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NON-DETERMINISTIC LINEAR HYPERSUBSTITUTIONS

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Abstract

A non-deterministic hypersubstitution maps operation symbols to sets of terms of the corresponding arity. A non-deterministic hypersubstitution of type τ is said to be linear if it maps any operation symbol to a set of linear terms of the corresponding arity. We show that the extension of non-deterministic linear hypersubstitutions of type τ map sets of linear terms to sets of linear terms. As a consequence, the collection of all non-deterministic linear hypersubstitutions forms a monoid. Non-deterministic linear hypersubstitutions can be applied to identities and to algebras of type τ.

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1. Introduction

In 2008, K. Denecke, P. Glubudom and J. Koppitz [3] studied non-deterministic hypersubstitutions and considered the extensions of such mappings. They also showed that the set of all non-deterministic hypersubstitutions forms a monoid under a certain binary operation.

The concept of linear terms has a long history as old as the concept of terms. In 2012, M. Couceiro and E. Lehtonen [2] gave a sufficient and necessary condition that a set of operations is the set of linear term operations of some algebra.

In this paper, we define non-deterministic linear hypersubstitutions and we show that the set of all non-deterministic linear hypersubstitutions forms a monoid.

Let $n \geq 1$ be a natural number. Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set. The set X_n is called an *alphabet* and its elements are called *variables*. Let

$\{f_i : i \in I\}$ be the set of *operation symbols*, indexed by the set I. The sets X_n and $\{f_i : i \in I\}$ have to be disjoint. To every operation symbol f_i, we assign a natural number $n_i \geq 1$, called the *arity* of f_i. As in the definition of algebra, the sequence $\tau = (n_i)_{i \in I}$ of all the arities is called the *type*. With this notation for operation symbols and variables, we can define the terms of type τ, (see also [5]).

The *n-ary terms* of type τ are defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an n-ary term.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

(iii) The set $W_\tau(X_n) = W_\tau(x_1, \ldots, x_n)$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

We denote by $W_\tau(X)$ the set of all terms of type τ over the countably infinite alphabet X, that is,

$$W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n).$$

Let t be a term. We denote the set of variables occurring in the term t by $var(t)$.

A term in which each variables occurs at most once, is said to be linear. For a formal definition of n-ary linear terms we replace condition (ii) in the definition of terms by a slightly different condition.

Definition [2]**.** An *n-ary linear term* of type τ is defined in the following inductive way:

(i) For any $j \in \{1, \ldots, n\}$, $x_j \in X_n$ is an n-ary linear term (of type τ).

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary linear terms and if $var(t_j) \cap var(t_k) = \emptyset$ for all $1 \leq j < k \leq n_i$, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary linear term.

(iii) The set $W_\tau^{\text{lin}}(X_n)$ of all n-ary linear terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

The set of all linear terms of type τ over the countably infinite alphabet X is defined by

$$W_\tau^{\text{lin}}(X) := \bigcup_{n \geq 1} W_\tau^{\text{lin}}(X_n).$$

The set $W_\tau(X)$ of all terms of type τ is closed under substitution. This is not true for linear terms as the following example shows: Let $\tau = (2)$ and let f be a binary operation symbol. Then $f(x_1, x_2)$ and $f(x_2, x_1)$ are linear, but if we substitute

$f(x_1, x_2)$ for x_1 and $f(x_2, x_1)$ for x_2 in $f(x_1, x_2)$, we obtain $f(f(x_1, x_2), f(x_2, x_1))$, which is not a linear.

One of the most interesting operations on terms is the superposition. Let $W_\tau(X_n)$ and $W_\tau(X_m)$ be the set of all n-ary and m-ary terms, respectively. Then the *superposition*

$$S^n_m : W_\tau(X_n) \times (W_\tau(X_m))^n \to W_\tau(X_m)$$

is defined inductively as follows:

(i) $S^n_m(x_j, t_1, \ldots, t_n) := t_j$, $x_j \in X_n$ and $t_i \in W_\tau(X_m)$.

(ii) $S^n_m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_{n_i}) :=$
$f_i(S^n_m(s_1, t_1, \ldots, t_{n_i}), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_{n_i}))$.

We can extend the superposition operation S^n_m to sets of terms by the following: Let m, n be natural numbers. We define

$$\hat{S}^n_m : \mathcal{P}(W_\tau(X_n)) \times (\mathcal{P}(W_\tau(X_m)))^n \to \mathcal{P}(W_\tau(X_m))$$

inductively as follows. Let $B \in \mathcal{P}(W_\tau(X_n)), B_1, \ldots, B_n \in \mathcal{P}(W_\tau(X_m))$.

(i) If $B = \{x_j\}$ for $1 \le j \le n$, then $\hat{S}^n_m(\{x_j\}, B_1, \ldots, B_n) := B_j$.

(ii) If $B = \{f_i(t_1, \ldots, t_{n_i})\}$ and if we suppose that the sets $\hat{S}^n_m(\{t_j\}, B_1, \ldots, B_n)$ for $1 \le j \le n_i$ are already defined, then $\hat{S}^n_m(\{f_i(t_1, \ldots, t_n)\}, B_1, \ldots, B_n) := \{f_i(r_1, \ldots, r_{n_i}) : r_j \in \hat{S}^n_m(\{t_j\}, B_1, \ldots, B_n), 1 \le j \le n_i\}$.

(iii) If B is an arbitrary non-empty subset of $W_\tau(X_n)$, we define

$$\hat{S}^n_m(B, B_1, \ldots, B_n) := \bigcup_{b \in B} \hat{S}^n_m(\{b\}, B_1, \ldots, B_n).$$

If one of the sets $B, B_1, \ldots, B_n$ is empty, we define $\hat{S}^n_m(B, B_1, \ldots, B_n) = \emptyset$.

Let $\tau = (n_i)_{i \in I}$ be a type and let $(f_i)_{i \in I}$ be an indexed set of operation symbols of type τ. Any mapping

$$\sigma : \{f_i : i \in I\} \to \mathcal{P}(W_\tau(X))$$

with $\sigma(f_i) \subseteq W_\tau(X_{n_i})$ for $i \in I$ is called a *non-deterministic hypersubstitution* of type τ [3]. For short we write non-deterministic hypersubstitution as nd-hypersubstitution. Every nd-hypersubstitution σ of type τ induces a mapping $\hat{\sigma} : \mathcal{P}(W_\tau(X)) \to \mathcal{P}(W_\tau(X))$ by the following inductive definition [3]:

(i) $\hat{\sigma}[\emptyset] := \emptyset$,

(ii) $\hat{\sigma}[\{x\}] := \{x\}$ for every variable $x \in X$,

(iii) For $t = f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X)$ we set

$$\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] := \hat{S}^{n_i}_m(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])$$

if we inductively assume that $\hat{\sigma}[\{t_j\}], 1 \leq j \leq n_i$ are already defined. Here n_i is the arity of f_i.

(iv) $\hat{\sigma}[B] := \bigcup\{\hat{\sigma}[\{t\}] : t \in B \subseteq W_\tau(X)\}$.

We denote by $Hyp^{nd}(\tau)$ the set of all non-deterministic hypersubstitutions of type τ.

In [3], the authors used the mapping $\hat{\sigma}$ for a nd-hypersubstitution σ on the set $Hyp^{nd}(\tau)$ to define a binary operation

$$\circ_{nd} : Hyp^{nd}(\tau) \times Hyp^{nd}(\tau) \to Hyp^{nd}(\tau)$$

by $\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Hyp^{nd}(\tau)$. The nd-hypersubstitution σ_{id} with $\sigma_{id}(f_i) := \{f_i(x_1, \ldots, x_{n_i})\}$, for all $i \in I$, is an identity element. They have shown that the algebra $(Hyp^{nd}(\tau); \circ_{nd}, \sigma_{id})$ is a monoid.

2. Non-deterministic linear hypersubstitutions

Non-deterministic linear hypersubstitution (for short, *nd-linear hypersupstitution*) map operation symbols to sets of linear terms of the corresponding arity. Formally, we define nd-linear hypersubstitutions in the following way:

Definition. A *non-deterministic linear hypersubstitution* of type τ is a mapping

$$\sigma : \{f_i \mid i \in I\} \to \mathcal{P}(W^{\text{lin}}_\tau(X))$$

with $\sigma(f_i) \subseteq W^{\text{lin}}_\tau(X_{n_i})$ for $i \in I$.

We denote $Hyp^{nd}_{\text{lin}}(\tau)$ by the set of all non-deterministic linear hypersubstitutions. For the extension of an nd-linear hypersubstitution σ the following holds:

Lemma 1 [1]**.** *For any linear hypersubstitution σ and any linear term t we have*

$$var(t) \supseteq var(\hat{\sigma}[t]).$$

Lemma 2. *For any nd-linear hypersubstitution σ and any set of linear terms T we have*

$$var(T) \supseteq var(\hat{\sigma}[T]).$$

Proof. If T is a one-element set, then we will give a proof by induction on the complexity of the linear term which forms the only element of the one-element set T.

1. If $T = \{x_j\}$, where $x_j \in X$, then

$$\begin{aligned} var(T) &= var(\{x_j\}) \\ &= var(\hat{\sigma}[\{x_j\}]) \\ &= var(\hat{\sigma}[T]). \end{aligned}$$

2. If $T = \{f_i(t_1, \ldots, t_{n_i})\}$ and we assume that

$$var(\{t_j\}) \supseteq var(\hat{\sigma}[\{t_j\}]),$$

for all $1 \leq j \leq n_i$, then

$$\begin{aligned} var(T) &= var(\{f_i(t_1, \ldots, t_{n_i})\}) \\ &= \bigcup_{j=1}^{n_i} var(\{t_j\}) \\ &\supseteq \bigcup_{j=1}^{n_i} var(\hat{\sigma}[\{t_j\}]) \\ &\supseteq var(\hat{S}^{n_i}_{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}])) \\ &= var(\hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}]) \\ &= var(\hat{\sigma}[T]). \end{aligned}$$

3. If T is an arbitrary non-empty subset of $W_\tau^{\text{lin}}(X)$, then

$$\begin{aligned} var(T) &= \bigcup_{t \in T} var(\{t\}) \\ &\supseteq \bigcup_{t \in T} var(\hat{\sigma}[\{t\}]) \\ &= var(\bigcup_{t \in T} \hat{\sigma}[\{t\}]) \\ &= var(\hat{\sigma}[T]). \end{aligned}$$

4. If T is the empty set, then $\emptyset = var(T) = var(\hat{\sigma}[\emptyset]) = var(\emptyset) = \emptyset$.

Therefore we have $var(T) \supseteq var(\hat{\sigma}[T])$. ∎

Lemma 3. *For a set of linear terms of the form* $T = \{f_i(t_1, \ldots, t_{n_i})\}$ *and an nd-linear hypersubstitution* σ *we have*

$$var(\hat{\sigma}[\{t_j\}]) \cap var(\hat{\sigma}[\{t_k\}]) = \emptyset$$

for all $1 \leq j < k \leq n_i$.

Proof. By the previous lemma we have $var(\{t_l\}) \supseteq var(\hat{\sigma}[\{t_l\}])$ for any $1 \leq l \leq n_i$ and thus

$$\emptyset = var(\{t_j\}) \cap var(\{t_k\}) \supseteq var(\hat{\sigma}[\{t_j\}]) \cap var(\hat{\sigma}[\{t_k\}]).$$

Therefore $var(\hat{\sigma}[\{t_j\}]) \cap var(\hat{\sigma}[\{t_k\}]) = \emptyset$. ■

Proposition 4. *The extension of any nd-linear hypersubstitution maps non-empty sets of linear terms to non-empty sets of linear terms.*

Proof. Let T be an element in $\mathcal{P}(W_\tau^{\mathrm{lin}}(X))$ and let $\sigma \in Hyp_{\mathrm{lin}}^{nd}(\tau)$.

1. If T is a one-element set, then we will give a proof by induction on the complexity of the linear term which forms the only element of the one-element set T.

   (a) If $T = \{x_j\}$, where $x_j \in X$, then

   $$\hat{\sigma}[T] = \hat{\sigma}[\{x_j\}] = \{x_j\},$$

   is a set of linear terms.

   (b) If $T = \{f_i(t_1, \ldots, t_{n_j})\}$, by the previous lemma we have $var(\hat{\sigma}[\{t_j\}]) \cap var(\hat{\sigma}[\{t_k\}]) = \emptyset$ for all $1 \leq j < k \leq n_i$, and if we assume that $\hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]$ are sets of linear terms, then

   $$\begin{aligned} \hat{\sigma}[T] &= \hat{\sigma}[\{f_i(t_1, \ldots, t_{n_i})\}] \\ &= \hat{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \ldots, \hat{\sigma}[\{t_{n_i}\}]), \end{aligned}$$

   is a set of linear terms.

2. If T is an arbitrary non-empty subset of $W_\tau^{\mathrm{lin}}(X)$, then $\hat{\sigma}[T] = \bigcup_{t \in T} \hat{\sigma}[\{t\}]$ is a non-empty set of linear terms.

Thus, the extension of an nd-linear hypersubstitution maps non-empty sets of linear terms to non-empty sets of linear terms. ■

Since the extension of an nd-linear hypersubstitution of type τ maps $\mathcal{P}(W_\tau^{\text{lin}}(X))$ to $\mathcal{P}(W_\tau^{\text{lin}}(X))$ we may define a product $\sigma_1 \circ_{nd} \sigma_2$, by

$$\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2.$$

Here $\circ$ is the usual composition of mappings. By the previous lemma $(\sigma_1 \circ_{nd} \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)]$ is a set of linear terms.

From the above facts we obtain the following theorem.

Theorem 5. *The set of all nd-linear hypersubstitutions is a submonoid of the set of all nd-hypersubstitution. That is,* $(Hyp^{nd}_{\text{lin}}(\tau), \circ_{nd}, \sigma_{id})$ *is a submonoid of the monoid* $(Hyp^{nd}(\tau), \circ_{nd}, \sigma_{id})$.

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