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PSEUDO-BCH-ALGEBRAS

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Abstract

The notion of pseudo-BCH-algebras is introduced, and some of their properties are investigated. Conditions for a pseudo-BCH-algebra to be a pseudo-BCI-algebra are given. Ideals and minimal elements in pseudo-BCH-algebras are considered.

Keywords: (pseudo-)BCK/BCI/BCH-algebra, minimal element, (closed) ideal, centre.

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1. Introduction

In 1966, Y. Imai and K. Iséki ([10, 11]) introduced BCK- and BCI-algebras. In 1983, Q.P. Hu and X. Li ([9]) introduced BCH-algebras. It is known that BCK- and BCI-algebras are contained in the class of BCH-algebras. J. Neggers and H.S. Kim ([16]) defined d-algebras which are a generalization of BCK-algebras.

In 2001, G. Georgescu and A. Iorgulescu ([8]) introduced the pseudo-BCK-algebras as an extension of BCK-algebras. In 2008, W.A. Dudek and Y.B. Jun ([3]) introduced pseudo-BCI-algebras as a natural generalization of BCI-algebras and of pseudo-BCK-algebras. These algebras have also connections with other algebras of logic such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgescu and A. Iorgulescu in [6] and [7], respectively. Those algebras were investigated by several authors in [4, 5, 14] and [15]. As a generalization of d-algebras, Y.B. Jun, H.S. Kim and J. Neggers ([13]) introduced pseudo-d-algebras. Recently, R.A. Borzooei *et al.* ([1]) defined pseudo-BE-algebras.

In this paper we introduce pseudo-BCH-algebras as an extension of BCH-algebras. We give basic properties of pseudo-BCH-algebras and provide some conditions for a pseudo-BCH-algebra to be a pseudo-BCI-algebra. Moreover we study the set Cen$\mathfrak{X}$ of all minimal elements of a pseudo-BCH-algebra $\mathfrak{X}$, the so-called centre of $\mathfrak{X}$. We also consider ideals in pseudo-BCH-algebras and establish a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre. Finally we show that the centre of a pseudo-BCH-algebra $\mathfrak{X}$ defines a regular congruence on $\mathfrak{X}$.

2. Definition and examples of pseudo-BCH-algebras

We recall that an algebra $\mathfrak{X} = (X; *, 0)$ of type $(2, 0)$ is called a *BCH-algebra* if it satisfies the following axioms:

(BCH-1) $x * x = 0$;
(BCH-2) $(x * y) * z = (x * z) * y$;
(BCH-3) $x * y = y * x = 0 \Longrightarrow x = y$.

A BCH-algebra $\mathfrak{X}$ is said to be a *BCI-algebra* if it satisfies the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$.

A *BCK-algebra* is a BCI-algebra $\mathfrak{X}$ satisfying the law $0 * x = 0$.

Definition 2.1 ([3]). A pseudo-BCI-algebra is a structure $\mathfrak{X} = (X; \leq, *, \diamond, 0)$, where "$\leq$" is a binary relation on the set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, satisfying the axioms:

(pBCI-1) $(x * y) \diamond (x * z) \leq z * y$, $(x \diamond y) * (x \diamond z) \leq z \diamond y$;
(pBCI-2) $x * (x \diamond y) \leq y$, $x \diamond (x * y) \leq y$;
(pBCI-3) $x \leq x$;
(pBCI-4) $x \leq y,\ y \leq x \Longrightarrow x = y$;
(pBCI-5) $x \leq y \Longleftrightarrow x * y = 0 \Longleftrightarrow x \diamond y = 0$.

A pseudo-BCI-algebra $\mathfrak{X}$ is called a *pseudo-BCK-algebra* if it satisfies the identities

(pBCK) $0 * x = 0 \diamond x = 0$.

Definition 2.2. A pseudo-BCH-algebra is an algebra $\mathfrak{X} = (X; *, \diamond, 0)$ of type $(2, 2, 0)$ satisfying the axioms:

(pBCH-1) $x * x = x \diamond x = 0$;
(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;
(pBCH-3) $x * y = y \diamond x = 0 \Longrightarrow x = y$;
(pBCH-4) $x * y = 0 \Longleftrightarrow x \diamond y = 0$.

Remark 2.3. Observe that if $(X;*,0)$ is a BCH-algebra, then letting $x \diamond y := x * y$, produces a pseudo-BCH-algebra $(X;*,\diamond,0)$. Therefore, every BCH-algebra is a pseudo-BCH-algebra in a natural way. It is easy to see that if $(X;*,\diamond,0)$ is a pseudo-BCH-algebra, then $(X;\diamond,*,0)$ is also a pseudo-BCH-algebra. From Proposition 3.2 of [3] we conclude that if $(X;\leq,*,\diamond,0)$ is a pseudo-BCI-algebra, then $(X;*,\diamond,0)$ is a pseudo-BCH-algebra.

We say that a pseudo-BCH-algebra $\mathfrak{X}$ is *proper* if $* \neq \diamond$ and it is not a pseudo-BCI-algebra.

Remark 2.4. The class of all pseudo-BCH-algebras is a quasi-variety. Therefore, if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two pseudo-BCH-algebras, then the direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is also a pseudo-BCH-algebra. In the case when at least one of $\mathfrak{X}_1$ and $\mathfrak{X}_2$ is proper, then $\mathfrak{X}$ is proper.

Example 2.5. Let $X_1 = \{0, a, b, c\}$. We define the binary operations $*_1$ and $\diamond_1$ on X_1 as follows:

$*_1$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	c	0

and

$\diamond_1$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	c	a	0

It is easy to check that $\mathfrak{X}_1 = (X_1;*_1,\diamond_1,0)$ is a pseudo-BCH-algebra. On the set $X_2 = \{0,1,2,3\}$ consider the operation $*_2$ given by the following table:

$*_2$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

By simple calculation we can get that $\mathfrak{X}_2 = (X_2;*_2,*_2,0)$ is a (pseudo)-BCH-algebra. The direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a pseudo-BCH-algebra. Observe that $\mathfrak{X}$ is proper. Let $x = (a,1)$, $y = (a,3)$ and $z = (a,2)$. Then $(x*y)\diamond(x*z) = (0,1)\diamond(0,0) = (0,1)$ and $z*y = (0,0)$. Since $(0,1) \nleq (0,0)$, we conclude that $\mathfrak{X}$ is not a pseudo-BCI-algebra, and therefore it is a proper pseudo-BCH-algebra.

Proposition 2.6. *Any* (*proper*) *pseudo-BCH-algebra satisfying* (pBCK) *can be extended to a* (*proper*) *pseudo-BCH-algebra containing one element more.*

Proof. Let $\mathfrak{X} = (X;*,\diamond,0)$ be a pseudo-BCH-algebra satisfying (pBCK) and let $\delta \notin X$. On the set $Y = X \cup \{\delta\}$ consider the operations:

$$x *' y = \begin{cases} x * y & \text{if} \quad x, y \in X, \\ \delta & \text{if} \quad x = \delta \text{ and } y \in X, \\ 0 & \text{if} \quad x \in Y \text{ and } y = \delta, \end{cases}$$

and

$$x \diamond' y = \begin{cases} x \diamond y & \text{if} \quad x, y \in X, \\ \delta & \text{if} \quad x = \delta \text{ and } y \in X, \\ 0 & \text{if} \quad x \in Y \text{ and } y = \delta. \end{cases}$$

Obviously, $(Y; *', \diamond', 0)$ satisfies the axioms (pBCH-1), (pBCH-3), and (pBCH-4). Further, the axiom (pBCH-2) is easily satisfied for all $x, y, z \in X$. Moreover, by routine calculation we can verify it in the case when at least one of x, y, z is equal to δ. Thus, by definition, $(Y; *', \diamond', 0)$ is a pseudo-BCH-algebra. Clearly, if $\mathfrak{X}$ is a proper pseudo-BCH-algebra, then $(Y; *', \diamond', 0)$ is also a proper pseudo-BCH-algebra. ■

From Example 2.5 and Proposition 2.6 we conclude that there are infinite many proper pseudo-BCH-algebras.

3. Properties of pseudo-BCH-algebras

Let $\mathfrak{X} = (X; *, \diamond, 0)$ be a pseudo-BCH-algebra. Define the relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$ (or equivalently, $x \diamond y = 0$).

For any $x \in X$ and $n = 0, 1, 2, \ldots$, we put

$$0 *^0 x = 0 \quad \text{and} \quad 0 *^{n+1} x = (0 *^n x) * x;$$
$$0 \diamond^0 x = 0 \quad \text{and} \quad 0 \diamond^{n+1} x = (0 \diamond^n x) \diamond x.$$

Proposition 3.1. *In a pseudo-BCH-algebra $\mathfrak{X}$ the following properties hold (for all $x, y, z \in X$):*

(P1) $x \leq y,\ y \leq x \Longrightarrow x = y$;

(P2) $x \leq 0 \Longrightarrow x = 0$;

(P3) $x * (x \diamond y) \leq y, \quad x \diamond (x * y) \leq y$;

(P4) $x * 0 = x = x \diamond 0$;

(P5) $0 * x = 0 \diamond x$;

(P6) $x \leq y \Longrightarrow 0 * x = 0 \diamond y$;

(P7) $0 \diamond (0 * (0 \diamond x)) = 0 \diamond x, \quad 0 * (0 \diamond (0 * x)) = 0 * x$;

(P8) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$;

(P9) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$.

Proof. (P1) follows from (pBCH-3).
(P2) Let $x \le 0$. Then $x * 0 = 0$. Applying (pBCH-2) and (pBCH-1) we obtain

$$0 \diamond x = (x * 0) \diamond x = (x \diamond x) * 0 = 0 * 0 = 0,$$

that is, $0 \le x$. Therefore $x = 0$ by (P1).

(P3) Using (pBCH-2) and (pBCH-1) we have $(x*(x\diamond y))\diamond y = (x\diamond y)*(x\diamond y) = 0$. Hence $x * (x \diamond y) \le y$. Similarly, $x \diamond (x * y) \le y$.

(P4) Putting $y = 0$ in (P3), we have $x * (x \diamond 0) \le 0$ and $x \diamond (x * 0) \le 0$. From (P2) we obtain $x * (x \diamond 0) = 0$ and $x \diamond (x * 0) = 0$. Thus $x \le x \diamond 0$ and $x \le x * 0$.

On the other hand, $(x \diamond 0) * x = (x * x) \diamond 0 = 0 \diamond 0 = 0$ and $(x * 0) \diamond x = (x \diamond x) * 0 = 0 * 0 = 0$, and so $x \diamond 0 \le x$ and $x * 0 \le x$. By (P1), $x * 0 = x = x \diamond 0$.

(P5) Applying (pBCH-1) and (pBCH-2) we get $0*x = (x\diamond x)*x = (x*x)\diamond x = 0 \diamond x$.

(P6) Let $x \le y$. Then $x\diamond y = 0$ and therefore $0*x = (x\diamond y)*x = (x*x)\diamond y = 0\diamond y$.

(P7) From (P3) it follows that $0 * (0 \diamond x) \le x$ and $0 \diamond (0 * x) \le x$. Hence, using (P5) and (P6) we obtain (P7).

(P8) Applying (pBCH-1) and (pBCH-2) we have

$$\begin{aligned}
(0 \diamond x) \diamond (0 * y) &= (((x * y) * (x * y)) \diamond x) \diamond (0 * y) \\
&= (((x * y) \diamond x) * (x * y)) \diamond (0 * y) \\
&= (((x \diamond x) * y) * (x * y)) \diamond (0 * y) \\
&= ((0 * y) * (x * y)) \diamond (0 * y) \\
&= ((0 * y) \diamond (0 * y)) * (x * y) \\
&= 0 * (x * y).
\end{aligned}$$

(P9) The proof is similar to the proof of (P8). ■

From (P1) and (P3) we get

Corollary 3.2. *Every pseudo-BCH-algebra satisfies* (pBCI-2)–(pBCI-5).

Remark 3.3. In any pseudo-BCI-algebra the relation $\le$ is transitive (see [3], Proposition 3.2). However, in the pseudo-BCH-algebra $\mathfrak{X}$ from Example 2.5 we have $(a, 1) \le (a, 2)$ and $(a, 2) \le (a, 3)$ but $(a, 1) \not\le (a, 3)$.

Theorem 3.4. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then $\mathfrak{X}$ is a pseudo-BCI-algebra if and only if it satisfies the following implication:*

$$x \le y \Longrightarrow x * z \le y * z,\ x \diamond z \le y \diamond z. \tag{3.1}$$

Proof. If $\mathfrak{X}$ is a pseudo-BCI-algebra, then $\mathfrak{X}$ satisfies (3.1) by Proposition 3.2 (b7) of [3]. Conversely, let (3.1) hold in $\mathfrak{X}$ and let $x, y, z \in X$. By (P3), $x \diamond (x * z) \leq z$ and $x * (x \diamond z) \leq z$. Hence $(x \diamond (x * z)) * y \leq z * y$ and $(x * (x \diamond z)) \diamond y \leq z \diamond y$, and so $(x * y) \diamond (x * z) \leq z * y$ and $(x \diamond y) * (x \diamond z) \leq z \diamond y$. Therefore, $\mathfrak{X}$ satisfies (pBCI-1). Consequently, $\mathfrak{X}$ is a pseudo-BCI-algebra. ■

Theorem 3.5. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. The following statements are equivalent:*

(i) $x * (y * z) = (x * y) * z$ *for all* $x, y, z \in X$;

(ii) $0 * x = x = 0 \diamond x$ *for every* $x \in X$;

(iii) $x * y = x \diamond y = y * x$ *for all* $x, y \in X$;

(iv) $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ *for all* $x, y, z \in X$.

Proof. (i) $\Longrightarrow$ (ii). Let $x \in X$. We have $x = x * 0 = x * (x * x) = (x * x) * x = 0 * x$. By (P5), $0 \diamond x = x$.

(iv) $\Longrightarrow$ (ii). The proof is similar to the above proof.

(ii) $\Longrightarrow$ (iii). Let (ii) hold and $x, y \in X$. Applying (P8) and (pBCH-2) we obtain

$$\begin{aligned} x * y &= 0 * (x * y) = (0 \diamond x) \diamond (0 * y) \\ &= x \diamond y \\ &= (0 * x) \diamond y = (0 \diamond y) * x = y * x. \end{aligned}$$

(iii) $\Longrightarrow$ (i). Let $x, y, z \in X$. Using (iii) and (pBCH-2) we get

$$x * (y * z) = (y \diamond z) * x = (y * x) \diamond z = (x * y) * z.$$

(iii) $\Longrightarrow$ (iv) has a proof similar to the proof of implication (iii) $\Longrightarrow$ (i). Hence all the conditions are equivalent. ■

Corollary 3.6. *If $\mathfrak{X}$ is a pseudo-BCH-algebra satisying the idendity $0 * x = x$, then $(X; *, 0)$ is an Abelian group each element of which has order 2 (that is, a Boolean group).*

4. The centre of a pseudo-BCH-algebra. Ideals

An element a of a pseudo-BCH-algebra $\mathfrak{X}$ is said to be *minimal* if for every $x \in X$ the following implication

$$x \leq a \Longrightarrow x = a$$

holds.

Proposition 4.1. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then the following conditions are equivalent (for every $x \in X$):*

(i) *a is minimal;*

(ii) $x \diamond (x * a) = a$*;*

(iii) $0 \diamond (0 * a) = a$*;*

(iv) $a * x = (0 * x) \diamond (0 * a)$*;*

(v) $a * x = 0 \diamond (x * a)$.

Proof. (i) $\Longrightarrow$ (ii). By (P2), $x \diamond (x * a) \leq a$ for all $x \in X$. Since a is minimal, we get (ii).

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (iv). We have $a * x = (0 \diamond (0 * a)) * x = (0 * x) \diamond (0 * a)$.

(iv) $\Longrightarrow$ (v). Applying (P5) and (P8) we see that

$$0 \diamond (x * a) = 0 * (x * a) = (0 \diamond x) \diamond (0 * a) = (0 * x) \diamond (0 * a) = a * x.$$

(v) $\Longrightarrow$ (i). Let $x \leq a$. Then $x * a = 0$ and hence $a * x = 0 \diamond (x * a) = 0$. Thus $a \leq x$. Consequently, $x = a$. ■

Replacing $*$ by $\diamond$ and $\diamond$ by $*$ in Proposition 4.1 we obtain

Proposition 4.2. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then for every $x \in X$ the following conditions are equivalent:*

(i) *a is minimal;*

(ii) $x * (x \diamond a) = a$*;*

(iii) $0 * (0 \diamond a) = a$*;*

(iv) $a \diamond x = (0 \diamond x) * (0 \diamond a)$*;*

(v) $a \diamond x = 0 * (x \diamond a)$.

Proposition 4.3. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $a \in X$. Then a is minimal if and only if there is an element $x \in X$ such that $a = 0 * x$.*

Proof. Let a be a minimal element of $\mathfrak{X}$. By Proposition 4.2, $a = 0 * (0 \diamond a)$. If we set $x = 0 \diamond a$, then $a = 0 * x$.

Conversely, suppose that $a = 0 * x$ for some $x \in X$. Using (P7) we get

$$0 * (0 \diamond a) = 0 * (0 \diamond (0 * x)) = 0 * x = a.$$

From Proposition 4.2 it follows that a is minimal. ■

For $x \in X$, set

$$\overline{x} = 0 \diamond (0 * x).$$

By (P5), $\overline{x} = 0 * (0 * x) = 0 \diamond (0 \diamond x) = 0 * (0 \diamond x)$.

Proposition 4.4. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. For any $x, y \in X$ we have:*

(a) $\overline{x * y} = \bar{x} * \bar{y}$;

(b) $\overline{x \diamond y} = \bar{x} \diamond \bar{y}$;

(c) $\bar{\bar{x}} = \bar{x}$.

Proof. (a) Applying (P8) and (P9) we get

$$\begin{aligned} \overline{x * y} &= 0 \diamond (0 * (x * y)) = 0 \diamond [(0 \diamond x) \diamond (0 * y)] \\ &= [0 * (0 \diamond x)] * [0 \diamond (0 * y)] = \overline{x} * \overline{y}. \end{aligned}$$

(b) has a proof similar to (a).

(c) By (P7), $0 * (0 \diamond (0 * x)) = 0 * x$, that is, $0 * \overline{x} = 0 * x$. Hence $\overline{\overline{x}} = 0 \diamond (0 * \overline{x}) = 0 \diamond (0 * x) = \overline{x}$. ■

Following the terminology from BCH-algebras (see [2], Definition 5) the set $\{x \in X : x = \overline{x}\}$ will be called the *centre* of $\mathfrak{X}$. We shall denote it by Cen$\mathfrak{X}$. By Proposition 4.1, Cen$\mathfrak{X}$ is the set of all minimal elements of $\mathfrak{X}$. We have

$$\text{Cen}\mathfrak{X} = \{\overline{x} : x \in X\}. \tag{4.1}$$

Define $\Phi : \mathfrak{X} \to \text{Cen}\mathfrak{X}$ by $\Phi(x) = \overline{x}$ for all $x \in X$. By Proposition 4.4, Φ is a homomorphism from $\mathfrak{X}$ onto Cen$\mathfrak{X}$. We also obtain

Proposition 4.5. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then* Cen$\mathfrak{X}$ *is a subalgebra of $\mathfrak{X}$.*

Proposition 4.6. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $x, y \in$ Cen$\mathfrak{X}$. Then for every $z \in X$ we have*

$$x \diamond (z * y) = y * (z \diamond x). \tag{4.2}$$

Proof. Let $z \in X$. Using Propositions 4.2 and 4.1 we obtain

$$x \diamond (z * y) = [z * (z \diamond x)] \diamond (z * y) = [z \diamond (z * y)] * (z \diamond x) = y * (z \diamond x),$$

that is, (4.2) holds. ■

Following [5], a pseudo-BCI-algebra $(X; \leq, *, \diamond, 0)$ is said to be *p-semisimple* if it satisfies for all $x \in X$,

$$0 \leq x \Longrightarrow x = 0.$$

From Theorem 3.1 of [5] it follows that if $\mathfrak{X} = (X; \leq, *, \diamond, 0)$ is a pseudo-BCI-algebra, then $\mathfrak{X}$ is p-semisimple if and only if $x = \overline{x}$ for every $x \in X$ (that is, $\mathrm{Cen}\mathfrak{X} = \mathfrak{X}$).

Theorem 4.7. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then* $\mathrm{Cen}\mathfrak{X}$ *is a p-semisimple pseudo-BCI-algebra.*

Proof. Since $\mathrm{Cen}\mathfrak{X}$ is a subalgebra of $\mathfrak{X}$, $\mathrm{Cen}\mathfrak{X}$ is a pseudo-BCH-algebra. Let $x, y, z \in \mathrm{Cen}\mathfrak{X}$ and let $x \leq y$. Since x and y are minimal elements of $\mathfrak{X}$, we get $x = y$. Hence $x * z \leq y * z$ and $x \diamond z \leq y \diamond z$. Then, by Theorem 3.4, $\mathrm{Cen}\mathfrak{X}$ is a pseudo-BCI-algebra. Obviously, $x = \overline{x}$ for every $x \in \mathrm{Cen}\mathfrak{X}$, and therefore $\mathrm{Cen}\mathfrak{X}$ is p-semisimple. ∎

Remark 4.8. From Theorem 3.6 of [5] we deduce that $(\mathrm{Cen}\mathfrak{X}; +, 0)$ is a group, where $x + y$ is $x * (0 \diamond y)$ for all $x, y \in \mathrm{Cen}\mathfrak{X}$.

Definition 4.9. Let X be a pseudo-BCH-algebra. A subset I of X is called an ideal of X if it satisfies for all $x, y \in X$

(I1) $0 \in I$;

(I2) if $x * y \in I$ and $y \in I$, then $x \in I$.

We will denote by $\mathrm{Id}(\mathfrak{X})$ the set of all ideals of $\mathfrak{X}$. Obviously, $\{0\}, X \in \mathrm{Id}(\mathfrak{X})$.

Proposition 4.10. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $I \in \mathrm{Id}(\mathfrak{X})$. For any $x, y \in X$, if $y \in I$ and $x \leq y$, then $x \in I$.*

Proof. Straightforward. ∎

Proposition 4.11. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and I be a subset of X satisfying* (I1)*. Then I is an ideal of $\mathfrak{X}$ if and only if for all $x, y \in X$,*

(I2') *if $x \diamond y \in I$ and $y \in I$, then $x \in I$.*

Proof. Let I be an ideal of $\mathfrak{X}$. Suppose that $x \diamond y \in I$ and $y \in I$. By (P3), $x * (x \diamond y) \leq y$ and from Proposition 4.10 it follows that $x * (x \diamond y) \in I$. Therefore, since $x \diamond y \in I$ and I satisfies (I2), we obtain $x \in I$, that is, (I2') holds. The proof of the implication (I2') $\Rightarrow$ (I2) is analogous. ∎

Example 4.12. Let $X = \{0, a, b, c, d\}$. Define binary operations $*$ and $\diamond$ on X by the following tables:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	b	c	0	d
d	d	d	d	d	0

$\diamond$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	a	0	d
b	b	b	0	0	d
c	c	c	a	0	d
d	d	d	d	d	0

By routine calculation, $\mathfrak{X} = (X; *, \diamond, 0)$ is a pseudo-BCH-algebra. It is easy to see that $\mathrm{Id}(\mathfrak{X}) = \{\{0\}, \{0, a\}, \{0, b\}, \{0, a, b, c\}, X\}$.

The following two propositions give the homomorphic properties of ideal.

Proposition 4.13. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras. If $\varphi : \mathfrak{X} \to \mathfrak{Y}$ is a homomorphism and $J \in \mathrm{Id}(\mathfrak{Y})$, then the inverse image $\varphi^{-1}(J)$ of J is an ideal of $\mathfrak{X}$.*

Proof. Straightforward. ∎

Proposition 4.14. *Let $\varphi : \mathfrak{X} \to \mathfrak{Y}$ be a surjective homomorphism. If I is an ideal of $\mathfrak{X}$ containing $\varphi^{-1}(0)$, then $\varphi(I)$ is an ideal of $\mathfrak{Y}$.*

Proof. Since $0 \in I$, we have $0 = \varphi(0) \in \varphi(I)$. Let $x, y \in Y$ and suppose that $x * y$, $y \in \varphi(I)$. Then there are $a \in X$ and $b, c \in I$ such that $x = \varphi(a)$, $y = \varphi(b)$ and $x * y = \varphi(c)$. We have $\varphi(a * b) = \varphi(c)$ and hence $(a * b) * c \in \varphi^{-1}(0) \subseteq I$. By the definition of an ideal, $a \in I$. Consequently, $x = \varphi(a) \in \varphi(I)$. This means that $\varphi(I)$ is an ideal of $\mathfrak{Y}$. ∎

Definition 4.15. An ideal I of a pseudo-BCH-algebra $\mathfrak{X}$ is said to be *closed* if $0 * x \in I$ for every $x \in I$.

Theorem 4.16. *An ideal I of a pseudo-BCH-algebra $\mathfrak{X}$ is closed if and only if I is a subalgebra of $\mathfrak{X}$.*

Proof. Suppose that I is a closed ideal of $\mathfrak{X}$ and let $x, y \in I$. By (pBCH-2) and (pBCH-1),

$$\begin{aligned} [(x * y) * (0 * y)] \diamond x &= [(x * y) \diamond x] * (0 * y) \\ &= [(x \diamond x) * y] * (0 * y) \\ &= (0 * y) * (0 * y) = 0. \end{aligned}$$

Hence $[(x * y) * (0 * y)] \diamond x \in I$. Since $x, 0 * y \in I$, we have $x * y \in I$. Similarly, $x \diamond y \in I$. Conversely, if I is a subalgebra of $\mathfrak{X}$, then $x \in I$ and $0 \in I$ imply $0 * x \in I$. ∎

Theorem 4.17. *Every ideal of a finite pseudo-BCH-algebra is closed.*

Proof. Let I be an ideal of a finite pseudo-BCH-algebra $\mathfrak{X}$ and let $a \in I$. Suppose that $|X| = n$ for some $n \in \mathbb{N}$. At least two of the $n + 1$ elements:

$$0, 0 * a, 0 *^2 a, \ldots, 0 *^n a$$

are equal, for instance, $0 *^r a = 0 *^s a$, where $0 \le s < r \le n$. Hence

$$0 = (0 *^r a) \diamond (0 *^s a) = [(0 *^s a) \diamond (0 *^s a)] *^{r-s} a = 0 *^{r-s} a.$$

Therefore $0 *^{r-s} a \in I$. Since $a \in I$, by definition, $0 * a \in I$. Consequently, I is a closed ideal of $\mathfrak{X}$. ■

For any pseudo-BCH-algebra $\mathfrak{X}$, we set

$$\mathrm{K}(\mathfrak{X}) = \{x \in X : 0 \leq x\}.$$

Observe that $\mathrm{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X}) = \{0\}$. Indeed, $0 \in \mathrm{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$ and if $x \in \mathrm{Cen}\mathfrak{X} \cap \mathrm{K}(\mathfrak{X})$, then $x = 0 \diamond (0 * x) = 0 \diamond 0 = 0$.

In Example 4.12, $\mathrm{Cen}\mathfrak{X} = \{0, d\}$ and $\mathrm{K}(\mathfrak{X}) = \{0, a, b, c\}$.

It is easy to see that

$$x \in \mathrm{K}(\mathfrak{X}) \Longleftrightarrow \overline{x} = 0 \Longleftrightarrow x \in \Phi^{-1}(0).$$

Thus

$$\mathrm{K}(\mathfrak{X}) = \Phi^{-1}(0). \tag{4.3}$$

Proposition 4.18. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then $\mathrm{K}(\mathfrak{X})$ is a closed ideal of $\mathfrak{X}$.*

Proof. By (4.3) and Proposition 4.13, $\mathrm{K}(\mathfrak{X})$ is an ideal of $\mathfrak{X}$. Let $x \in \mathrm{K}(\mathfrak{X})$. Then $\overline{x} = 0$ and hence $\Phi(0 * x) = 0 * \overline{x} = 0$. Consequently, $0 * x \in \mathrm{K}(\mathfrak{X})$. Thus $\mathrm{K}(\mathfrak{X})$ is a closed ideal. ■

Corollary 4.19. *For any pseudo-BCH-algebra $\mathfrak{X}$ the set $\mathrm{K}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$, and so it is a pseudo-BCH-algebra.*

Proposition 4.20. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BCH-algebras. Then:*

(a) $\mathrm{Cen}(\mathfrak{X} \times \mathfrak{Y}) = \mathrm{Cen}(\mathfrak{X}) \times \mathrm{Cen}(\mathfrak{Y})$;
(b) $\mathrm{K}(\mathfrak{X} \times \mathfrak{Y}) = \mathrm{K}(\mathfrak{X}) \times \mathrm{K}(\mathfrak{Y})$.

Proof. This is immediate from definitions. ■

For any element a of a pseudo-BCH-algebra $\mathfrak{X}$, we define a subset $\mathrm{V}(a)$ of X as

$$\mathrm{V}(a) = \{x \in X : a \leq x\}.$$

Note that $\mathrm{V}(a) \neq \emptyset$, because $a \leq a$ gives $a \in \mathrm{V}(a)$. Furthermore, $\mathrm{V}(0) = \mathrm{K}(\mathfrak{X})$.

Proposition 4.21. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then for each $x \in X$ there exists a unique element $a \in \mathrm{Cen}\mathfrak{X}$ such that $a \leq x$.*

Proof. Let $x \in X$. Take $a = \overline{x}$, that is, $a = 0 \diamond (0 * x)$. By (P3), $a \leq x$. From (4.1) it follows that $a \in \mathrm{Cen}\mathfrak{X}$. To prove uniqueness, let $b \in \mathrm{Cen}\mathfrak{X}$ be such that $b \leq x$. Then $b \diamond x = 0$. Therefore,

$$0 * b = (b \diamond x) * b = (b * b) \diamond x = 0 \diamond x = 0 * x$$

and hence $b = \overline{b} = 0 \diamond (0 * b) = 0 \diamond (0 * x) = \overline{x} = a$. ■

Lemma 4.22. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and $a \in \mathrm{Cen}\mathfrak{X}$. Then*

$$\mathrm{V}(a) = \Phi^{-1}(a).$$

Proof. Suppose that $x \in \mathrm{V}(a)$, that is, $a \leq x$. We have $\overline{x} \leq x$. Since $a, \overline{x} \in \mathrm{Cen}\mathfrak{X}$, by Proposition 4.21, $a = \overline{x}$, that is, $x \in \Phi^{-1}(a)$.
Conversely, if $a = \overline{x}$, then $a \leq x$ by (P3). Hence $x \in \mathrm{V}(a)$. ■

Proposition 4.23. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra. Then:*

(a) $X = \bigcup\limits_{a \in \mathrm{Cen}\mathfrak{X}} \mathrm{V}(a)$;

(b) *if $a, b \in \mathrm{Cen}\mathfrak{X}$ and $a \neq b$, then $\mathrm{V}(a) \cap \mathrm{V}(b) = \emptyset$.*

Proof. (a) Clearly, $\bigcup_{a \in \mathrm{Cen}\mathfrak{X}} \mathrm{V}(a) \subseteq X$ and let $x \in X$. Obviously, $x \in \mathrm{V}(\overline{x})$ and $\overline{x} \in \mathrm{Cen}\mathfrak{X}$. Therefore, $x \in \bigcup_{a \in \mathrm{Cen}\mathfrak{X}} \mathrm{V}(a)$.

(b) Let $a, b \in \mathrm{Cen}(\mathfrak{X})$ and $a \neq b$. On the contrary suppose that $\mathrm{V}(a) \cap \mathrm{V}(b) \neq \emptyset$. Let $x \in \mathrm{V}(a) \cap \mathrm{V}(b)$. Then $a \leq x$ and $b \leq x$. From Proposition 4.21 it follows that $a = b$, a contradition. ■

We now establish a relationship between the ideals of a pseudo-BCH-algebra and the ideals of its centre.

Proposition 4.24. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $A \subseteq \mathrm{Cen}\mathfrak{X}$. The following statements are equivalent:*

(i) *A is an ideal of $\mathrm{Cen}\mathfrak{X}$;*

(ii) $\bigcup\limits_{a \in A} \mathrm{V}(a)$ *is an ideal of $\mathfrak{X}$.*

Proof. Let $I = \bigcup_{a \in A} \mathrm{V}(a)$. From Lemma 4.22 we have $I = \bigcup_{a \in A} \Phi^{-1}(a) = \Phi^{-1}(A)$.

(i) $\Rightarrow$ (ii). Let $A \in \mathrm{Id}(\mathrm{Cen}\mathfrak{X})$. By Proposition 4.13, I is an ideal of $\mathfrak{X}$.

(ii) $\Rightarrow$ (i). Since $I = \Phi^{-1}(A)$, we conclude that $A = \Phi(I)$. Obviously, $0 \in A$ and hence $\Phi^{-1}(0) \subseteq I$. Applying Proposition 4.14 we deduce that A is an ideal of $\mathrm{Cen}\mathfrak{X}$. ■

Theorem 4.25. *There is a one-to-one correspondence between ideals of a pseudo-BCH-algebra $\mathfrak{X}$ containing $\mathrm{K}(\mathfrak{X})$ and ideals of $\mathrm{Cen}\mathfrak{X}$.*

Proof. Set $\mathcal{I} = \{I \in \mathrm{Id}(\mathfrak{X}) : I \supseteq \mathrm{K}(\mathfrak{X})\}$ and $\mathcal{C} = \mathrm{Id}(\mathrm{Cen}\mathfrak{X})$. We consider two functions:

$$f : I \in \mathcal{I} \to \{\overline{x} : x \in I\} \quad \text{and} \quad g : A \in \mathcal{C} \to \bigcup_{a \in A} \mathrm{V}(a).$$

Since $f(I) = \Phi(I)$, from Proposition 4.14 we conclude that f maps $\mathcal{I}$ into $\mathcal{C}$. By Proposition 4.24, $g(A) = \bigcup_{a\in A} \mathrm{V}(a) \in \mathcal{I}$ for all $A \in \mathcal{C}$, and therefore g maps $\mathcal{C}$ into $\mathcal{I}$. We have

$$(f \circ g)(A) = \Phi(\Phi^{-1}(A)) = A \qquad \text{for all} \quad A \in \mathcal{C}. \tag{4.4}$$

Obviously, $I \subseteq \Phi^{-1}(\Phi(I))$. Let now $x \in \Phi^{-1}(\Phi(I))$, that is, $\overline{x} = \overline{a}$ for some $a \in I$. Then $\Phi(x * a) = 0$, and hence $x * a \in \Phi^{-1}(0)$. Therefore, $x * a \in I$ (since $\Phi^{-1}(0) = \mathrm{K}(\mathfrak{X}) \subseteq I$). By definition, $x \in I$. Thus $\Phi^{-1}(\Phi(I)) = I$. Consequently,

$$(g \circ f)(I) = \Phi^{-1}(\Phi(I)) = I \qquad \text{for all} \quad I \in \mathcal{I}. \tag{4.5}$$

We conclude from (4.4) and (4.5) that $f \circ g = \mathrm{id}_{\mathcal{C}}$ and $g \circ f = \mathrm{id}_{\mathcal{I}}$, hence that f and g are inverse bijections between $\mathcal{I}$ and $\mathcal{C}$. ■

Example 4.26. Let $\mathfrak{X}_1 = (\{0, a, b, c\}; *_1, \diamond_1, 0)$ be the pseudo-BCH-algebra from our Example 2.5. Consider the set $X_2 = \{0, 1, 2, 3, 4\}$ with the operation $*_2$ defined by the following table:

$*_2$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

From Example 3 of [17] it follows that $\mathfrak{X}_2 = (X_2; *_2, *_2, 0)$ is a (pseudo)-BCH-algebra. The direct product $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ is a pseudo-BCH-algebra. From Proposition 4.20 we have $\mathrm{Cen}\mathfrak{X} = \{0\} \times \{0, 2, 3, 4\}$ and $\mathrm{K}(\mathfrak{X}) = X_1 \times \{0, 1\}$. It is easy to see that $\mathrm{Id}(\mathrm{Cen}\mathfrak{X}) = \{\{(0, 0)\}, \{(0, 0), (0, 3)\}, \mathrm{Cen}\mathfrak{X}\}$. Then, by Theorem 4.25, $\mathfrak{X}$ has three ideals containing $\mathrm{K}(\mathfrak{X})$, namely: $\mathrm{K}(\mathfrak{X})$, $\mathrm{K}(\mathfrak{X}) \cup \{(0, 3), (a, 3), (b, 3), (c, 3)\}$ and $\mathfrak{X}$.

Now we shall show that the centre $\mathrm{Cen}\mathfrak{X}$ defines a regular congruence on a pseudo-BCH-algebra $\mathfrak{X}$. Let $\mathrm{Con}\mathfrak{X}$ denote the set of all congruences on $\mathfrak{X}$ and let

$\theta \in \mathrm{Con}\mathfrak{X}$. For $x \in X$, we write x/θ for the congruence class containing x, that is, $x/\theta = \{y \in X : y\,\theta\,x\}$. Set $X/\theta = \{x/\theta : x \in X\}$. It is easy to see that the factor algebra $\mathfrak{X}/\theta = \langle X/\theta; *, \diamond, 0/\theta\rangle$ satisfies (pBCH-1) and (pBCH-2). The axioms (pBCH-3) and (pBCH-4) are not necessarity satisfied. If $\mathfrak{X}/\theta$ is a pseudo-BCH-algebra, then we say that θ is *regular.*

Remark 4.27. A. Wroński has shown that non-regular congruences exist in BCK-algebras (see [18]) and hence in pseudo-BCH-algebras.

Theorem 4.28. *Let $\mathfrak{X}$ be a pseudo-BCH-algebra and let $\theta_c = \{(x, y) \in X^2 : \overline{x} = \overline{y}\}$. Then θ_c is a regular congruence on $\mathfrak{X}$ and $\mathfrak{X}/\theta_c \cong \mathrm{Cen}\mathfrak{X}$.*

Proof. The mapping Φ is a homomorphism from $\mathfrak{X}$ onto $\mathrm{Cen}\mathfrak{X}$. Moreover we have

$$\mathrm{Ker}\Phi = \{(x, y) \in X^2 : \Phi(x) = \Phi(y)\} = \theta_c.$$

By the Isomorphism Theorem we get $\mathfrak{X}/\theta_c \cong \mathrm{Cen}\mathfrak{X}$, and therefore θ_c is a regular congruence on $\mathfrak{X}$. ■

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