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ENUMERATION OF Γ-GROUPS OF FINITE ORDER

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Abstract

The concept of Γ-semigroups is a generalization of semigroups. In this paper, we consider Γ-groups and prove that every Γ-group is derived from a group then, we give the number of Γ-groups of small order.

Keywords: Γ-semigroup, Γ-group.

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1. Introduction

The concept of Γ-semigroups was introduced by Sen in [14] and [15] that is a generalization of a semigroups. Many classical notions of semigroups have been extended to Γ-semigroups (see, for example, [6, 10, 13, 16] and [17]). Dutta and Adhikari have found operator semigroups of a Γ-semigroup to be a very effective tool in studying Γ-semigroups [5]. Recently, Davvaz *et al.* introduced the notion of Γ-semihypergroups as a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups [2, 8, 9].

The determination of all groups of a given order up to isomorphism is a very old question in group theory. It was introduced by Cayley who constructed the groups of order 4 and 6 in 1854, see [4]. In this paper, we prove that a Γ-group is derived from a group. Also, we give the number of Γ-groups of small order.

2. Preliminaries

We begin this section by the definition of a Γ-semigroup.

Definition [14]**.** Let S and Γ be nonempty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S$, written (a, γ, b) by $a\gamma b$, such that satisfies the identities $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Let S be a Γ-semigroup and α be a fixed element in Γ. We define $a.b = a\alpha b$, for all $a, b \in S$. It is easy to check that $(S, .)$ is a semigroup and we denote this semigroup by S_α.

Let A and B be subsets of a Γ-semigroup S and $\Delta \subseteq \Gamma$. Then $A\Delta B$ is defined as follows

$$A\Delta B = \{a\delta b \mid a \in A, b \in B, \delta \in \Delta\}.$$

For simplicity we write $a\Delta B$ and $A\Delta b$ instead of $\{a\}\Delta B$ and $A\Delta\{b\}$, respectively. Also, we write $A\delta B$ in place of $A\{\delta\}B$.

Let S be an arbitrary semigroup and Γ any nonempty set. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus a semigroup can be considered to be a Γ-semigroup.

In the following some examples of Γ-semigroups are presented.

Example 1. Let $S = \{i, 0, -i\}$ and $\Gamma = S$. Then S is a Γ-semigroup under the multiplication over complex number while S is not a semigroup under complex number multiplication.

Example 2. Let S be the set of all $m \times n$ matrices with entries from a field F and Γ be a set of $n \times m$ matrices with entries from F. Then S is a Γ-semigroup with the usual product of matrices.

Example 3. Let $(S, \leq)$ be a totally ordered set and Γ be a nonempty subset of S. We define

$$x\gamma y = \max\{x, \gamma, y\},$$

for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semigroup.

Example 4. Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$ we define $x\gamma y = \frac{xy}{\gamma}$. Then, for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$(x\alpha y)\beta z = \frac{xyz}{\alpha\beta} = x\alpha(y\beta z).$$

This means that S is a Γ-semigroup.

A nonempty subset T of a Γ-semigroup S is said to be a Γ-subsemigroup of S if $T\Gamma T \subseteq T$.

Definition. A nonempty subset I of Γ-semigroup S is called a left (right) Γ-closed subset if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). A Γ-semigroup S is called a left (right) simple Γ-semigroup if it has no proper left (right) Γ-closed subset. Also, S is called a simple Γ-semigroup if it has no proper Γ-closed subset both left and right.

3. Enumeration of Γ-groups of finite order

Definition. A Γ-semigroup S is called a Γ-group if S_α is a group, for every $\alpha \in \Gamma$.

Example 5. Let $S = \{a, b, c, d, e, f\}$ and $\Gamma = \{\alpha, \beta\}$. Define the operations α and β as the following tables

α	a	b	c	d	e	f
a	b	c	d	e	f	a
b	c	d	e	f	a	b
c	d	e	f	a	b	c
d	e	f	a	b	c	d
e	f	a	b	c	d	e
f	a	b	c	d	e	f

β	a	b	c	d	e	f
a	c	d	e	f	a	b
b	d	e	f	a	b	c
c	e	f	a	b	c	d
d	f	a	b	c	d	e
e	a	b	c	d	e	f
f	b	c	d	e	f	a

Then S is a Γ-group. One can see that f and e are the neutral elements of S_α and S_β, respectively.

Theorem 6. *Let S be a Γ-semigroup. Then S is a simple Γ-semigroup if and only if S_α is a group, for every $\alpha \in \Gamma$.*

Proof. Let S be a simple Γ-semigroup and $\alpha \in \Gamma$, we show that S_α is a group. Let $I = a\alpha S$, where $a \in S$. Then, I is a right Γ-closed subset of S, indeed

$$I\Gamma S = (a\alpha S)\Gamma S \subseteq a\alpha S = I.$$

Since S has no proper right Γ-closed subset, we have $I = a\alpha S = S$. Similarly, we can prove that $S\alpha a = S$. Therefore, S_α is a group.

Conversely, let $I \neq \phi$ be a left Γ-closed subset of S, $s \in S$ and $a \in I$. Since S_α is a group, there exists $t \in S$ such that $s = t\alpha a \subseteq S\alpha I \subseteq I$. So $S = I$. Similarly, we can prove that S has no proper right Γ-closed subset. Therefore, S is simple. ■

Corollary 7. *Let S be a Γ-semigroup. If S_α is a group, for some $\alpha \in \Gamma$, then S_β is a group, for every $\beta \in \Gamma$.*

Proof. Since S_α is a group, previous theorem implies that S is a simple Γ-group. Thus, for every $\beta \in \Gamma$, S_β is a group. ■

Corollary 8. *Let S be a Γ-semigroup. If S_α is a group, for some $\alpha \in \Gamma$, then S is a Γ-group.*

Proof. By Corollary 7, it is trivial. ■

Theorem 9. *Let S be a Γ-group and $\alpha, \beta \in \Gamma$. Then there exists $b \in S$ such that $x\beta y = x\alpha b\alpha y$, for every $x, y \in S$.*

Proof. It is sufficient to put $b = e_\alpha \beta e_\alpha$, where e_α is the neutral element of S_α. Then, for every $x, y \in S$, we have

$$\begin{aligned} x\beta y &= (x\alpha e_\alpha)\beta(e_\alpha \alpha y) \\ &= x\alpha(e_\alpha \beta e_\alpha)\alpha y \\ &= x\alpha b\alpha y. \end{aligned}$$

■

By the previous theorem, we conclude that every Γ-group is derived from a group. Therefore, if S is a Γ-group, then we can consider $(S, .)$ as a group and $\Gamma \subseteq S$, so $x\alpha y$ is a product in $(S, .)$, for every $x, y \in S$ and $\alpha \in \Gamma$. Also, Theorem 9 implies that the groups S_α and S_β are isomorphic, for every $\alpha, \beta \in \Gamma$.

Definition. Let S be a Γ-group and S' be a Γ'-group. If there exist mappings $\varphi_\gamma : S \longrightarrow S'$, for every $\gamma \in \Gamma$, and $f : \Gamma \longrightarrow \Gamma'$ such that

$$\varphi_\gamma(x\gamma y) = \varphi_\gamma(x) f(\gamma) \varphi_\gamma(y),$$

for all $x, y \in S$, then we say $(\{\varphi_\gamma\}_{\gamma\in\Gamma}, f)$ is a homomorphism between S and S'. Also, if f and φ_γ, for every $\gamma \in \Gamma$, are bijections, then $(\{\varphi_\gamma\}_{\gamma\in\Gamma}, f)$ is called an isomorphism, and S and S' are called isomorphic.

Lemma 10. *Let S be a Γ-group and S' be a Γ'-group. Then S and S' are isomorphic if and only if S and S' are isomorphic group and $|\Gamma| = |\Gamma'|$.*

Proof. If S and S' are isomorphic, then by the previous definition, for every $\alpha \in \Gamma$, the groups S_α and $S'_{\alpha'}$ are isomorphic where $f : S \longrightarrow S'$ is a bijection and $f(\alpha) = \alpha'$. ■

Theorem 11. *The number of* Γ*-groups of order* n *is* nk*, up to isomorphism, where* k *is the number of isomorphism classes of groups of order* n.

Proof. Suppose that $(S, \cdot)$ is a group and Γ and Γ' be two subsets of S such that $|\Gamma| = |\Gamma'|$. Then by previous lemma, there exists only one Γ-group derived from $(S, .)$, up to isomorphism. So, for every $m \leq n$ there exists only one Γ-group, where Γ is a subset of S such that $|\Gamma| = m$. Thus, the number of Γ-groups derived from $(S, .)$ is n, up to isomorphism. Therefore, if there exist k groups of order n, then we have nk Γ-groups of order n, up to isomorphism. ■

Corollary 12. *Suppose that* $n > 1$ *is an integer with decomposition into primes as* $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$*. If* n *is prime to*

$$\prod_{j=1}^{r} (p_j^{e_j} - 1)$$

and $e_j \leq 2$*, then the number of* Γ*-groups of order* n *is* $n2^m$*, where* m *is the number of* j*'s with* $e_j = 2$.

Proof. By a result of Rédei [12], all such groups of order n are abelian. Thus, the number of isomorphism types of abelian groups of order n is given by

$$\prod_{j=1}^{r} p(e_j) = 2^m,$$

where $p(e_j)$ is the number of partitions of $e_j \leq 2$ and $p(1) = 1, p(2) = 2$. The proof is completed by applying Theorem 11. ■

The case $m = 0$ of the Corollary 12 was studied by Szele [18]. In connection with this, Erdös [7] showed that the number of $n \leq x$ such that $(n, \varphi(n)) = 1$ ($\varphi(n)$ is Euler's phi function) is asymptotic to

$$\frac{e^{-\gamma} x}{\text{logloglog} x}$$

where γ is Euler's constant. For additional results on the asymptotic of $n \leq x$ satisfying Rédei's condition and asymptotic enumeration of finite abelian groups see [1, 11, 19].

In the following table we give the number of Γ-groups of order less than 30.

Order	*Number of* $\Gamma-$ *groups*	*Order*	*Number of* $\Gamma-$ *groups*
1	1	16	224
2	2	17	17
3	3	18	90
4	8	19	19
5	5	20	100
6	12	21	42
7	7	22	44
8	40	23	23
9	18	24	360
10	20	25	50
11	11	26	52
12	60	27	135
13	13	28	112
14	28	29	29
15	15	30	120

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