# PSEUDO-BCH-ALGEBRAS 

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#### Abstract

The notion of pseudo-BCH-algebras is introduced, and some of their properties are investigated. Conditions for a pseudo- BCH -algebra to be a pseudo-BCI-algebra are given. Ideals and minimal elements in pseudo-BCHalgebras are considered.


Keywords: (pseudo-)BCK/BCI/BCH-algebra, minimal element, (closed) ideal, centre.
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## 1. Introduction

In 1966, Y. Imai and K. Iséki ([10, 11]) introduced BCK- and BCI-algebras. In 1983, Q.P. Hu and X. Li ([9]) introduced BCH-algebras. It is known that BCKand BCI-algebras are contained in the class of BCH-algebras. J. Neggers and H.S. Kim ([16]) defined d-algebras which are a generalization of BCK-algebras.

In 2001, G. Georgescu and A. Iorgulescu ([8]) introduced the pseudo-BCKalgebras as an extension of BCK-algebras. In 2008, W.A. Dudek and Y.B. Jun ([3]) introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu in [6] and [7], respectively. Those algebras were investigated by several authors in $[4,5,14]$ and [15]. As a generalization of dalgebras, Y.B. Jun, H.S. Kim and J. Neggers ([13]) introduced pseudo-d-algebras. Recently, R.A. Borzooei et al. ([1]) defined pseudo-BE-algebras.

In this paper we introduce pseudo-BCH-algebras as an extension of BCH algebras. We give basic properties of pseudo- BCH -algebras and provide some
conditions for a pseudo-BCH-algebra to be a pseudo-BCI-algebra. Moreover we study the set Cen $\mathfrak{X}$ of all minimal elements of a pseudo-BCH-algebra $\mathfrak{X}$, the socalled centre of $\mathfrak{X}$. We also consider ideals in pseudo-BCH-algebras and establish a relationship between the ideals of a pseudo- BCH -algebra and the ideals of its centre. Finally we show that the centre of a pseudo-BCH-algebra $\mathfrak{X}$ defines a regular congruence on $\mathfrak{X}$.

## 2. Definition and examples of pseudo-BCH-algebras

We recall that an algebra $\mathfrak{X}=(X ; *, 0)$ of type $(2,0)$ is called a $B C H$-algebra if it satisfies the following axioms:
(BCH-1) $\quad x * x=0$;
(BCH-2) $\quad(x * y) * z=(x * z) * y$;
(BCH-3) $\quad x * y=y * x=0 \Longrightarrow x=y$.
A BCH-algebra $\mathfrak{X}$ is said to be a BCI-algebra if it satisfies the identity
(BCI) $\quad((x * y) *(x * z)) *(z * y)=0$.
A BCK-algebra is a BCI-algebra $\mathfrak{X}$ satisfying the law $0 * x=0$.
Definition 2.1 ([3]). A pseudo-BCI-algebra is a structure $\mathfrak{X}=(X ; \leq, *, \diamond, 0)$, where " $\leq$ " is a binary relation on the set $X, " * "$ and " $\diamond$ " are binary operations on $X$ and " 0 " is an element of $X$, satisfying the axioms:

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(pBCI-1) \(\quad(x * y) \diamond(x * z) \leq z * y, \quad(x \diamond y) *(x \diamond z) \leq z \diamond y ;\)
\((\) pBCI-2) \(\quad x *(x \diamond y) \leq y, \quad x \diamond(x * y) \leq y ;\)
(pBCI-3) \(\quad x \leq x\);
(pBCI-4) \(\quad x \leq y, y \leq x \Longrightarrow x=y\);
(pBCI-5) \(\quad x \leq y \Longleftrightarrow x * y=0 \Longleftrightarrow x \diamond y=0\).
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A pseudo-BCI-algebra $\mathfrak{X}$ is called a pseudo-BCK-algebra if it satisfies the identities
(pBCK) $0 * x=0 \diamond x=0$.
Definition 2.2. A pseudo-BCH-algebra is an algebra $\mathfrak{X}=(X ; *, \diamond, 0)$ of type $(2,2,0)$ satisfying the axioms:

$$
\begin{array}{ll}
\text { (pBCH-1) } & x * x=x \diamond x=0 ; \\
\text { (pBCH-2) } & (x * y) \diamond z=(x \diamond z) * y ; \\
\text { (pBCH-3) } & x * y=y \diamond x=0 \Longrightarrow x=y ; \\
\text { (pBCH-4) } & x * y=0 \Longleftrightarrow x \diamond y=0 .
\end{array}
$$

Remark 2.3. Observe that if $(X ; *, 0)$ is a BCH-algebra, then letting $x \diamond y:=$ $x * y$, produces a pseudo- BCH -algebra ( $X ; *, \diamond, 0$ ). Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if ( $X ; *, \diamond, 0$ ) is a pseudo-BCH-algebra, then $(X ; \diamond, *, 0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [3] we conclude that if $(X ; \leq, *, \diamond, 0)$ is a pseudo-BCI-algebra, then $(X ; *, \diamond, 0)$ is a pseudo-BCH-algebra.

We say that a pseudo-BCH-algebra $\mathfrak{X}$ is proper if $* \neq \diamond$ and it is not a pseudo-BCI-algebra.

Remark 2.4. The class of all pseudo-BCH-algebras is a quasi-variety. Therefore, if $\mathfrak{X}_{1}$ and $\mathfrak{X}_{2}$ are two pseudo-BCH-algebras, then the direct product $\mathfrak{X}=\mathfrak{X}_{1} \times \mathfrak{X}_{2}$ is also a pseudo-BCH-algebra. In the case when at least one of $\mathfrak{X}_{1}$ and $\mathfrak{X}_{2}$ is proper, then $\mathfrak{X}$ is proper.

Example 2.5. Let $X_{1}=\{0, a, b, c\}$. We define the binary operations $*_{1}$ and $\diamond_{1}$ on $X_{1}$ as follows:

| $*_{1}$ | 0 | $a$ | $b$ | $c$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 |  |
| $a$ | $a$ | 0 | $a$ | 0 |  |
| $b$ | $b$ | $b$ | 0 | 0 |  |
| $c$ | $c$ | $b$ | $c$ | 0 |  |


| $\diamond_{1}$ | 0 | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | 0 |
| $b$ | $b$ | $b$ | 0 | 0 |
| $c$ | $c$ | $c$ | $a$ | 0 |

It is easy to check that $\mathfrak{X}_{1}=\left(X_{1} ; *_{1}, \diamond_{1}, 0\right)$ is a pseudo-BCH-algebra. On the set $X_{2}=\{0,1,2,3\}$ consider the operation $*_{2}$ given by the following table:

| $*_{2}$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

By simple calculation we can get that $\mathfrak{X}_{2}=\left(X_{2} ; *_{2}, *_{2}, 0\right)$ is a (pseudo)-BCHalgebra. The direct product $\mathfrak{X}=\mathfrak{X}_{1} \times \mathfrak{X}_{2}$ is a pseudo-BCH-algebra. Observe that $\mathfrak{X}$ is proper. Let $x=(a, 1), y=(a, 3)$ and $z=(a, 2)$. Then $(x * y) \diamond(x * z)=$ $(0,1) \diamond(0,0)=(0,1)$ and $z * y=(0,0)$. Since $(0,1) \not \leq(0,0)$, we conclude that $\mathfrak{X}$ is not a pseudo-BCI-algebra, and therefore it is a proper pseudo-BCH-algebra.

Proposition 2.6. Any (proper) pseudo-BCH-algebra satisfying ( pBCK ) can be extended to a (proper) pseudo-BCH-algebra containing one element more.

Proof. Let $\mathfrak{X}=(X ; *, \diamond, 0)$ be a pseudo-BCH-algebra satisfying (pBCK) and let $\delta \notin X$. On the set $Y=X \cup\{\delta\}$ consider the operations:

$$
x *^{\prime} y= \begin{cases}x * y & \text { if } \quad x, y \in X \\ \delta & \text { if } \quad x=\delta \text { and } y \in X \\ 0 & \text { if } \quad x \in Y \text { and } y=\delta\end{cases}
$$

and

$$
x \diamond^{\prime} y= \begin{cases}x \diamond y & \text { if } \quad x, y \in X \\ \delta & \text { if } \quad x=\delta \text { and } y \in X \\ 0 & \text { if } \quad x \in Y \text { and } y=\delta\end{cases}
$$

Obviously, $\left(Y ; *^{\prime}, \diamond^{\prime}, 0\right)$ satisfies the axioms $(\mathrm{pBCH}-1),(\mathrm{pBCH}-3)$, and $(\mathrm{pBCH}-4)$. Further, the axiom ( $\mathrm{pBCH}-2$ ) is easily satisfied for all $x, y, z \in X$. Moreover, by routine calculation we can verify it in the case when at least one of $x, y, z$ is equal to $\delta$. Thus, by definition, $\left(Y ; *^{\prime}, \diamond^{\prime}, 0\right)$ is a pseudo-BCH-algebra. Clearly, if $\mathfrak{X}$ is a proper pseudo-BCH-algebra, then $\left(Y ; *^{\prime}, \diamond^{\prime}, 0\right)$ is also a proper pseudo-BCH-algebra.

From Example 2.5 and Proposition 2.6 we conclude that there are infinite many proper pseudo-BCH-algebras.

## 3. Properties of pseudo-BCH-ALGEbras

Let $\mathfrak{X}=(X ; *, \diamond, 0)$ be a pseudo-BCH-algebra. Define the relation $\leq$ on $X$ by $x \leq y$ if and only if $x * y=0$ (or equivalently, $x \diamond y=0$ ).

For any $x \in X$ and $n=0,1,2, \ldots$, we put

$$
\begin{aligned}
& 0 *^{0} x=0 \quad \text { and } \quad 0 *^{n+1} x=\left(0 *^{n} x\right) * x \\
& 0 \diamond^{0} x=0 \quad \text { and } \quad 0 \diamond^{n+1} x=\left(0 \diamond^{n} x\right) \diamond x
\end{aligned}
$$

Proposition 3.1. In a pseudo-BCH-algebra $\mathfrak{X}$ the following properties hold (for all $x, y, z \in X$ ):
(P1) $x \leq y, y \leq x \Longrightarrow x=y$;
(P2) $x \leq 0 \Longrightarrow x=0$;
$(\mathrm{P} 3) x *(x \diamond y) \leq y, \quad x \diamond(x * y) \leq y$;
(P4) $x * 0=x=x \diamond 0$;
(P5) $0 * x=0 \diamond x$;
(P6) $x \leq y \Longrightarrow 0 * x=0 \diamond y$;
$(\mathrm{P} 7) 0 \diamond(0 *(0 \diamond x))=0 \diamond x, \quad 0 *(0 \diamond(0 * x))=0 * x$;
(P8) $0 *(x * y)=(0 \diamond x) \diamond(0 * y)$;
$(\mathrm{P} 9) 0 \diamond(x \diamond y)=(0 * x) *(0 \diamond y)$.

Proof. (P1) follows from ( $\mathrm{pBCH}-3$ ).
(P2) Let $x \leq 0$. Then $x * 0=0$. Applying ( $\mathrm{pBCH}-2$ ) and ( $\mathrm{pBCH}-1$ ) we obtain

$$
0 \diamond x=(x * 0) \diamond x=(x \diamond x) * 0=0 * 0=0
$$

that is, $0 \leq x$. Therefore $x=0$ by ( P 1 ).
(P3) Using ( $\mathrm{pBCH}-2)$ and $(\mathrm{pBCH}-1)$ we have $(x *(x \diamond y)) \diamond y=(x \diamond y) *(x \diamond y)=0$. Hence $x *(x \diamond y) \leq y$. Similarly, $x \diamond(x * y) \leq y$.
(P4) Putting $y=0$ in (P3), we have $x *(x \diamond 0) \leq 0$ and $x \diamond(x * 0) \leq 0$. From (P2) we obtain $x *(x \diamond 0)=0$ and $x \diamond(x * 0)=0$. Thus $x \leq x \diamond 0$ and $x \leq x * 0$.

On the other hand, $(x \diamond 0) * x=(x * x) \diamond 0=0 \diamond 0=0$ and $(x * 0) \diamond x=$ $(x \diamond x) * 0=0 * 0=0$, and so $x \diamond 0 \leq x$ and $x * 0 \leq x$. By ( P 1 ), $x * 0=x=x \diamond 0$.
(P5) Applying (pBCH-1) and (pBCH-2) we get $0 * x=(x \diamond x) * x=(x * x) \diamond x=$ $0 \diamond x$.
(P6) Let $x \leq y$. Then $x \diamond y=0$ and therefore $0 * x=(x \diamond y) * x=(x * x) \diamond y=0 \diamond y$.
(P7) From (P3) it follows that $0 *(0 \diamond x) \leq x$ and $0 \diamond(0 * x) \leq x$. Hence, using (P5) and (P6) we obtain (P7).
(P8) Applying ( $\mathrm{pBCH}-1$ ) and ( $\mathrm{pBCH}-2$ ) we have

$$
\begin{aligned}
(0 \diamond x) \diamond(0 * y) & =(((x * y) *(x * y)) \diamond x) \diamond(0 * y) \\
& =(((x * y) \diamond x) *(x * y)) \diamond(0 * y) \\
& =(((x \diamond x) * y) *(x * y)) \diamond(0 * y) \\
& =((0 * y) *(x * y)) \diamond(0 * y) \\
& =((0 * y) \diamond(0 * y)) *(x * y) \\
& =0 *(x * y) .
\end{aligned}
$$

(P9) The proof is similar to the proof of (P8).
From (P1) and (P3) we get
Corollary 3.2. Every pseudo-BCH-algebra satisfies (pBCI-2)-(pBCI-5).
Remark 3.3. In any pseudo-BCI-algebra the relation $\leq$ is transitive (see [3], Proposition 3.2). However, in the pseudo-BCH-algebra $\mathfrak{X}$ from Example 2.5 we have $(a, 1) \leq(a, 2)$ and $(a, 2) \leq(a, 3)$ but $(a, 1) \not \leq(a, 3)$.

Theorem 3.4. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then $\mathfrak{X}$ is a pseudo-BCI-algebra if and only if it satisfies the following implication:

$$
\begin{equation*}
x \leq y \Longrightarrow x * z \leq y * z, x \diamond z \leq y \diamond z \tag{3.1}
\end{equation*}
$$

Proof. If $\mathfrak{X}$ is a pseudo-BCI-algebra, then $\mathfrak{X}$ satisfies (3.1) by Proposition 3.2 (b7) of [3]. Conversely, let (3.1) hold in $\mathfrak{X}$ and let $x, y, z \in X$. By (P3), $x \diamond(x * z) \leq z$ and $x *(x \diamond z) \leq z$. Hence $(x \diamond(x * z)) * y \leq z * y$ and $(x *(x \diamond z)) \diamond y \leq z \diamond y$, and so $(x * y) \diamond(x * z) \leq z * y$ and $(x \diamond y) *(x \diamond z) \leq z \diamond y$. Therefore, $\mathfrak{X}$ satisfies (pBCI-1). Consequently, $\mathfrak{X}$ is a pseudo-BCI-algebra.

Theorem 3.5. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. The following statements are equivalent:
(i) $x *(y * z)=(x * y) * z \quad$ for all $\quad x, y, z \in X$;
(ii) $0 * x=x=0 \diamond x \quad$ for every $\quad x \in X$;
(iii) $x * y=x \diamond y=y * x \quad$ for all $\quad x, y \in X$;
(iv) $x \diamond(y \diamond z)=(x \diamond y) \diamond z \quad$ for all $\quad x, y, z \in X$.

Proof. (i) $\Longrightarrow$ (ii). Let $x \in X$. We have $x=x * 0=x *(x * x)=(x * x) * x=0 * x$. By (P5), $0 \diamond x=x$.
(iv) $\Longrightarrow$ (ii). The proof is similar to the above proof.
(ii) $\Longrightarrow$ (iii). Let (ii) hold and $x, y \in X$. Applying (P8) and (pBCH-2) we obtain

$$
\begin{aligned}
x * y & =0 *(x * y)=(0 \diamond x) \diamond(0 * y) \\
& =x \diamond y \\
& =(0 * x) \diamond y=(0 \diamond y) * x=y * x .
\end{aligned}
$$

(iii) $\Longrightarrow$ (i). Let $x, y, z \in X$. Using (iii) and (pBCH-2) we get

$$
x *(y * z)=(y \diamond z) * x=(y * x) \diamond z=(x * y) * z .
$$

(iii) $\Longrightarrow$ (iv) has a proof similar to the proof of implication (iii) $\Longrightarrow$ (i).

Hence all the conditions are equivalent.
Corollary 3.6. If $\mathfrak{X}$ is a pseudo- $B C H$-algebra satisying the idendity $0 * x=x$, then $(X ; *, 0)$ is an Abelian group each element of which has order 2 (that is, a Boolean group).

## 4. The centre of a pseudo-BCH-algebra. Ideals

An element $a$ of a pseudo-BCH-algebra $\mathfrak{X}$ is said to be minimal if for every $x \in X$ the following implication

$$
x \leq a \Longrightarrow x=a
$$

holds.

Proposition 4.1. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then the following conditions are equivalent (for every $x \in X$ ):
(i) a is minimal;
(ii) $x \diamond(x * a)=a$;
(iii) $0 \diamond(0 * a)=a$;
(iv) $a * x=(0 * x) \diamond(0 * a)$;
(v) $a * x=0 \diamond(x * a)$.

Proof. (i) $\Longrightarrow$ (ii). By (P2), $x \diamond(x * a) \leq a$ for all $x \in X$. Since $a$ is minimal, we get (ii).
(ii) $\Longrightarrow$ (iii). Obvious.
(iii) $\Longrightarrow$ (iv). We have $a * x=(0 \diamond(0 * a)) * x=(0 * x) \diamond(0 * a)$.
(iv) $\Longrightarrow$ (v). Applying (P5) and (P8) we see that

$$
0 \diamond(x * a)=0 *(x * a)=(0 \diamond x) \diamond(0 * a)=(0 * x) \diamond(0 * a)=a * x
$$

$(\mathrm{v}) \Longrightarrow(\mathrm{i})$. Let $x \leq a$. Then $x * a=0$ and hence $a * x=0 \diamond(x * a)=0$. Thus $a \leq x$. Consequently, $x=a$.

Replacing $*$ by $\diamond$ and $\diamond$ by $*$ in Proposition 4.1 we obtain
Proposition 4.2. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then for every $x \in X$ the following conditions are equivalent:
(i) $a$ is minimal;
(ii) $x *(x \diamond a)=a$;
(iii) $0 *(0 \diamond a)=a$;
(iv) $a \diamond x=(0 \diamond x) *(0 \diamond a)$;
(v) $a \diamond x=0 *(x \diamond a)$.

Proposition 4.3. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then $a$ is minimal if and only if there is an element $x \in X$ such that $a=0 * x$.

Proof. Let $a$ be a minimal element of $\mathfrak{X}$. By Proposition 4.2, $a=0 *(0 \diamond a)$. If we set $x=0 \diamond a$, then $a=0 * x$.

Conversely, suppose that $a=0 * x$ for some $x \in X$. Using (P7) we get

$$
0 *(0 \diamond a)=0 *(0 \diamond(0 * x))=0 * x=a
$$

From Proposition 4.2 it follows that $a$ is minimal.
For $x \in X$, set

$$
\bar{x}=0 \diamond(0 * x)
$$

$\mathrm{By}(\mathrm{P} 5), \bar{x}=0 *(0 * x)=0 \diamond(0 \diamond x)=0 *(0 \diamond x)$.
Proposition 4.4. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. For any $x, y \in X$ we have:
(a) $\overline{x * y}=\bar{x} * \bar{y}$;
(b) $\overline{x \diamond y}=\bar{x} \diamond \bar{y}$;
(c) $\overline{\bar{x}}=\bar{x}$.

Proof. (a) Applying (P8) and (P9) we get

$$
\begin{aligned}
\overline{x * y} & =0 \diamond(0 *(x * y))=0 \diamond[(0 \diamond x) \diamond(0 * y)] \\
& =[0 *(0 \diamond x)] *[0 \diamond(0 * y)]=\bar{x} * \bar{y} .
\end{aligned}
$$

(b) has a proof similar to (a).
(c) By $(\mathrm{P} 7), 0 *(0 \diamond(0 * x))=0 * x$, that is, $0 * \bar{x}=0 * x$. Hence $\overline{\bar{x}}=0 \diamond(0 * \bar{x})=$ $0 \diamond(0 * x)=\bar{x}$.

Following the terminology from BCH-algebras (see [2], Definition 5) the set $\{x \in$ $X: x=\bar{x}\}$ will be called the centre of $\mathfrak{X}$. We shall denote it by Cen $\mathfrak{X}$. By Proposition 4.1, Cen $\mathfrak{X}$ is the set of all minimal elements of $\mathfrak{X}$. We have

$$
\begin{equation*}
\text { Cen } \mathfrak{X}=\{\bar{x}: x \in X\} . \tag{4.1}
\end{equation*}
$$

Define $\Phi: \mathfrak{X} \rightarrow$ Cen $\mathfrak{X}$ by $\Phi(x)=\bar{x}$ for all $x \in X$. By Proposition 4.4, $\Phi$ is a homomorphism from $\mathfrak{X}$ onto Cen $\mathfrak{X}$. We also obtain

Proposition 4.5. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then Cen $\mathfrak{X}$ is a subalgebra of $\mathfrak{X}$.

Proposition 4.6. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $x, y \in C e n \mathfrak{X}$. Then for every $z \in X$ we have

$$
\begin{equation*}
x \diamond(z * y)=y *(z \diamond x) \tag{4.2}
\end{equation*}
$$

Proof. Let $z \in X$. Using Propositions 4.2 and 4.1 we obtain

$$
x \diamond(z * y)=[z *(z \diamond x)] \diamond(z * y)=[z \diamond(z * y)] *(z \diamond x)=y *(z \diamond x)
$$

that is, (4.2) holds.
Following [5], a pseudo-BCI-algebra $(X ; \leq, *, \diamond, 0)$ is said to be $p$-semisimple if it satisfies for all $x \in X$,

$$
0 \leq x \Longrightarrow x=0
$$

From Theorem 3.1 of [5] it follows that if $\mathfrak{X}=(X ; \leq, *, \diamond, 0)$ is a pseudo-BCIalgebra, then $\mathfrak{X}$ is $p$-semisimple if and only if $x=\bar{x}$ for every $x \in X$ (that is, Cen $\mathfrak{X}=\mathfrak{X})$.

Theorem 4.7. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then Cen $\mathfrak{X}$ is a p-semisimple pseudo-BCI-algebra.
Proof. Since Cen $\mathfrak{X}$ is a subalgebra of $\mathfrak{X}$, Cen $\mathfrak{X}$ is a pseudo-BCH-algebra. Let $x, y, z \in \operatorname{Cen} \mathfrak{X}$ and let $x \leq y$. Since $x$ and $y$ are minimal elements of $\mathfrak{X}$, we get $x=y$. Hence $x * z \leq y * z$ and $x \diamond z \leq y \diamond z$. Then, by Theorem 3.4, Cen $\mathfrak{X}$ is a pseudo-BCI-algebra. Obviously, $x=\bar{x}$ for every $x \in$ Cen $\mathfrak{X}$, and therefore Cen $\mathfrak{X}$ is $p$-semisimple.

Remark 4.8. From Theorem 3.6 of [5] we deduce that (Cen $\mathfrak{X} ;+, 0$ ) is a group, where $x+y$ is $x *(0 \diamond y)$ for all $x, y \in \operatorname{Cen} \mathfrak{X}$.

Definition 4.9. Let $X$ be a pseudo-BCH-algebra. A subset $I$ of $X$ is called an ideal of $X$ if it satisfies for all $x, y \in X$
(I1) $0 \in I$;
(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.
We will denote by $\operatorname{Id}(\mathfrak{X})$ the set of all ideals of $\mathfrak{X}$. Obviously, $\{0\}, X \in \operatorname{Id}(\mathfrak{X})$.
Proposition 4.10. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $I \in \operatorname{Id}(\mathfrak{X})$. For any $x, y \in X$, if $y \in I$ and $x \leq y$, then $x \in I$.
Proof. Straightforward.
Proposition 4.11. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and $I$ be a subset of $X$ satisfying (I1). Then $I$ is an ideal of $\mathfrak{X}$ if and only if for all $x, y \in X$,
(I2') if $x \diamond y \in I$ and $y \in I$, then $x \in I$.
Proof. Let $I$ be an ideal of $\mathfrak{X}$. Suppose that $x \diamond y \in I$ and $y \in I$. By (P3), $x *(x \diamond y) \leq y$ and from Proposition 4.10 it follows that $x *(x \diamond y) \in I$. Therefore, since $x \diamond y \in I$ and $I$ satisfies (I2), we obtain $x \in I$, that is, (I2') holds. The proof of the implication (I2') $\Rightarrow$ (I2) is analogous.

Example 4.12. Let $X=\{0, a, b, c, d\}$. Define binary operations $*$ and $\diamond$ on $X$ by the following tables:

| $*$ | 0 | $a$ | $b$ | $c$ | $d$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 | 0 | $d$ |
| $a$ | $a$ | 0 | $a$ | 0 | $d$ |
| $b$ | $b$ | $b$ | 0 | 0 | $d$ |
| $c$ | $c$ | $b$ | $c$ | 0 | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ | 0 |


| $\diamond$ | 0 | $a$ | $b$ | $c$ | $d$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | $d$ |
| $a$ | $a$ | 0 | $a$ | 0 | $d$ |
| $b$ | $b$ | $b$ | 0 | 0 | $d$ |
| $c$ | $c$ | $c$ | $a$ | 0 | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ | 0 |

By routine calculation, $\mathfrak{X}=(X ; *, \diamond, 0)$ is a pseudo- BCH -algebra. It is easy to see that $\operatorname{Id}(\mathfrak{X})=\{\{0\},\{0, a\},\{0, b\},\{0, a, b, c\}, X\}$.

The following two propositions give the homomorphic properties of ideal.
Proposition 4.13. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras. If $\varphi: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a homomorphism and $J \in \operatorname{Id}(\mathfrak{Y})$, then the inverse image $\varphi^{-1}(J)$ of $J$ is an ideal of $\mathfrak{X}$.

Proof. Straightforward.
Proposition 4.14. Let $\varphi: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a surjective homomorphism. If $I$ is an ideal of $\mathfrak{X}$ containing $\varphi^{-1}(0)$, then $\varphi(I)$ is an ideal of $\mathfrak{Y}$.
Proof. Since $0 \in I$, we have $0=\varphi(0) \in \varphi(I)$. Let $x, y \in Y$ and suppose that $x * y, y \in \varphi(I)$. Then there are $a \in X$ and $b, c \in I$ such that $x=\varphi(a), y=\varphi(b)$ and $x * y=\varphi(c)$. We have $\varphi(a * b)=\varphi(c)$ and hence $(a * b) * c \in \varphi^{-1}(0) \subseteq I$. By the definition of an ideal, $a \in I$. Consequently, $x=\varphi(a) \in \varphi(I)$. This means that $\varphi(I)$ is an ideal of $\mathfrak{Y}$.

Definition 4.15. An ideal $I$ of a pseudo-BCH-algebra $\mathfrak{X}$ is said to be closed if $0 * x \in I$ for every $x \in I$.

Theorem 4.16. An ideal $I$ of a pseudo-BCH-algebra $\mathfrak{X}$ is closed if and only if $I$ is a subalgebra of $\mathfrak{X}$.
Proof. Suppose that $I$ is a closed ideal of $\mathfrak{X}$ and let $x, y \in I$. By ( $\mathrm{pBCH}-2$ ) and (pBCH-1),

$$
\begin{aligned}
{[(x * y) *(0 * y)] \diamond x } & =[(x * y) \diamond x] *(0 * y) \\
& =[(x \diamond x) * y] *(0 * y) \\
& =(0 * y) *(0 * y)=0 .
\end{aligned}
$$

Hence $[(x * y) *(0 * y)] \diamond x \in I$. Since $x, 0 * y \in I$, we have $x * y \in I$. Similarly, $x \diamond y \in I$. Conversely, if $I$ is a subalgebra of $\mathfrak{X}$, then $x \in I$ and $0 \in I$ imply $0 * x \in I$.

Theorem 4.17. Every ideal of a finite pseudo-BCH-algebra is closed.
Proof. Let $I$ be an ideal of a finite pseudo-BCH-algebra $\mathfrak{X}$ and let $a \in I$. Suppose that $|X|=n$ for some $n \in \mathbb{N}$. At least two of the $n+1$ elements:

$$
0,0 * a, 0 *^{2} a, \ldots, 0 *^{n} a
$$

are equal, for instance, $0 *^{r} a=0 *^{s} a$, where $0 \leq s<r \leq n$. Hence

$$
0=\left(0 *^{r} a\right) \diamond\left(0 *^{s} a\right)=\left[\left(0 *^{s} a\right) \diamond\left(0 *^{s} a\right)\right] *^{r-s} a=0 *^{r-s} a .
$$

Therefore $0 *^{r-s} a \in I$. Since $a \in I$, by definition, $0 * a \in I$. Consequently, $I$ is a closed ideal of $\mathfrak{X}$.

For any pseudo-BCH-algebra $\mathfrak{X}$, we set

$$
\mathrm{K}(\mathfrak{X})=\{x \in X: 0 \leq x\} .
$$

Observe that Cen $\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})=\{0\}$. Indeed, $0 \in \operatorname{Cen} \mathfrak{X} \cap \mathrm{~K}(\mathfrak{X})$ and if $x \in$ Cen $\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$, then $x=0 \diamond(0 * x)=0 \diamond 0=0$.

In Example 4.12, Cen $\mathfrak{X}=\{0, d\}$ and $\mathrm{K}(\mathfrak{X})=\{0, a, b, c\}$.
It is easy to see that

$$
x \in \mathrm{~K}(\mathfrak{X}) \Longleftrightarrow \bar{x}=0 \Longleftrightarrow x \in \Phi^{-1}(0) .
$$

Thus

$$
\begin{equation*}
\mathrm{K}(\mathfrak{X})=\Phi^{-1}(0) . \tag{4.3}
\end{equation*}
$$

Proposition 4.18. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then $\mathrm{K}(\mathfrak{X})$ is a closed ideal of $\mathfrak{X}$.

Proof. By (4.3) and Proposition 4.13, $\mathrm{K}(\mathfrak{X})$ is an ideal of $\mathfrak{X}$. Let $x \in \mathrm{~K}(\mathfrak{X})$. Then $\bar{x}=0$ and hence $\Phi(0 * x)=0 * \bar{x}=0$. Consequently, $0 * x \in \mathrm{~K}(\mathfrak{X})$. Thus $\mathrm{K}(\mathfrak{X})$ is a closed ideal.

Corollary 4.19. For any pseudo-BCH-algebra $\mathfrak{X}$ the set $\mathrm{K}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$, and so it is a pseudo-BCH-algebra.

Proposition 4.20. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras. Then:
(a) $\operatorname{Cen}(\mathfrak{X} \times \mathfrak{Y})=\operatorname{Cen}(\mathfrak{X}) \times \operatorname{Cen}(\mathfrak{Y})$;
(b) $K(\mathfrak{X} \times \mathfrak{Y})=K(\mathfrak{X}) \times K(\mathfrak{Y})$.

Proof. This is immediate from definitions.
For any element $a$ of a pseudo- BCH -algebra $\mathfrak{X}$, we define a subset $\mathrm{V}(a)$ of $X$ as

$$
\mathrm{V}(a)=\{x \in X: a \leq x\}
$$

Note that $\mathrm{V}(a) \neq \emptyset$, because $a \leq a$ gives $a \in \mathrm{~V}(a)$. Furthermore, $\mathrm{V}(0)=\mathrm{K}(\mathfrak{X})$.
Proposition 4.21. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then for each $x \in X$ there exists a unique element $a \in$ Cen $\mathfrak{X}$ such that $a \leq x$.

Proof. Let $x \in X$. Take $a=\bar{x}$, that is, $a=0 \diamond(0 * x)$. By (P3), $a \leq x$. From (4.1) it follows that $a \in$ Cen $\mathfrak{X}$. To prove uniqueness, let $b \in$ Cen $\mathfrak{X}$ be such that $b \leq x$. Then $b \diamond x=0$. Therefore,

$$
0 * b=(b \diamond x) * b=(b * b) \diamond x=0 \diamond x=0 * x
$$

and hence $b=\bar{b}=0 \diamond(0 * b)=0 \diamond(0 * x)=\bar{x}=a$.
Lemma 4.22. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and $a \in \operatorname{Cen} \mathfrak{X}$. Then

$$
\mathrm{V}(a)=\Phi^{-1}(a) .
$$

Proof. Suppose that $x \in \mathrm{~V}(a)$, that is, $a \leq x$. We have $\bar{x} \leq x$. Since $a, \bar{x} \in$ Cen $\mathfrak{X}$, by Proposition 4.21, $a=\bar{x}$, that is, $x \in \Phi^{-1}(a)$.
Conversely, if $a=\bar{x}$, then $a \leq x$ by (P3). Hence $x \in \mathrm{~V}(a)$.
Proposition 4.23. Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then:
(a) $X=\underset{a \in \operatorname{Cen} X}{ } \mathrm{~V}(a)$;
(b) if $a, b \in \mathrm{CenX}$ and $a \neq b$, then $\mathrm{V}(a) \cap \mathrm{V}(b)=\emptyset$.

Proof. (a) Clearly, $\bigcup_{a \in \operatorname{Cen} \mathfrak{X}} \mathrm{~V}(a) \subseteq X$ and let $x \in X$. Obviously, $x \in \mathrm{~V}(\bar{x})$ and $\bar{x} \in$ Cen $\mathfrak{X}$. Therefore, $x \in \bigcup_{a \in \operatorname{Cen} \mathfrak{X}} \mathrm{~V}(a)$.
(b) Let $a, b \in \operatorname{Cen}(\mathfrak{X})$ and $a \neq b$. On the contrary suppose that $\mathrm{V}(a) \cap \mathrm{V}(b) \neq \emptyset$. Let $x \in \mathrm{~V}(a) \cap \mathrm{V}(b)$. Then $a \leq x$ and $b \leq x$. From Proposition 4.21 it follows that $a=b$, a contradition.

We now establish a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre.

Proposition 4.24. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $A \subseteq$ Cen $\mathfrak{X}$. The following statements are equivalent:
(i) $A$ is an ideal of Cen $\mathfrak{X}$;
(ii) $\bigcup_{a \in A} \mathrm{~V}(a)$ is an ideal of $\mathfrak{X}$.

Proof. Let $I=\bigcup_{a \in A} \mathrm{~V}(a)$. From Lemma 4.22 we have $I=\bigcup_{a \in A} \Phi^{-1}(a)=$ $\Phi^{-1}(A)$.
(i) $\Rightarrow$ (ii). Let $A \in \operatorname{Id}(\operatorname{Cen} \mathfrak{X})$. By Proposition 4.13, $I$ is an ideal of $\mathfrak{X}$.
(ii) $\Rightarrow$ (i). Since $I=\Phi^{-1}(A)$, we conclude that $A=\Phi(I)$. Obviously, $0 \in A$ and hence $\Phi^{-1}(0) \subseteq I$. Applying Proposition 4.14 we deduce that $A$ is an ideal of Cen $\mathfrak{X}$.

Theorem 4.25. There is a one-to-one correspondence between ideals of a pseudo-BCH-algebra $\mathfrak{X}$ containing $\mathrm{K}(\mathfrak{X})$ and ideals of Cen $\mathfrak{X}$.

Proof. Set $\mathcal{I}=\{I \in \operatorname{Id}(\mathfrak{X}): I \supseteq \mathrm{~K}(\mathfrak{X})\}$ and $\mathcal{C}=\operatorname{Id}($ Cen $\mathfrak{X})$. We consider two functions:

$$
f: I \in \mathcal{I} \rightarrow\{\bar{x}: x \in I\} \quad \text { and } \quad g: A \in \mathcal{C} \rightarrow \bigcup_{a \in A} \mathrm{~V}(a)
$$

Since $f(I)=\Phi(I)$, from Proposition 4.14 we conclude that $f$ maps $\mathcal{I}$ into $\mathcal{C}$. By Proposition 4.24, $g(A)=\bigcup_{a \in A} \mathrm{~V}(a) \in \mathcal{I}$ for all $A \in \mathcal{C}$, and therefore $g$ maps $\mathcal{C}$ into $\mathcal{I}$. We have

$$
\begin{equation*}
(f \circ g)(A)=\Phi\left(\Phi^{-1}(A)\right)=A \quad \text { for all } \quad A \in \mathcal{C} \tag{4.4}
\end{equation*}
$$

Obviously, $I \subseteq \Phi^{-1}(\Phi(I))$. Let now $x \in \Phi^{-1}(\Phi(I))$, that is, $\bar{x}=\bar{a}$ for some $a \in I$. Then $\Phi(x * a)=0$, and hence $x * a \in \Phi^{-1}(0)$. Therefore, $x * a \in I$ (since $\left.\Phi^{-1}(0)=\mathrm{K}(\mathfrak{X}) \subseteq I\right)$. By definition, $x \in I$. Thus $\Phi^{-1}(\Phi(I))=I$. Consequently,

$$
\begin{equation*}
(g \circ f)(I)=\Phi^{-1}(\Phi(I))=I \quad \text { for all } \quad I \in \mathcal{I} \tag{4.5}
\end{equation*}
$$

We conclude from (4.4) and (4.5) that $f \circ g=\mathrm{id}_{\mathcal{C}}$ and $g \circ f=\mathrm{id}_{\mathcal{I}}$, hence that $f$ and $g$ are inverse bijections between $\mathcal{I}$ and $\mathcal{C}$.

Example 4.26. Let $\mathfrak{X}_{1}=\left(\{0, a, b, c\} ; *_{1}, \diamond_{1}, 0\right)$ be the pseudo-BCH-algebra from our Example 2.5. Consider the set $X_{2}=\{0,1,2,3,4\}$ with the operation $*_{2}$ defined by the following table:

| $*_{2}$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 4 | 3 | 2 |
| 1 | 1 | 0 | 4 | 3 | 2 |
| 2 | 2 | 2 | 0 | 4 | 3 |
| 3 | 3 | 3 | 2 | 0 | 4 |
| 4 | 4 | 4 | 3 | 2 | 0 |

From Example 3 of [17] it follows that $\mathfrak{X}_{2}=\left(X_{2} ; *_{2}, *_{2}, 0\right)$ is a (pseudo)-BCHalgebra. The direct product $\mathfrak{X}=\mathfrak{X}_{1} \times \mathfrak{X}_{2}$ is a pseudo-BCH-algebra. From Proposition 4.20 we have Cen $\mathfrak{X}=\{0\} \times\{0,2,3,4\}$ and $\mathrm{K}(\mathfrak{X})=X_{1} \times\{0,1\}$. It is easy to see that $\operatorname{Id}(\operatorname{Cen} \mathfrak{X})=\{\{(0,0)\},\{(0,0),(0,3)\}$, Cen $\mathfrak{X}\}$. Then, by Theorem $4.25, \mathfrak{X}$ has three ideals containing $\mathrm{K}(\mathfrak{X})$, namely: $\mathrm{K}(\mathfrak{X}), \mathrm{K}(\mathfrak{X}) \cup\{(0,3),(a, 3),(b, 3),(c, 3)\}$ and $\mathfrak{X}$.

Now we shall show that the centre Cen $\mathfrak{X}$ defines a regular congruence on a pseudo-BCH-algebra $\mathfrak{X}$. Let Con $\mathfrak{X}$ denote the set of all congruences on $\mathfrak{X}$ and let
$\theta \in \operatorname{Con} \mathfrak{X}$. For $x \in X$, we write $x / \theta$ for the congruence class containing $x$, that is, $x / \theta=\{y \in X: y \theta x\}$. Set $X / \theta=\{x / \theta: x \in X\}$. It is easy to see that the factor algebra $\mathfrak{X} / \theta=\langle X / \theta ; *, \diamond, 0 / \theta\rangle$ satisfies ( $\mathrm{pBCH}-1$ ) and ( $\mathrm{pBCH}-2$ ). The axioms ( $\mathrm{pBCH}-3$ ) and ( $\mathrm{pBCH}-4$ ) are not necessarity satisfied. If $\mathfrak{X} / \theta$ is a pseudo- BCH -algebra, then we say that $\theta$ is regular.

Remark 4.27. A. Wroński has shown that non-regular congruences exist in BCK-algebras (see [18]) and hence in pseudo-BCH-algebras.

Theorem 4.28. Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $\theta_{c}=\left\{(x, y) \in X^{2}\right.$ : $\bar{x}=\bar{y}\}$. Then $\theta_{c}$ is a regular congruence on $\mathfrak{X}$ and $\mathfrak{X} / \theta_{c} \cong$ Cen $\mathfrak{X}$.

Proof. The mapping $\Phi$ is a homomorphism from $\mathfrak{X}$ onto Cen $\mathfrak{X}$. Moreover we have

$$
\operatorname{Ker} \Phi=\left\{(x, y) \in X^{2}: \Phi(x)=\Phi(y)\right\}=\theta_{c}
$$

By the Isomorphism Theorem we get $\mathfrak{X} / \theta_{c} \cong$ Cen $\mathfrak{X}$, and therefore $\theta_{c}$ is a regular congruence on $\mathfrak{X}$.

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