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SOME RESULTS OF REVERSE DERIVATION ON PRIME AND SEMIPRIME Γ-RINGS

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Abstract

In the present paper, it is introduced the definition of a reverse derivation on a Γ-ring M. It is shown that a mapping derivation on a semiprime Γ-ring M is central if and only if it is reverse derivation. Also it is shown that M is commutative if for all $a, b \in I$ (I is an ideal of M) satisfying $d(a) \in Z(M)$, and $d(a \circ b) = 0$.

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1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [6] (which is presently known as a Γ_N-ring), more general than a ring, and afterwards it was generalized by Barnes [1]. This generalization states that every Γ_N-ring is a a Γ-ring, but the converse is not necessarily true. After these two authors many mathematicians made works on Γ-ring as will as (Kyuno [4], Luh [5]), were obtained some important properties of Γ-ring.

The gamma ring is defined by Barnes in [1] as follows: Let M and Γ be two additive abelian groups. If there exists a mapping M×Γ×M⟶ M (sending (x, α, y) in to $x\alpha y$) for all $x, y, z \in$ M and $\alpha, \beta \in \Gamma$, satisfying the following conditions:

(i) $x\alpha y \in$ M,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha+\beta)y = x\alpha y + x\beta y$,
$x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a Γ-ring (in the sense of Barnes).

We may note that it follows from (i)–(iii) that $0\alpha x = x0y = 0\alpha x = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$.

An additive subgroup I of M is called a left (right) ideal of M if MΓI⊆I (IΓM⊆I). If I is both left and right ideal of M, then we say I is an ideal of M. Besides a Γ-ring M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for $x \in$ M. M is called a prime Γ-ring if for any two elements $x, y \in$ M, $x\Gamma$M $\Gamma y = 0$ implies either $x = 0$ or $y = 0$, and M is called semiprime if $x\Gamma$M$\Gamma x = 0$ with $x \in$ M implies $x = 0$. Note that every prime Γ-ring is semiprime. Furthermore, the set Z(M)=$\{x \in$ M; $x\alpha y = y\alpha x$ for all $x, y \in$ M and $\alpha \in \Gamma\}$ is called the center of M. The commutator $x\alpha y - y\alpha x$ will be denoted by $[x, y]_\alpha$.

The notion of derivation in Γ-ring have been introduced by Sapanci and Nakajima [7] as follows: An additive mapping d : M$\longrightarrow$ M is called a derivation if $d(x\alpha y) = d(x)\alpha y + xd(y)$ for all $x, y \in$ M and $\alpha \in \Gamma$. The notion of a reverse derivation in a ring R was introduced by Bresar and Vukman [2]. An additive mapping $d : R \longrightarrow R$ is called a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. Inspired by the definition in [2], we introduce the definition of revsesr derivation on a Γ-ring M as follows: An additive mapping d : M$\longrightarrow$M is called a reverse derivation if $d(x\alpha y) = d(y)\alpha x + y\alpha d(x)$ for all $x, y \in$ M and $\alpha \in \Gamma$.

Throughout this paper, we shall use (*) for $x\alpha y\beta z = x\beta y\alpha z$, for all x, y, z $\in$ M and $\alpha, \beta \in \Gamma$, we show that for a semiprime Γ-ring M, any reverse derivation is a derivation mapping M into its center and we will show that the derivation and the reverse derivation are not coincide by the following examples.

Example 1.1. Let R be a ring and

$$M = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | \, x, y \in R\}, \quad \text{where} \quad R^2 \neq 0,$$

$$\text{and} \quad \Gamma = \{\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix} | \; n \text{ is an integre .}\}$$

Then it is easy to show that M is a Γ-ring. Let d :M$\longrightarrow$M defined by

$$d(A) = d\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}.$$

It is easy to show that d is derivation but not reverse derivation.

Example 1.2. Let R be a ring and

$$M = \{ \begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in R \},$$

$$\text{and} \quad \Gamma = \{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & n \end{pmatrix} \mid n \text{ is an integre } \}.$$

Then it is easy to show that M is a Γ-ring. Let d :M$\longrightarrow$M defined by

$$d(A) = d\left(\begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 & -z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that d is reverse derivation but not derivation.

Lemma 1.3 (3, Lemma 2.3). *Let M be a semiprime Γ-ring satisfying the assumption $(*)$ and $a \in M$ such that $a\beta[a, x]_\alpha = 0$, for all $x \in$M, then $a \in Z(M)$, the center of M.*

The following result shows that a reverse derivation is a derivation on semiprime Γ-rings.

Theorem 1.4. *If M is a semiprime Γ-ring satisfying the assumption $(*)$ and d is a nonzero derivation, then d is central if and only if d is reverse derivation.*

Proof. Suppose that d is central derivation, then it is clear that d is reverse derivation. Now we suppose that d is reverse derivation, then we have

$$d(x\alpha y) = d(y)\alpha x + y\alpha d(x)$$

Replacing y by $y\beta y$, we get for all $x, y \in$ M and $\alpha, \beta \in \Gamma$.

$$\begin{aligned} d(x\alpha(y\beta y)) &= d(y\beta y)\alpha x + y\beta y\alpha d(x) \\ &= d(y)\beta y\alpha x + y\beta d(y)\alpha x + y\beta y\alpha d(x). \end{aligned} \tag{1.1}$$

On the other hand, we obtain

$$\begin{aligned} d((x\alpha y)\beta y) &= d(y)\beta x\alpha y + y\beta d(x\alpha y) \\ &= d(y)\beta x\alpha\beta y + y\beta d(y)\alpha x + y\beta y\alpha d(x). \end{aligned} \tag{1.2}$$

From (1.1) and (1.2) we get

$$d(y)\beta y\alpha x = d(y)\beta x\alpha y$$

This implies

$$d(y)\beta[x,y]_\alpha = 0, \quad \text{for all } \ x,y \in M \ \text{ and } \ \alpha,\beta \in \Gamma. \tag{1.3}$$

Linearization (1.3) with respect to y and using (1.3), we have

$$\begin{aligned} 0 &= d(y+z)\beta[x,y+z]_\alpha \\ &= d(y)\beta[x,z]_\alpha + d(z)\beta[x,y]_\alpha \ \text{ for all } \ x,y,z \in M \ \text{ and } \ \alpha,\beta \in \Gamma. \end{aligned}$$

That is

$$d(y)\beta[x,z]_\alpha = -d(z)\beta[x,y]_\alpha = d(z)\beta[y,x]_\alpha. \tag{1.4}$$

Replacing x by $w\gamma x$ in (1.3), we get

$$d(y)\beta w\gamma[x,z]_\alpha = 0 \ \text{ for all } \ x,y,w \in M \ \text{ and } \ \alpha,\beta,\gamma \in \Gamma. \tag{1.5}$$

Replacing w by $[x,z]_\alpha\delta w\beta d(z)$ in (1.5) and using (1.4), we get

$$0 = d(y)\beta[x,z]_\alpha\delta w\beta d(z)\gamma[x,y]_\alpha = -d(z)\beta[x,y]_\alpha\delta w\beta d(z)\gamma[x,y]_\alpha.$$

Hence

$$d(z)\beta[x,y]_\alpha\delta w\beta d(z)\gamma[x,y]_\alpha = 0, \quad \text{for all } \ x,y,z,w \in M \ \text{ and } \ \alpha,\beta,\gamma,\delta \in \Gamma.$$

By semiprimeness we obtain $d(z)\beta[x,y]_\alpha = 0$. By Lemma 1.3 we have $d(z) \in Z(M)$, for all $z \in M$.

Hence $d(x\alpha y) = d(y)\alpha x + y\alpha d(x) = x\alpha d(y) + d(x)\alpha y$. ∎

Form the theorem we can get the following corollaries

Corollary 1.5. *A mapping d on a semiprime Γ-ring M is reverse derivation if and only if it is left derivation.*

Corollary 1.6. *Let M be a prime Γ-ring. If M admits a nonzero reverse derivation, then M is commutative.*

Lemma 1.7 (8, Lemma 2). *Let M be a 2-torsion free prime Γ-ring and I be a nonzero ideal of M. For $a, b \in M$, if $a\Gamma I\Gamma b = 0$, then either $a = 0$ or $b = 0$.*

Theorem 1.8. *Let d be a nonzero reverse derivation of a prime Γ-ring M satisfying the assumption $(*)$ and I be an ideal of M. If $d(a) \in Z(M)$, for all $a \in I$, then M is commutative.*

Proof. Since $d(a) \in Z(M)$, then

$$[d(a), y]_\alpha = 0, \quad \text{for all} \quad a \in I \quad \text{and} \quad y \in M. \tag{1.6}$$

Replacing a by $a\beta x$, we get

$$[d(a\beta x), y]_\alpha = 0, \quad \text{for all} \quad a \in I \quad \text{and} \quad x, y \in M.$$

Hence we obtain

$$\begin{aligned} 0 &= [d(x)\beta a + x\beta d(a), y]_\alpha \\ &= [d(x), y]_\alpha \beta a + d(x)\beta[a, y]_\alpha + x\beta[d(a), y]_\alpha + [x, y]_\alpha \beta d(a) \\ &= [d(x), y]_\alpha \beta a + d(x)\beta[a, y]_\alpha + [x, y]_\alpha \beta d(a). \end{aligned}$$

Put $y = x$, we obtain

$$[d(x), x]_\alpha \beta a + d(x)\beta[a, x]_\alpha = 0. \tag{1.7}$$

By expanding equation (1.7), we get

$$\begin{aligned} 0 &= d(x)\alpha x\beta a - x\alpha d(x)\beta a + d(x)\beta a\alpha x - d(x)\beta x\alpha a \\ &= -x\alpha d(x)\beta a + d(x)\beta a\alpha x. \end{aligned}$$

That is

$$d(x)\beta a\alpha x = x\alpha d(x)\beta a. \tag{1.8}$$

Hence

$$d(x)\beta a\alpha x\gamma z = x\alpha d(x)\beta a\gamma z. \tag{1.9}$$

Replacing a by az in (1.8), we get

$$d(x)\beta a\gamma z\alpha x = x\alpha d(x)\beta a\gamma z. \tag{1.10}$$

Comparing (1.9) and (1.10) we obtain

$$d(x)\beta a\gamma z\alpha x = d(x)\beta a\alpha x\gamma z.$$

By using property $(*)$ we get $d(x)\beta a\gamma[z,x]_\alpha = 0$, for all $a \in I$ and $x, z \in$ M. Therefore by Lemma 1.7 we get $d(x) = 0$ or $[z,x]_\alpha = 0$, but $d(x) \neq 0$, hence M is commutative. ■

References

[1] W.E. Barness, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (3) (1966) 411–422.

[2] M. Bresar and J. Vukman, *On some additive mappings in rings with involution*, Aequation Math. **38** (1989) 178–185. doi:10.1007/BF01840003

[3] Md. F. Hoque and A.C. Paul, *On centralizers of semiprime gamma rings*, Intr. Math. Forum **6** (13) (2011) 627–638.

[4] S. Kyuno, *On prime gamma rings*, Pacific J. Math. **75** (1978) 185–190.

[5] L. Luh, *On the theory of simple Gamma rings*, Michigan Math. J. **16** (1969) 576–584. doi:10.1307/mmj/1029000167

[6] N. Nobusawa, *On a generalization of the ring theory*, Osaka J. Math. **1** (1964) 81–89.

[7] M. Sapanci and A. Nakajima, *A note on gamma rings*, Turkish J. Math. **20** (1996) 463–465.

[8] M. Soytürk, *The commutativity in prime gamma rings with Derivation*, Turkish J. Math. **18** (1994) 149–155.

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