of [5] it follows that if $\mathfrak{X} = (X; \leq, *, \diamond, 0)$ is a pseudo-BCI-algebra, then \mathfrak{X} is p -semisimple if and only if $x = \bar{x}$ for every $x \in X$ (that is, $\text{Cen}\mathfrak{X} = \mathfrak{X}$).

Theorem 4.7. *Let \mathfrak{X} be a pseudo-BCH-algebra. Then $\text{Cen}\mathfrak{X}$ is a p -semisimple pseudo-BCI-algebra.*

Proof. Since $\text{Cen}\mathfrak{X}$ is a subalgebra of \mathfrak{X} , $\text{Cen}\mathfrak{X}$ is a pseudo-BCH-algebra. Let $x, y, z \in \text{Cen}\mathfrak{X}$ and let $x \leq y$. Since x and y are minimal elements of \mathfrak{X} , we get $x = y$. Hence $x * z \leq y * z$ and $x \diamond z \leq y \diamond z$. Then, by Theorem 3.4, $\text{Cen}\mathfrak{X}$ is a pseudo-BCI-algebra. Obviously, $x = \bar{x}$ for every $x \in \text{Cen}\mathfrak{X}$, and therefore $\text{Cen}\mathfrak{X}$ is p -semisimple. ■

Remark 4.8. From Theorem 3.6 of [5] we deduce that $(\text{Cen}\mathfrak{X}; +, 0)$ is a group, where $x + y$ is $x * (0 \diamond y)$ for all $x, y \in \text{Cen}\mathfrak{X}$.

Definition 4.9. Let X be a pseudo-BCH-algebra. A subset I of X is called an ideal of X if it satisfies for all $x, y \in X$

(I1) $0 \in I$;

(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\text{Id}(\mathfrak{X})$ the set of all ideals of \mathfrak{X} . Obviously, $\{0\}, X \in \text{Id}(\mathfrak{X})$.

Proposition 4.10. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $I \in \text{Id}(\mathfrak{X})$. For any $x, y \in X$, if $y \in I$ and $x \leq y$, then $x \in I$.*

Proof. Straightforward. ■

Proposition 4.11. *Let \mathfrak{X} be a pseudo-BCH-algebra and I be a subset of X satisfying (I1). Then I is an ideal of \mathfrak{X} if and only if for all $x, y \in X$,*

(I2') if $x \diamond y \in I$ and $y \in I$, then $x \in I$.

Proof. Let I be an ideal of \mathfrak{X} . Suppose that $x \diamond y \in I$ and $y \in I$. By (P3), $x * (x \diamond y) \leq y$ and from Proposition 4.10 it follows that $x * (x \diamond y) \in I$. Therefore, since $x \diamond y \in I$ and I satisfies (I2), we obtain $x \in I$, that is, (I2') holds. The proof of the implication (I2') \Rightarrow (I2) is analogous. ■

Example 4.12. Let $X = \{0, a, b, c, d\}$. Define binary operations $*$ and \diamond on X by the following tables:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	b	c	0	d
d	d	d	d	d	0

\diamond	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	c	a	0	d
d	d	d	d	d	0

By routine calculation, $\mathfrak{X} = (X; *, \diamond, 0)$ is a pseudo-BCH-algebra. It is easy to see that $\text{Id}(\mathfrak{X}) = \{\{0\}, \{0, a\}, \{0, b\}, \{0, a, b, c\}, X\}$.

The following two propositions give the homomorphic properties of ideal.

Proposition 4.13. *Let \mathfrak{X} and \mathfrak{Y} be pseudo-BCH-algebras. If $\varphi : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a homomorphism and $J \in \text{Id}(\mathfrak{Y})$, then the inverse image $\varphi^{-1}(J)$ of J is an ideal of \mathfrak{X} .*

Proof. Straightforward. ■

Proposition 4.14. *Let $\varphi : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a surjective homomorphism. If I is an ideal of \mathfrak{X} containing $\varphi^{-1}(0)$, then $\varphi(I)$ is an ideal of \mathfrak{Y} .*

Proof. Since $0 \in I$, we have $0 = \varphi(0) \in \varphi(I)$. Let $x, y \in Y$ and suppose that $x * y, y \in \varphi(I)$. Then there are $a \in X$ and $b, c \in I$ such that $x = \varphi(a)$, $y = \varphi(b)$ and $x * y = \varphi(c)$. We have $\varphi(a * b) = \varphi(c)$ and hence $(a * b) * c \in \varphi^{-1}(0) \subseteq I$. By the definition of an ideal, $a \in I$. Consequently, $x = \varphi(a) \in \varphi(I)$. This means that $\varphi(I)$ is an ideal of \mathfrak{Y} . ■

Definition 4.15. An ideal I of a pseudo-BCH-algebra \mathfrak{X} is said to be *closed* if $0 * x \in I$ for every $x \in I$.

Theorem 4.16. *An ideal I of a pseudo-BCH-algebra \mathfrak{X} is closed if and only if I is a subalgebra of \mathfrak{X} .*

Proof. Suppose that I is a closed ideal of \mathfrak{X} and let $x, y \in I$. By (pBCH-2) and (pBCH-1),

$$\begin{aligned} [(x * y) * (0 * y)] \diamond x &= [(x * y) \diamond x] * (0 * y) \\ &= [(x \diamond x) * y] * (0 * y) \\ &= (0 * y) * (0 * y) = 0. \end{aligned}$$

Hence $[(x * y) * (0 * y)] \diamond x \in I$. Since $x, 0 * y \in I$, we have $x * y \in I$. Similarly, $x \diamond y \in I$. Conversely, if I is a subalgebra of \mathfrak{X} , then $x \in I$ and $0 \in I$ imply $0 * x \in I$. ■

Theorem 4.17. *Every ideal of a finite pseudo-BCH-algebra is closed.*

Proof. Let I be an ideal of a finite pseudo-BCH-algebra \mathfrak{X} and let $a \in I$. Suppose that $|X| = n$ for some $n \in \mathbb{N}$. At least two of the $n + 1$ elements:

$$0, 0 * a, 0 *^2 a, \dots, 0 *^n a$$

are equal, for instance, $0 *^r a = 0 *^s a$, where $0 \leq s < r \leq n$. Hence

$$0 = (0 *^r a) \diamond (0 *^s a) = [(0 *^s a) \diamond (0 *^s a)] *^{r-s} a = 0 *^{r-s} a.$$

Therefore $0 *^{r-s} a \in I$. Since $a \in I$, by definition, $0 * a \in I$. Consequently, I is a closed ideal of \mathfrak{X} . ■

For any pseudo-BCH-algebra \mathfrak{X} , we set

$$K(\mathfrak{X}) = \{x \in X : 0 \leq x\}.$$

Observe that $\text{Cen}\mathfrak{X} \cap K(\mathfrak{X}) = \{0\}$. Indeed, $0 \in \text{Cen}\mathfrak{X} \cap K(\mathfrak{X})$ and if $x \in \text{Cen}\mathfrak{X} \cap K(\mathfrak{X})$, then $x = 0 \diamond (0 * x) = 0 \diamond 0 = 0$.

In Example 4.12, $\text{Cen}\mathfrak{X} = \{0, d\}$ and $K(\mathfrak{X}) = \{0, a, b, c\}$.

It is easy to see that

$$x \in K(\mathfrak{X}) \iff \bar{x} = 0 \iff x \in \Phi^{-1}(0).$$

Thus

$$(4.3) \quad K(\mathfrak{X}) = \Phi^{-1}(0).$$

Proposition 4.18. *Let \mathfrak{X} be a pseudo-BCH-algebra. Then $K(\mathfrak{X})$ is a closed ideal of \mathfrak{X} .*

Proof. By (4.3) and Proposition 4.13, $K(\mathfrak{X})$ is an ideal of \mathfrak{X} . Let $x \in K(\mathfrak{X})$. Then $\bar{x} = 0$ and hence $\Phi(0 * x) = 0 * \bar{x} = 0$. Consequently, $0 * x \in K(\mathfrak{X})$. Thus $K(\mathfrak{X})$ is a closed ideal. ■

Corollary 4.19. *For any pseudo-BCH-algebra \mathfrak{X} the set $K(\mathfrak{X})$ is a subalgebra of \mathfrak{X} , and so it is a pseudo-BCH-algebra.*

Proposition 4.20. *Let \mathfrak{X} and \mathfrak{Y} be pseudo-BCH-algebras. Then:*

- (a) $\text{Cen}(\mathfrak{X} \times \mathfrak{Y}) = \text{Cen}(\mathfrak{X}) \times \text{Cen}(\mathfrak{Y})$;
- (b) $K(\mathfrak{X} \times \mathfrak{Y}) = K(\mathfrak{X}) \times K(\mathfrak{Y})$.

Proof. This is immediate from definitions. ■

For any element a of a pseudo-BCH-algebra \mathfrak{X} , we define a subset $V(a)$ of X as

$$V(a) = \{x \in X : a \leq x\}.$$

Note that $V(a) \neq \emptyset$, because $a \leq a$ gives $a \in V(a)$. Furthermore, $V(0) = K(\mathfrak{X})$.

Proposition 4.21. *Let \mathfrak{X} be a pseudo-BCH-algebra. Then for each $x \in X$ there exists a unique element $a \in \text{Cen}\mathfrak{X}$ such that $a \leq x$.*

Proof. Let $x \in X$. Take $a = \bar{x}$, that is, $a = 0 \diamond (0 * x)$. By (P3), $a \leq x$. From (4.1) it follows that $a \in \text{Cen}\mathfrak{X}$. To prove uniqueness, let $b \in \text{Cen}\mathfrak{X}$ be such that $b \leq x$. Then $b \diamond x = 0$. Therefore,

$$0 * b = (b \diamond x) * b = (b * b) \diamond x = 0 \diamond x = 0 * x$$

and hence $b = \bar{b} = 0 \diamond (0 * b) = 0 \diamond (0 * x) = \bar{x} = a$. ■

Lemma 4.22. *Let \mathfrak{X} be a pseudo-BCH-algebra and $a \in \text{Cen}\mathfrak{X}$. Then*

$$V(a) = \Phi^{-1}(a).$$

Proof. Suppose that $x \in V(a)$, that is, $a \leq x$. We have $\bar{x} \leq x$. Since $a, \bar{x} \in \text{Cen}\mathfrak{X}$, by Proposition 4.21, $a = \bar{x}$, that is, $x \in \Phi^{-1}(a)$. Conversely, if $a = \bar{x}$, then $a \leq x$ by (P3). Hence $x \in V(a)$. ■

Proposition 4.23. *Let \mathfrak{X} be a pseudo-BCH-algebra. Then:*

- (a) $X = \bigcup_{a \in \text{Cen}\mathfrak{X}} V(a)$;
- (b) if $a, b \in \text{Cen}\mathfrak{X}$ and $a \neq b$, then $V(a) \cap V(b) = \emptyset$.

Proof. (a) Clearly, $\bigcup_{a \in \text{Cen}\mathfrak{X}} V(a) \subseteq X$ and let $x \in X$. Obviously, $x \in V(\bar{x})$ and $\bar{x} \in \text{Cen}\mathfrak{X}$. Therefore, $x \in \bigcup_{a \in \text{Cen}\mathfrak{X}} V(a)$.

(b) Let $a, b \in \text{Cen}(\mathfrak{X})$ and $a \neq b$. On the contrary suppose that $V(a) \cap V(b) \neq \emptyset$. Let $x \in V(a) \cap V(b)$. Then $a \leq x$ and $b \leq x$. From Proposition 4.21 it follows that $a = b$, a contradiction. ■

We now establish a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre.

Proposition 4.24. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $A \subseteq \text{Cen}\mathfrak{X}$. The following statements are equivalent:*

- (i) A is an ideal of $\text{Cen}\mathfrak{X}$;
- (ii) $\bigcup_{a \in A} V(a)$ is an ideal of \mathfrak{X} .

Proof. Let $I = \bigcup_{a \in A} V(a)$. From Lemma 4.22 we have $I = \bigcup_{a \in A} \Phi^{-1}(a) = \Phi^{-1}(A)$.

(i) \Rightarrow (ii). Let $A \in \text{Id}(\text{Cen}\mathfrak{X})$. By Proposition 4.13, I is an ideal of \mathfrak{X} .

(ii) \Rightarrow (i). Since $I = \Phi^{-1}(A)$, we conclude that $A = \Phi(I)$. Obviously, $0 \in A$ and hence $\Phi^{-1}(0) \subseteq I$. Applying Proposition 4.14 we deduce that A is an ideal of $\text{Cen}\mathfrak{X}$. ■

Theorem 4.25. *There is a one-to-one correspondence between ideals of a pseudo-BCH-algebra \mathfrak{X} containing $K(\mathfrak{X})$ and ideals of $\text{Cen}\mathfrak{X}$.*

Proof. Set $\mathcal{I} = \{I \in \text{Id}(\mathfrak{X}) : I \supseteq K(\mathfrak{X})\}$ and $\mathcal{C} = \text{Id}(\text{Cen}\mathfrak{X})$. We consider two functions:

$$f : I \in \mathcal{I} \rightarrow \{\bar{x} : x \in I\} \quad \text{and} \quad g : A \in \mathcal{C} \rightarrow \bigcup_{a \in A} V(a).$$

Since $f(I) = \Phi(I)$, from Proposition 4.14 we conclude that f maps \mathcal{I} into \mathcal{C} . By Proposition 4.24, $g(A) = \bigcup_{a \in A} V(a) \in \mathcal{I}$ for all $A \in \mathcal{C}$, and therefore g maps \mathcal{C} into \mathcal{I} . We have

$$(4.4) \quad (f \circ g)(A) = \Phi(\Phi^{-1}(A)) = A \quad \text{for all } A \in \mathcal{C}.$$

Obviously, $I \subseteq \Phi^{-1}(\Phi(I))$. Let now $x \in \Phi^{-1}(\Phi(I))$, that is, $\bar{x} = \bar{a}$ for some $a \in I$. Then $\Phi(x * a) = 0$, and hence $x * a \in \Phi^{-1}(0)$. Therefore, $x * a \in I$ (since $\Phi^{-1}(0) = K(\mathfrak{X}) \subseteq I$). By definition, $x \in I$. Thus $\Phi^{-1}(\Phi(I)) = I$. Consequently,

$$(4.5) \quad (g \circ f)(I) = \Phi^{-1}(\Phi(I)) = I \quad \text{for all } I \in \mathcal{I}.$$

We conclude from (4.4) and (4.5) that $f \circ g = \text{id}_{\mathcal{C}}$ and $g \circ f = \text{id}_{\mathcal{I}}$, hence that f and g are inverse bijections between \mathcal{I} and \mathcal{C} . ■

Example 4.26. Let $\mathfrak{X}_1 = (\{0, a, b, c\}; *_1, \diamond_1, 0)$ be the pseudo-BCH-algebra from our Example 2.5. Consider the set $X_2 = \{0, 1, 2, 3, 4\}$ with the operation $*_2$ defined by the following table:

$*_2$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

From Example 3 of [17] it follows that $\mathfrak{X}_2 = (X_2; *_2, *_2, 0)$ is a (pseudo)-BCH-algebra. The direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a pseudo-BCH-algebra. From Proposition 4.20 we have $\text{Cen}\mathfrak{X} = \{0\} \times \{0, 2, 3, 4\}$ and $K(\mathfrak{X}) = X_1 \times \{0, 1\}$. It is easy to see that $\text{Id}(\text{Cen}\mathfrak{X}) = \{\{(0, 0)\}, \{(0, 0), (0, 3)\}, \text{Cen}\mathfrak{X}\}$. Then, by Theorem 4.25, \mathfrak{X} has three ideals containing $K(\mathfrak{X})$, namely: $K(\mathfrak{X})$, $K(\mathfrak{X}) \cup \{(0, 3), (a, 3), (b, 3), (c, 3)\}$ and \mathfrak{X} .

Now we shall show that the centre $\text{Cen}\mathfrak{X}$ defines a regular congruence on a pseudo-BCH-algebra \mathfrak{X} . Let $\text{Con}\mathfrak{X}$ denote the set of all congruences on \mathfrak{X} and let

$\theta \in \text{Con}\mathfrak{X}$. For $x \in X$, we write x/θ for the congruence class containing x , that is, $x/\theta = \{y \in X : y \theta x\}$. Set $X/\theta = \{x/\theta : x \in X\}$. It is easy to see that the factor algebra $\mathfrak{X}/\theta = \langle X/\theta; *, \diamond, 0/\theta \rangle$ satisfies (pBCH-1) and (pBCH-2). The axioms (pBCH-3) and (pBCH-4) are not necessarily satisfied. If \mathfrak{X}/θ is a pseudo-BCH-algebra, then we say that θ is *regular*.

Remark 4.27. A. Wroński has shown that non-regular congruences exist in BCK-algebras (see [18]) and hence in pseudo-BCH-algebras.

Theorem 4.28. *Let \mathfrak{X} be a pseudo-BCH-algebra and let $\theta_c = \{(x, y) \in X^2 : \bar{x} = \bar{y}\}$. Then θ_c is a regular congruence on \mathfrak{X} and $\mathfrak{X}/\theta_c \cong \text{Cen}\mathfrak{X}$.*

Proof. The mapping Φ is a homomorphism from \mathfrak{X} onto $\text{Cen}\mathfrak{X}$. Moreover we have

$$\text{Ker}\Phi = \{(x, y) \in X^2 : \Phi(x) = \Phi(y)\} = \theta_c.$$

By the Isomorphism Theorem we get $\mathfrak{X}/\theta_c \cong \text{Cen}\mathfrak{X}$, and therefore θ_c is a regular congruence on \mathfrak{X} . ■

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