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GRADED HILBERT-SYMBOL EQUIVALENCE OF NUMBER FIELDS

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Abstract

We present a new criterion for the existence of Hilbert-symbol equivalence of two number fields. In principle, we show that the system of local conditions for this equivalence may be expressed in terms of Clifford invariants in place of Hilbert-symbols, shifting the focus from Brauer groups to Brauer-Wall groups.

Keywords: Brauer group, Brauer-Wall group, Hilbert symbol equivalence, Witt equivalence, graded quaternion algebras.

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1. Introduction and notation

The *Witt functor* is a covariant functor from the category of fields into the category of commutative rings, that assigns to a field K its *Witt ring* WK. This is a set of similarity classes of non-degenerate symmetric bilinear forms over K, equipped with an addition induced by an orthogonal sum and a multiplication induced by a tensor product. Thus, in a certain sense, the Witt ring encodes information about all possible orthogonal geometries over K. It is natural to wonder to what extend arithmetic of a field determines possible geometries over it. This leads to the notion of *Witt equivalence*—a concept introduced in 1970 by D. Harrison (see [8]). Two fields are said to be Witt equivalent, when there is an isomorphism between their Witt rings. Hence, in a sense, Witt equivalent fields admit the same classes of orthogonal geometries, notwithstanding the differences in their underlying arithmetics. The basic tool for studying Witt equivalence of global fields is a so called *Hilbert-symbol equivalence* (HSE for short) introduced[1] in the beginning of 1990s by R. Perlis, K. Szymiczek, P.E. Conner and R. Litherland (see [12]). Since then, HSE has been studied in numerous papers (see e.g. [1, 2, 16, 17]) and generalized to higher degree forms ([3, 6, 7, 13]) and also to other classes of fields. In fact, recently Hilbert-symbol equivalence has became an independent research subject by itself for a number of authors including: A. Czogała and B. Rothkegel ([4, 5]), T.C. Palfrey ([11]), M. Somodi ([14, 15]).

For any field K, by $\mathrm{Br}(K)$ we denote the Brauer group of similarity classes of central simple algebras over K and by $\mathrm{BW}(K)$ the Brauer-Wall group of classes of central simple *graded* K-algebras (for details see e.g. [10, Chapter IV]). It is well known, that the subgroup of $\mathrm{Br}(K)$ consisting of elements of order 2 is generated by classes of quaternion algebras. We shall denote this group by $\mathrm{Q}(K)$. Its counterpart—a subgroup of $\mathrm{BW}(K)$ generated by classes of graded quaternion algebras will be denoted $\mathrm{GQ}(K)$.

Let K and L be two number fields. Denote by Ω_K (respectively Ω_L) the set of all primes of K (resp. L), i.e. the set of classes of non-trivial valuations on K (resp. L). A pair of maps (t, T), where $t\colon \dot{K}/\dot{K}^2 \xrightarrow{\sim} \dot{L}/\dot{L}^2$ is an isomorphism of the square class groups and $T\colon \Omega_K \xrightarrow{\sim} \Omega_L$ is a bijection of their sets of primes, is called a Hilbert-symbol equivalence (c.f. [18, §6.4] or [4, p. 14]), if for every prime $\mathfrak{p} \in \Omega_K$ the map

$$\left(\frac{a,b}{K_{\mathfrak{p}}}\right) \mapsto \left(\frac{ta,tb}{L_{T\mathfrak{p}}}\right) \tag{1}$$

induces a group isomorphism $\mathrm{Q}(K_{\mathfrak{p}}) \xrightarrow{\sim} \mathrm{Q}(L_{T\mathfrak{p}})$.

Remark 1. Since for every prime $\mathfrak{p}$ of a number field K, the group $\mathrm{Q}(K_{\mathfrak{p}})$ either consists of two elements $\{\pm 1\}$ or is trivial (when $K_{\mathfrak{p}} \cong \mathbb{C}$), thus the condition in the above definition of HSE is usually stated in the form:

$$\underset{a,b\in \dot{K}/\dot{K}^2}{\forall} \left(\frac{a,b}{K_{\mathfrak{p}}}\right) = 1 \iff \left(\frac{ta,tb}{L_{T\mathfrak{p}}}\right) = 1, \tag{2}$$

for every prime $\mathfrak{p} \in \Omega_K$.

In a nutshell, HSE is a pair of maps compatible with splitting of local quaternion algebras or, in other words, with splitting of 2-fold Pfister forms. The aim of this paper is to prove a new criterion for two number fields to be Hilbert-symbol equivalent, that slightly relaxes this condition, by moving the focus from classes

[1] When introduced, Hilbert-symbol equivalence was first called "reciprocity equivalence", the present term was introduced later.

of central simple algebras to classes of central simple graded algebras. That is, instead of controlling splitting of 2-fold Pfister forms, we demand the control only over binary forms, yet still achieving the same overall effect. To this end, we shall say that a pair of maps (t, T) as above is a *graded Hilbert-symbol equivalence* (or GHSE in short), if for every prime $\mathfrak{p} \in \Omega_K$ the map

$$\left\langle \frac{a, b}{K_{\mathfrak{p}}} \right\rangle \mapsto \left\langle \frac{ta, tb}{L_{T\mathfrak{p}}} \right\rangle \tag{3}$$

induces a group isomorphism $\Lambda_{\mathfrak{p}}\colon \operatorname{GQ}(K_{\mathfrak{p}}) \xrightarrow{\sim} \operatorname{GQ}(L_{T\mathfrak{p}})$.

As mentioned above, in HSE we control the splitting of 2-fold Pfister forms, while in GHSE we control only binary forms. Therefore, it does not come as a surprise that every HSE is a GHSE (see Proposition 6). Nevertheless, in this paper we prove that for any two given number fields HSE exists if and only if GHSE exists (Theorem 13). Moreover, if any of the fields in question has a single dyadic point, then every GHSE is in fact a HSE (Proposition 14).

Remark 2. A similar problem was treated earlier in [9] for function fields. In the former paper, we defined a "graded *quaternion*-symbol equivalence" by the condition

$$\underset{a,b\in \dot K/\dot K^2}{\forall} \left\langle \frac{a, b}{K_{\mathfrak{p}}} \right\rangle = 1 \iff \left\langle \frac{ta, tb}{L_{T\mathfrak{p}}} \right\rangle = 1$$

analogous to (2). For function fields, it turned out to be equivalent to our present condition (3). It is not clear, but rather unlikely, whether this would still hold for number fields. Hence, in t his paper we use the latter condition and to avoid confusion the new name "graded *Hilbert*-symbol equivalence".

2. Main result

In what follows, we freely use a "triple-notation" (see [10, Ch. V, §3] for details) for elements of a Brauer-Wall group. In this notation, a class of a graded quaternion algebra $\langle \frac{a,b}{F} \rangle$ is represented by a triple $\big((\frac{a,b}{F}), 0, -ab\big)$. If (t, T) is a GHSE, then by definition for every prime $\mathfrak{p} \in \Omega_K$, the map $\Lambda_{\mathfrak{p}}\colon \operatorname{GQ}(K_{\mathfrak{p}}) \xrightarrow{\sim} \operatorname{GQ}(L_{T\mathfrak{p}})$ is an isomorphism. In the triple notation

$$\Lambda_{\mathfrak{p}}\Big(\Big(\frac{a, b}{K_{\mathfrak{p}}}\Big), 0, -ab\Big) = \Big(\Big(\frac{ta, tb}{L_{T\mathfrak{p}}}\Big), 0, -tatb\Big).$$

We begin with a description of the group GQ. It is obvious that for any prime $\mathfrak{p} \in \Omega_K$ (resp. $\mathfrak{q} \in \Omega_L$) the group $\operatorname{GQ}(K_{\mathfrak{p}})$ (resp. $\operatorname{GQ}(L_{\mathfrak{q}})$) consists of all the triples $(A, 0, a)$ with $A \in \operatorname{Q}(K_{\mathfrak{p}})$ and $a \in \dot K_{\mathfrak{p}}/\dot K_{\mathfrak{p}}^2$ (reps. $A \in \operatorname{Q}(L_{\mathfrak{q}})$ and $a \in \dot L_{\mathfrak{q}}/\dot L_{\mathfrak{q}}^2$). Thus:

Observation 3. *There is a canonical bijection between* $\mathrm{GQ}(K_\mathfrak{p})$ *and* $\mathrm{Q}(K_\mathfrak{p}) \times {}^{\dot K_\mathfrak{p}}/_{\dot K_\mathfrak{p}^2}$. *In particular:*

1. *if* $\mathfrak{p}$ *is an infinite complex prime, then* $|\,\mathrm{GQ}(K_\mathfrak{p})| = 1$*;*
2. *if* $\mathfrak{p}$ *is an infinite real prime, then* $|\,\mathrm{GQ}(K_\mathfrak{p})| = 4$*;*
3. *if* $\mathfrak{p}$ *is a finite non-dyadic prime, then* $|\,\mathrm{GQ}(K_\mathfrak{p})| = 8$*;*
4. *if* $\mathfrak{p}$ *is a finite dyadic prime, then* $|\,\mathrm{GQ}(K_\mathfrak{p})| = 2^{n+3}$, *where* $n = (K_\mathfrak{p} \colon \mathbb{Q}_2)$.

In general the bijection $\mathrm{GQ}(K_\mathfrak{p}) \to \mathrm{Q}(K_\mathfrak{p}) \times {}^{\dot K_\mathfrak{p}}/_{\dot K_\mathfrak{p}^2}$ is not a group isomorphism. For instance, if $K_\mathfrak{p} \cong \mathbb{R}$, then $\mathrm{GQ}(K_\mathfrak{p})$ is a cyclic group (the class $(1, 0, -1)$ being a generator), while $\mathrm{Q}(K_\mathfrak{p}) \times {}^{\dot K_\mathfrak{p}}/_{\dot K_\mathfrak{p}^2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Similarly, if $\mathfrak{p}$ is finite non-dyadic and the level of $K_\mathfrak{p}$ is $s(K_\mathfrak{p}) = 2$, then $\mathrm{GQ}(K_\mathfrak{p}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ but $\mathrm{Q}(K_\mathfrak{p}) \times {}^{\dot K_\mathfrak{p}}/_{\dot K_\mathfrak{p}^2} \cong \mathbb{Z}_2^3$. Nevertheless, the above observation has immediate consequences:

Corollary 4. *If* (t, T) *is a GHSE, then* T *maps complex primes of* K *to complex primes of* L*, real primes to real primes, finite non-dyadic primes to finite non-dyadic primes and dyadic primes to dyadic primes.*

Corollary 5. *If* $\mathfrak{p}$ *is a dyadic prime of* K*, then* $(K_\mathfrak{p} : \mathbb{Q}_2) = (L_{T\mathfrak{p}} : \mathbb{Q}_2)$.

As it was mentioned in the introduction, every HSE is a GHSE.

Proposition 6. *If* (t, T) *is a Hilbert-symbol equivalence, then it is a graded Hilbert-symbol equivalence.*

Proof. Let (t, T) be a HSE, we need to show that for every prime $\mathfrak{p} \in \Omega_K$, it induces a group isomorphism $\Lambda_\mathfrak{p} \colon \mathrm{GQ}(K_\mathfrak{p}) \xrightarrow{\sim} \mathrm{GQ}(L_{T\mathfrak{p}})$. In the triple-notation, an element of $\mathrm{GQ}(K_\mathfrak{p})$ has a form $(A, 0, a)$ with $A \in \mathrm{Q}(K_\mathfrak{p})$ and $a \in {}^{\dot K_\mathfrak{p}}/_{\dot K_\mathfrak{p}^2}$. Now, (t, T) is a HSE, thus t factors through the local square class groups by [16, Proposition 1.4] and, by definition, $\Gamma_\mathfrak{p}\bigl(\frac{a,b}{K_\mathfrak{p}}\bigr) := \bigl(\frac{ta,tb}{L_{T\mathfrak{p}}}\bigr)$ is an isomorphism of $\mathrm{Q}(K_\mathfrak{p})$ and $\mathrm{Q}(L_{T\mathfrak{p}})$. It follows that $\Lambda_\mathfrak{p}\bigl((A, 0, a)\bigr) := (\Gamma_\mathfrak{p} A, 0, ta)$ is a bijection. It is in fact it is a group isomorphism as:

$$\begin{aligned}
&\Lambda_\mathfrak{p}\bigl((A,0,a)(B,0,b)\bigr) = \Lambda_\mathfrak{p}\Bigl(\bigl(A \cdot B \cdot \bigl(\tfrac{a,b}{K_\mathfrak{p}}\bigr), 0, ab\bigr)\Bigr)\\
&= \Bigl(\Gamma_\mathfrak{p}\bigl(A \cdot B \cdot \bigl(\tfrac{a,b}{K_\mathfrak{p}}\bigr)\bigr), 0, t(ab)\Bigr) = \Bigl(\Gamma_\mathfrak{p} A \cdot \Gamma_\mathfrak{p} B \cdot \bigl(\tfrac{ta,tb}{L_{T\mathfrak{p}}}\bigr), 0, tatb\Bigr)\\
&= (\Gamma_\mathfrak{p} A, 0, ta)(\Gamma_\mathfrak{p} B, 0, tb) = \Lambda_\mathfrak{p}(A,0,a) \cdot \Lambda_\mathfrak{p}(B,0,b).
\end{aligned}$$

■

Lemma 7. *If* (t, T) *is a GHSE, then for every prime* $\mathfrak{p} \in \Omega_K$ *and every square class* $a \in {}^{\dot K}/_{\dot K^2}$*:*

1. *$a \in -\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^2$ if and only if $ta \in -\dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^2$;*
2. *$a \in \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^2$ if and only if $ta \in \dot{L}_{T\mathfrak{p}}/\dot{L}_{T\mathfrak{p}}^2$.*

Proof. The assertions are trivial, when $\mathfrak{p}$ is a complex prime. Therefore, without loss of generality, we may assume that $\mathfrak{p}$ is not complex. In particular $\mathrm{Q}(K_\mathfrak{p}) \cong \mathbb{Z}_2 \cong \mathrm{Q}(L_{T\mathfrak{p}})$. Fix a square class $a \in \dot{K}/\dot{K}^2$ and assume that a is a minus square in $K_\mathfrak{p}$, hence the graded quaternion algebra $\langle\frac{1,a}{K_\mathfrak{p}}\rangle$ splits and so does its image $\Lambda_\mathfrak{p}(\langle\frac{1,a}{K_\mathfrak{p}}\rangle) = \langle\frac{1,ta}{L_{T\mathfrak{p}}}\rangle$. Consequently ta is a minus square in $L_{T\mathfrak{p}}$. This proves the first assertion.

Once we know that local minus squares are preserved we may show that it preserves local squares, as well. In fact, this was already proved in [9, Observation 3.4] for arbitrary fields, but for sake of completeness, we repeat the argument. Take any $a \in \dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^2$, it is a square if and only if $\langle\frac{-1,a}{K_\mathfrak{p}}\rangle = 1$. Now, $\Lambda_\mathfrak{p}$ is an isomorphism, so that $\langle\frac{t(-1),ta}{L_{T\mathfrak{p}}}\rangle = 1$. We already know that t preserves minus squares, therefore $\langle\frac{-1,ta}{L_{T\mathfrak{p}}}\rangle = 1$. This means that ta is a square in $L_{T\mathfrak{p}}$ as claimed. ∎

Lemma 8. *If either $F = \mathbb{R}$ or F is a local field, then the group $\mathrm{GQ}(F)$ is a disjoint sum*

$$\mathrm{GQ}(F) = \{\langle\tfrac{a,b}{F}\rangle \colon a, b \in \dot{F}/\dot{F}^2\} \cup \{(-1,0,1)\}.$$

Proof. First we show that the triple $(-1,0,1) \in \mathrm{GQ}(F)$ is not a class of any graded quaternion algebra. Indeed, suppose a contrario, that there are $a, b \in \dot{F}/\dot{F}^2$ such that $\langle\frac{a,b}{F}\rangle = \big((\frac{a,b}{F}), 0, -ab\big) = (-1,0,1)$. Then $-ab$ is a square in F, hence the binary quadratic form $ax^2 + by^2$ is hyperbolic and consequently $(\frac{a,b}{F})$ splits.

Now, we need to show that every other element $(A, 0, a)$ of $\mathrm{GQ}(F)$ is a class of some graded quaternion algebra. To this end, fix a square class $a \in \dot{F}/\dot{F}^2$. Observe that $(1,0,a) = \big((\frac{1,-a}{F}), 0, a\big)$ is a class of $\langle\frac{1,-a}{F}\rangle$. Finally, when a is not a square, we construct the class $(-1,0,a)$ as follows. Since $a \notin \dot{F}^2$, then there is $b \in \dot{F}/\dot{F}^2$ such that $(\frac{a,b}{F}) = -1$ by [10, Theorem VI.2.16]. Take, thus, a graded quaternion algebra $\langle\frac{-ab,b}{F}\rangle$, then

$$\langle\tfrac{-ab,b}{F}\rangle = \big((\tfrac{-ab,b}{F}), 0, ab^2\big) = \big((\tfrac{a,b}{F})(\tfrac{-b,b}{F}), 0, a\big) = (-1,0,a).$$

∎

Now, since $\Lambda_\mathfrak{p}$ maps classes of graded quaternion algebras to classes of graded quaternion algebras, it follows that it must preserve the distinguished element $(-1,0,1)$.

Corollary 9. *If (t,T) is a GHSE, then for every prime $\mathfrak{p} \in \Omega_K$ the associated isomorphism $\Lambda_\mathfrak{p} \colon \mathrm{GQ}(K_\mathfrak{p}) \to \mathrm{GQ}(L_{T\mathfrak{p}})$ maps $(-1,0,1)$ to $(-1,0,1)$.*

Recall (see e.g. [10, Chapter XI]) that the *level* $s(F)$ of a field F is a minimal length of a sum of squares that represents -1, with the convention that $s(F) = \infty$, when F is formally real.

Lemma 10. *If F is a local field, then the level $s(F)$ of F equals the order, in the group* $\mathrm{GQ}(F)$*, of the class of the graded quaternion algebra $\langle \frac{1,1}{F} \rangle$.*

Proof. Since F is a local field, therefore its level is either 1, 2 or 4 (the last case is only possible when F is dyadic). Suppose that $s(F) = 1$, thus -1 is a square and consequently $\langle \frac{1,1}{F} \rangle = (1, 0, 1)$ is the unit element of the Brauer-Wall group of F. Now, let -1 be a sum of two squares but not a square itself. Then $\langle \frac{1,1}{F} \rangle = (1, 0, -1) \neq (1, 0, 1)$ while

$$\langle \tfrac{1,1}{F} \rangle^2 = (1, 0, -1)^2 = \left(1 \cdot 1 \cdot \left(\tfrac{-1,-1}{F}\right), 0, 1\right) = (1, 0, 1),$$

because $\left(\frac{-1,-1}{F}\right)$ splits by [10, Corollary III.2.8(3)]. Finally assume that $s(F) = 4$, thus $\left(\frac{-1,-1}{F}\right) = -1$, hence

$$\langle \tfrac{1,1}{F} \rangle^2 = (-1, 0, 1) \qquad \text{and} \qquad \langle \tfrac{1,1}{F} \rangle^4 = (-1, 0, 1)^2 = (1, 0, 1). \qquad \blacksquare$$

It follows that a graded Hilbert-symbol equivalence preserves not only local squares but local levels, as well.

Corollary 11. *If (t, T) is a GHSE, then for every prime $\mathfrak{p} \in \Omega_K$, the local levels $s(K_\mathfrak{p})$ and $s(L_{T\mathfrak{p}})$ are equal.*

Proposition 12. *If there is a graded Hilbert-symbol equivalence of two number fields K and L, then there is a Hilbert-symbol equivalence of K, L.*

Proof. It is known (see [17]) that there exists a Hilbert-symbol equivalence of two number fields if and only if their primes can be coupled in such a way that complex primes of K are matched with complex primes of L, real primes with real primes, finite non-dyadic primes with finite non-dyadic primes and finite dyadic primes with finite dyadic primes and all the local levels are preserved and, in addition, for dyadic primes the local degrees over $\mathbb{Q}_2$ are preserved. Observe that all these hold in our case: the matching is obtained by Corollary 4, the local levels of finite primes agree by the previous corollary (and for infinite primes they agree trivially), finally the local degrees of dyadic primes are also preserved by Observation 3.4. $\blacksquare$

Combining now Propositions 6 and 12 we may write our main result.

Theorem 13. *Let K, L be two number fields, then the following conditions are equivalent:*

1. *there exists a graded Hilbert-symbol equivalence of K and L;*
2. *there exists a Hilbert-symbol equivalence of K and L;*
3. *fields K and L are Witt equivalent.*

The equivalence of (2) and (3) above was actually the original reason for introducing HSE in the first place and is proved in [12]. The bottom line of the above theorem is that the fact whether two number fields admit the same classes of orthogonal geometries (Witt equivalence) is fully determined by the behavior of Clifford invariants over them (graded Hilbert-symbol equivalence). If we additionally assume that any (hence both in view of Corollary 4) of the two fields has a unique dyadic prime, then more can be said. Instead of the above "existential" result, we can prove a full converse of Proposition 6.

Proposition 14. *If a number field K has a unique dyadic prime, then every graded Hilbert-symbol equivalence is a Hilbert-symbol equivalence.*

Proof. Let (t, T) be a GHSE. It follows immediately form Corollary 4 that the field L has only one dyadic prime, say $\mathfrak{e} = T\mathfrak{d}$, where $\mathfrak{d}$ is the unique dyadic prime of K. First, take an infinite prime $\mathfrak{p}$ of K. Then $T\mathfrak{p} \in \Omega_L$ is also an infinite prime. If $\mathfrak{p}$ is complex (and so is $T\mathfrak{p}$), then all local quaternion algebras split at $\mathfrak{p}$ and the condition (2) is vacuously satisfied. Assume that $\mathfrak{p}$ is a real prime, then $\left(\frac{a,b}{K_\mathfrak{p}}\right)$ splits if and only if either a or b is a square in $K_\mathfrak{p}$. Now, GHSE preserves local squares by Lemma 7. Consequently $\left(\frac{ta,tb}{L_{T\mathfrak{p}}}\right)$ splits if and only if $\left(\frac{a,b}{K_\mathfrak{p}}\right)$ splits, as desired.

Next, take a finite non-dyadic prime $\mathfrak{p} \in \Omega_K$. Then $T\mathfrak{p}$ is also finite and non-dyadic. The local level $s(K_\mathfrak{p})$ is either 2 or 1. Suppose first that $s(K_\mathfrak{p}) = 2$. Corollary 11 asserts that also $s(L_{T\mathfrak{p}}) = 2$. The square class group $\dot{K}_\mathfrak{p}/\dot{K}_\mathfrak{p}^2$ consists of four elements: ± 1, $\pm\pi$, where π is a uniformizer of $\mathfrak{p}$. Denote $\rho := t\pi$ and $\mathfrak{q} := T\mathfrak{p}$. Now, t preserves local squares and local minus squares by Lemma 7, hence $\operatorname{ord}_\mathfrak{q} \pm\rho = 1$. The unique non-split quaternion algebra over $K_\mathfrak{p}$ is $\left(\frac{-1,\pm\pi}{K_\mathfrak{p}}\right) = \left(\frac{\pi,\pi}{K_\mathfrak{p}}\right) = \left(\frac{-\pi,-\pi}{K_\mathfrak{p}}\right)$. It is mapped to $\left(\frac{-1,\pm\rho}{L_\mathfrak{q}}\right) = \left(\frac{\rho,\rho}{L_\mathfrak{q}}\right) = \left(\frac{-\rho,-\rho}{L_\mathfrak{q}}\right)$, which again is a unique non-split quaternion algebra over $L_{T\mathfrak{p}}$. Hence, the condition (2) is met. Assume conversely that $s(K_\mathfrak{p}) = s(L_{T\mathfrak{p}}) = 1$, then $\left(\frac{a,b}{K_\mathfrak{p}}\right) = 1$ if and only if either $a \in K_\mathfrak{p}^2$ or $b \in K_\mathfrak{p}^2$ or $ab \in K_\mathfrak{p}^2$. But local squares are preserved, therefore $\left(\frac{a,b}{K_\mathfrak{p}}\right) = 1$ if and only if $\left(\frac{ta,tb}{L_{T\mathfrak{p}}}\right) = 1$.

Finally, consider the unique dyadic prime $\mathfrak{d}$ of K and take any two square-classes $a, b \in \dot{K}/\dot{K}^2$. Suppose that the local quaternion algebra $\left(\frac{a,b}{K_\mathfrak{d}}\right)$ splits. By the Hilbert's reciprocity formula we have

$$1 = \prod_{\mathfrak{p}\in\Omega_K} \left(\frac{a,b}{K_{\mathfrak{p}}}\right) = \prod_{\substack{\mathfrak{p}\in\Omega_K\\ \mathfrak{p}\neq\mathfrak{d}}} \left(\frac{a,b}{K_{\mathfrak{p}}}\right).$$

However, for all primes $\mathfrak{p} \neq \mathfrak{d}$ we have already shown that (2) holds. Consequently we can write

$$1 = \prod_{\substack{\mathfrak{p}\in\Omega_K\\ \mathfrak{p}\neq\mathfrak{d}}} \left(\frac{ta,tb}{L_{T\mathfrak{p}}}\right) = \prod_{\substack{\mathfrak{q}\in\Omega_L\\ \mathfrak{q}\neq\mathfrak{e}}} \left(\frac{ta,tb}{L_{\mathfrak{q}}}\right).$$

Using Hilbert's reciprocity law again, we see that $\left(\frac{ta,tb}{L_{\mathfrak{e}}}\right) = 1$. Analogously one shows that $\left(\frac{a,b}{K_{\mathfrak{d}}}\right) = -1$ implies $\left(\frac{ta,tb}{L_{\mathfrak{e}}}\right) = -1$. Thus we have proved (2) for all primes and so (t, T) is a HSE. ■

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