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ON A PERIOD OF ELEMENTS OF PSEUDO-BCI-ALGEBRAS

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Abstract

The notions of a period of an element of a pseudo-BCI-algebra and a periodic pseudo-BCI-algebra are defined. Some of their properties and characterizations are given.

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1. Introduction

In 1966 K. Iséki introduced the notion of BCI-algebra (see [10]). BCI-algebras have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. The name of BCI-algebra originates from the combinatories B, C, I in combinatory logic.

The concept of pseudo-BCI-algebra has been introduced in [1] as an extension of BCI-algebras. Pseudo-BCI-algebras are algebraic models of some extension of a noncommutative version of the BCI-logic (see [5] for details). These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras introduced by G. Georgescu and A. Iorgulescu in [6, 7] and [8], respectively. More about those algebras the reader can find in [9].

In this paper we define the notion of a period of an element of a pseudo-BCI-algebra. Some of its properties are also given. Finally, we study the concept of a periodic pseudo-BCI-algebra proving some of its interesting characterization. All necessary material needed in the sequel is presented in Section 2 making our exposition self-contained.

2. Preliminaries

A *pseudo-BCI-algebra* is a structure $\mathcal{X} = (X; \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is binary relation on a set X, $\rightarrow$ and $\rightsquigarrow$ are binary operations on X and 1 is an element of X such that for all $x, y, z \in X$, we have

(a1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,

(a2) $x \leq (x \rightarrow y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \rightarrow y$,

(a3) $x \leq x$,

(a4) if $x \leq y$ and $y \leq x$, then $x = y$,

(a5) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

It is obvious that any pseudo-BCI-algebra $(X; \leq, \rightarrow, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$. Note that every pseudo-BCI-algebra satisfying $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI-algebra satisfying $x \leq 1$ for all $x \in X$ is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Any pseudo-BCI-algebra $\mathcal{X} = (X; \leq, \rightarrow, \rightsquigarrow, 1)$ satisfies the following, for all $x, y, z \in X$,

(b1) if $1 \leq x$, then $x = 1$,

(b2) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,

(b4) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,

(b5) $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,

(b6) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), \quad x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$,

(b7) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(b8) $1 \rightarrow x = 1 \rightsquigarrow x = x$,

(b9) $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y, \quad ((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y$,

(b10) $x \rightarrow y \leq (y \rightarrow x) \rightsquigarrow 1, \quad x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightarrow 1$,

(b11) $(x \rightarrow y) \rightarrow 1 = (x \rightarrow 1) \rightsquigarrow (y \rightsquigarrow 1), \quad (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \rightarrow (y \rightarrow 1)$,

(b12) $x \rightarrow 1 = x \rightsquigarrow 1$.

If $(X;\leq,\to,\rightsquigarrow,1)$ is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1), $(X;\leq)$ is a poset with 1 as a maximal element.

Example 2.1 ([3])**.** Let $X=\{a,b,c,d,e,f,1\}$ and define binary operations $\to$ and $\rightsquigarrow$ on X by the following tables:

$\to$	a	b	c	d	e	f	1
a	1	d	e	b	c	a	a
b	c	1	a	e	d	b	b
c	e	a	1	c	b	d	d
d	b	e	d	1	a	c	c
e	d	c	b	a	1	e	e
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	c	b	e	d	a	a
b	d	1	e	a	c	b	b
c	b	e	1	c	a	d	d
d	e	a	d	1	b	c	c
e	c	d	a	b	1	e	e
f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1

Then $\mathcal{X}=(X;\to,\rightsquigarrow,1)$ is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because $a\nleq 1$.

Example 2.2 ([11])**.** Let $Y_1=(-\infty,0]$ and let $\leq$ be the usual order on Y_1. Define binary operations $\to$ and $\rightsquigarrow$ on Y_1 by

$$x\to y=\begin{cases}0 & \text{if} \quad x\leq y,\\ \frac{2y}{\pi}\arctan(\ln(\frac{y}{x})) & \text{if} \quad y<x,\end{cases}$$

$$x\rightsquigarrow y=\begin{cases}0 & \text{if} \quad x\leq y,\\ ye^{-\tan(\frac{\pi x}{2y})} & \text{if} \quad y<x\end{cases}$$

for all $x,y\in Y_1$. Then $\mathcal{Y}_1=(Y_1;\leq,\to,\rightsquigarrow,0)$ is a pseudo-BCK-algebra, and hence it is a nonproper pseudo-BCI-algebra.

Example 2.3 ([4])**.** Let $Y_2=\mathbb{R}^2$ and define binary operations $\to$ and $\rightsquigarrow$ and a binary relation $\leq$ on Y_2 by

$$(x_1,y_1)\to(x_2,y_2)=(x_2-x_1,(y_2-y_1)e^{-x_1}),$$
$$(x_1,y_1)\rightsquigarrow(x_2,y_2)=(x_2-x_1,y_2-y_1e^{x_2-x_1}),$$

$$(x_1,y_1)\leq(x_2,y_2)\Leftrightarrow(x_1,y_1)\to(x_2,y_2)=(0,0)=(x_1,y_1)\rightsquigarrow(x_2,y_2)$$

for all $(x_1,y_1),(x_2,y_2)\in Y_2$. Then $\mathcal{Y}_2=(Y_2;\leq,\to,\rightsquigarrow,(0,0))$ is a proper pseudo-BCI-algebra. Notice that $\mathcal{Y}_2$ is not a pseudo-BCK-algebra because there exists $(x,y)=(1,1)\in Y_2$ such that $(x,y)\nleq(0,0)$.

Example 2.4 ([4]). Let $\mathcal{Y}$ be the direct product of pseudo-BCI-algebras $\mathcal{Y}_1$ and $\mathcal{Y}_2$ from Examples 2.2 and 2.3, respectively. Then $\mathcal{Y}$ is a proper pseudo-BCI-algebra, where $Y = (-\infty, 0] \times \mathbb{R}^2$ and binary operations $\to$ and $\rightsquigarrow$ and binary relation $\leq$ are defined on Y by

$$(x_1, y_1, z_1) \to (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_1 \leq x_2, \\ (\frac{2x_2}{\pi}\arctan(\ln(\frac{x_2}{x_1})), y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \rightsquigarrow (x_2, y_2, z_2) = \begin{cases} (0, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_1 \leq x_2, \\ (x_2 e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_2 < x_1, \end{cases}$$

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 = y_2 \text{ and } z_1 = z_2.$$

Notice that $\mathcal{Y}$ is not a pseudo-BCK-algebra because there exists $(x, y, z) = (0, 1, 1) \in Y$ such that $(x, y, z) \not\leq (0, 0, 0)$.

Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Define

$$x \to^0 y = y,$$
$$x \to^n y = x \to (x \to^{n-1} y),$$

where $x, y \in X$ and $n = 1, 2, \ldots$. Similarly we define $x \rightsquigarrow^n y$ for any $n = 0, 1, 2, \ldots$.

Proposition 2.5. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent for any $x, y \in X$ and $n = 0, 1, 2, \ldots$,*

(i) $x \to^n y = 1$,

(ii) $x \rightsquigarrow^n y = 1$.

Proof. It follows by (a5) and (b4). ■

For any pseudo-BCI-algebra $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ the set

$$K(X) = \{x \in X : x \leq 1\}$$

is a subalgebra of $\mathcal{X}$ (called pseudo-BCK-part of $\mathcal{X}$, see [1]). Then $(K(X); \to, \rightsquigarrow, 1)$ is a pseudo-BCK-algebra. Note that if $\mathcal{X}$ is a pseudo-BCK-algebra, then $X = K(X)$.

It is easily seen that for the pseudo-BCI-algebras $\mathcal{X}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}$ from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have $K(X) = \{f, 1\}$, $K(Y_1) = Y_1$, $K(Y_2) = \{(0, 0)\}$ and $K(Y) = \{(x, 0, 0) : x \leq 0\}$.

An element a of a pseudo-BCI-algebra $\mathcal{X}$ is called a *maximal element* of $\mathcal{X}$ if for every $x \in X$ the following holds

$$\text{if } a \leq x, \text{ then } x = a.$$

We will denote by $M(X)$ the set of all maximal elements of $\mathcal{X}$. Obviously, $1 \in M(X)$. Notice that $M(X) \cap K(X) = \{1\}$. Indeed, if $a \in M(X) \cap K(X)$, then $a \leq 1$ and, by above implication, $a = 1$. Moreover, observe that 1 is the only maximal element of a pseudo-BCK-algebra. Therefore, for a pseudo-BCK-algebra $\mathcal{X}$, $M(X) = \{1\}$. In [2] there is shown that $M(X) = \{x \in X : x = (x \to 1) \to 1\}$. Moreover we have the following simple lemma.

Lemma 2.6. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $x, y \in X$. If $x \leq y$, then $x \to 1 = x \rightsquigarrow 1 = y \to 1 = y \rightsquigarrow 1$.*

Observe that for the pseudo-BCI-algebras $\mathcal{X}$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}$ from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have $M(X) = \{a, b, c, d, e, 1\}$, $M(Y_1) = \{0\}$, $M(Y_2) = Y_2$ and $M(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$.

Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then $\mathcal{X}$ is *p-semisimple* if it satisfies for all $x \in X$,

$$\text{if } x \leq 1, \text{ then } x = 1.$$

Note that if $\mathcal{X}$ is a p-semisimple pseudo-BCI-algebra, then $K(X) = \{1\}$. Hence, if $\mathcal{X}$ is a p-semisimple pseudo-BCK-algebra, then $X = \{1\}$. Moreover, as it is proved in [4], $M(X)$ is a p-semisimple pseudo-BCI-subalgebra of $\mathcal{X}$ and $\mathcal{X}$ is p-semisimple if and only if $X = M(X)$.

Proposition 2.7 ([4]). *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent:*

(i) *$\mathcal{X}$ is p-semisimple,*

(ii) *for all $x, y \in X$, $(x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$,*

(iii) *for all $x \in X$, $x = (x \to 1) \to 1$.*

It is not difficult to see that the pseudo-BCI-algebras $\mathcal{X}$, $\mathcal{Y}_1$ and $\mathcal{Y}$ from Examples 2.1, 2.2 and 2.4, respectively, are not p-semisimple, and the pseudo-BCI-algebra $\mathcal{Y}_2$ from Example 2.3 is a p-semisimple algebra.

Theorem 2.8. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent:*

(i) *$\mathcal{X}$ is p-semisimple,*

(ii) $X = \{x \to 1 : x \in X\}$.

Proof. (i)⇒(ii) Take $y \in X$. Since $\mathcal{X}$ is p-semisimple, $y = (y \to 1) \to 1$. Putting $x = y \to 1 \in X$, we get $y = x \to 1$.

(ii)⇒(i) Take $a \in X$. We show that a is a maximal element of $\mathcal{X}$, that is, $X = M(X)$. Suppose that $a = x \to 1$ for some $x \in X$. Let $y \in X$ be such that $a \leq y$. Then $(x \to 1) \to y = 1$ and, by (b4), (b9), (b11) and (b12), we have

$$\begin{aligned} y \to a &= y \to (x \to 1) = y \to (((x \to 1) \to 1) \rightsquigarrow 1) \\ &= ((x \to 1) \to 1) \rightsquigarrow (y \to 1) = ((x \to 1) \to y) \to 1 \\ &= 1 \to 1 = 1. \end{aligned}$$

Hence $y \leq a$. So, $y = a$, that is, $a \in M(X)$ and $\mathcal{X}$ is p-semisimple. ∎

For p-semisimple pseudo-BCI-algebras we have the following useful fact.

Theorem 2.9 [4]**.** *A pseudo-BCI-algebra $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ is p-semisimple if and only if $(X; \cdot, ^{-1}, 1)$ is a group, where $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$, $x^{-1} = x \to 1 = x \rightsquigarrow 1$, $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$ for any $x, y \in X$.*

Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. We say that a subset D of X is a *deductive system* of $\mathcal{X}$ if it satisfies: (i) $1 \in D$, (ii) for all $x, y \in X$, if $x \in D$ and $x \to y \in D$, then $y \in D$. Under this definition, $\{1\}$ and X are the simplest examples of deductive systems. Note that the condition (ii) can be replaced by (ii') for all $x, y \in X$, if $x \in D$ and $x \rightsquigarrow y \in D$, then $y \in D$. It can be easily proved that for any $x, y \in X$, if $x \in D$ and $x \leq y$, then $y \in D$. A deductive system D of a pseudo-BCI-algebra $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ is called *closed* if D is closed under operations $\to$ and $\rightsquigarrow$, that is, if D is a subalgebra of $\mathcal{X}$. It is not difficult to show (see [3]) that a deductive system D of a pseudo-BCI-algebra $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ is closed if and only if for any $x \in D$, $x \to 1 = x \rightsquigarrow 1 \in D$. Obviously, the pseudo-BCK-part $K(X)$ is a closed deductive system of $\mathcal{X}$.

Proposition 2.10 ([2])**.** *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $M(X)$ be finite. Then every deductive system of $\mathcal{X}$ is closed.*

Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. It is obvious that the intersection of arbitrary number of deductive systems is a deductive system. Hence, for any $A \subseteq X$ there exists the least deductive system containing A. Denote it by $D(A)$ and call it the deductive system *generated* by A. In particular, if $A = \{a_1, \ldots, a_n\}$, then we write $D(a_1, \ldots, a_n)$ instead of $D(\{a_1, \ldots, a_n\})$. It is also obvious that $D(\emptyset) = \{1\}$.

Proposition 2.11 ([3]). *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. For any $a \in X$,*

$$\begin{aligned} D(a) &= \{1\} \cup \{x \in X : a \to^n x = 1 \text{ for some } n \in \mathbb{N}\} \\ &= \{1\} \cup \{x \in X : a \rightsquigarrow^n x = 1 \text{ for some } n \in \mathbb{N}\}. \end{aligned}$$

3. Period of elements

Proposition 3.1. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following hold for any $x, y, z \in X$ and $m, n = 0, 1, 2, \ldots$,*

(i) $x \to^n 1 = x \rightsquigarrow^n 1$,

(ii) $x \to^n x = x \to^{n-1} 1, \quad x \rightsquigarrow^n x = x \rightsquigarrow^{n-1} 1$,

(iii) $(x \to 1) \to^n 1 = (x \to^n 1) \to 1, \quad (x \rightsquigarrow 1) \rightsquigarrow^n 1 = (x \rightsquigarrow^n 1) \rightsquigarrow 1$,

(iv) $x \to (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \to z), \quad x \rightsquigarrow (y \to^n z) = y \to^n (x \rightsquigarrow z)$,

(v) $x \to^m (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \to^m z)$,

(vi) $x \to^n 1 = ((x \to 1) \to 1) \to^n 1, \quad x \rightsquigarrow^n 1 = ((x \rightsquigarrow 1) \rightsquigarrow 1) \rightsquigarrow^n 1$.

Proof. (i) Follows from (b4) and (b12).

(ii) Obvious.

(iii) We prove first equation by induction. The proof of second equation is analogous. For $n = 0$ it is obvious. Assume it for $n = k$:

$$(x \to 1) \to^k 1 = (x \to^k 1) \to 1.$$

We have, by definition, assumption, (i), (b11) and (b12),

$$\begin{aligned} (x \to 1) \to^{k+1} 1 &= (x \to 1) \to ((x \to 1) \to^k 1) = (x \to 1) \to ((x \to^k 1) \to 1) \\ &= (x \rightsquigarrow 1) \to ((x \rightsquigarrow^k 1) \rightsquigarrow 1) = (x \rightsquigarrow (x \rightsquigarrow^k 1)) \rightsquigarrow 1 \\ &= (x \rightsquigarrow^{k+1} 1) \rightsquigarrow 1 = (x \to^{k+1} 1) \to 1. \end{aligned}$$

So, the equation holds for any $n = 0, 1, 2, \ldots$.

(iv) We prove first equation by induction. The proof of second equation is analogous. For $n = 0$ it is obvious. Assume it for $n = k$, that is,

$$x \to (y \rightsquigarrow^k z) = y \rightsquigarrow^k (x \to z).$$

We have, by definition, assumption and (b4),

$$\begin{aligned} x \to (y \rightsquigarrow^{k+1} z) &= x \to (y \rightsquigarrow (y \rightsquigarrow^k z)) = y \rightsquigarrow (x \to (y \rightsquigarrow^k z)) \\ &= y \rightsquigarrow (y \rightsquigarrow^k (x \to z)) = y \rightsquigarrow^{k+1} (x \to z). \end{aligned}$$

Hence, the equation holds for any $n = 0, 1, 2, \ldots$.

(v) We get it easily by (iv).

(vi) We prove first equation by induction. The proof of second equation is analogous. For $n = 0$ it is obvious. Assume it for $n = k$, that is,

$$x \to^k 1 = ((x \to 1) \to 1) \to^k 1.$$

We have, by definition, assumption, (i), (iv), (b9) and (b12),

$$\begin{aligned} ((x \to 1) \to 1) \to^{k+1} 1 &= ((x \to 1) \to 1) \to (((x \to 1) \to 1) \to^k 1) \\ &= ((x \to 1) \to 1) \to (x \to^k 1) \\ &= ((x \to 1) \to 1) \to (x \rightsquigarrow^k 1) \\ &= x \rightsquigarrow^k (((x \to 1) \to 1) \to 1) \\ &= x \rightsquigarrow^k (x \to 1) \\ &= x \to (x \rightsquigarrow^k 1) \\ &= x \to (x \to^k 1) \\ &= x \to^{k+1} 1. \end{aligned}$$

Hence, the equation holds for any $n = 0, 1, 2, \ldots$. ■

Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. For any $x \in X$, if there exists the least natural number n such that $x \to^n 1 = 1$, then n is called a *period of* x denoted $p(x)$. If, for any natural number n, $x \to^n 1 \neq 1$, then a period of x is called to be infinite and denoted $p(x) = \infty$. Obviously, $p(1) = 1$.

Proposition 3.2. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then $p(x) = p(x \to 1) = p(x \rightsquigarrow 1)$ for all $x \in X$.*

Proof. Obviously, $p(x \to 1) = p(x \rightsquigarrow 1)$. For any $x \in X$, by Proposition 3.1(iii,v), we have

$$x \to^k 1 = ((x \to 1) \to 1) \to^k 1 = ((x \to 1) \to^k 1) \to 1.$$

Since $(x \to 1) \to^k 1$ is a maximal element, we have that $x \to^k 1 = 1$ if and only if $(x \to 1) \to^k 1 = 1$. Thus, $p(x) = p(x \to 1)$. ■

Proposition 3.3. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $x, y \in X$. If $x \leq y$, then $p(x) = p(y)$.*

Proof. Let $x, y \in X$. By Lemma 2.6 and Proposition 3.2, if $x \leq y$, then $x \to 1 = y \to 1$ and $p(x) = p(x \to 1) = p(y \to 1) = p(y)$. ■

Theorem 3.4. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra and $(X; \cdot, ^{-1}, 1)$ be a group related with $\mathcal{X}$. Then $p(x) = o(x)$ for any $x \in X$, where $o(x)$ means an order of an element x in a group $(X; \cdot, ^{-1}, 1)$.*

Proof. Let $x \in X$. Since $x \to y = y \cdot x^{-1}$, it is not difficult to see that $(x \to 1) \to^k 1 = x^k$ for any $k = 0, 1, 2, \ldots$. Then,

$$(x \to 1) \to^k 1 = 1 \;\text{ iff }\; x^k = 1.$$

So, $p(x \to 1) = o(x)$. Thus, by Proposition 3.2, $p(x) = o(x)$. ■

Corollary 3.5. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra. Then the following hold for any $x, y \in X$,*

(i) $p(x \to y) = p(y \to x), \quad p(x \rightsquigarrow y) = p(y \rightsquigarrow x)$,

(ii) $p(x \to y) = p(x \rightsquigarrow y)$.

Now we prove that identities from Corollary 3.5 hold also for arbitrary pseudo-BCI-algebras.

Theorem 3.6. *Let $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then the following hold for any $x, y \in X$,*

(i) $p(x \to y) = p(y \to x), \quad p(x \rightsquigarrow y) = p(y \rightsquigarrow x)$,

(ii) $p(x \to y) = p(x \rightsquigarrow y)$.

Proof. (i) We show the first equation. The proof of the second one is analogous. Let $x, y \in X$. Then $x \to 1, y \to 1 \in M(X)$. By Proposition 3.2, (b11), (b12) and Corollary 3.5 we have

$$\begin{aligned} p(x \to y) &= p((x \to y) \to 1) = p((x \to 1) \rightsquigarrow (y \to 1)) \\ &= p((y \to 1) \rightsquigarrow (x \to 1)) = p((y \to x) \to 1) \\ &= p(y \to x). \end{aligned}$$

(ii) Similarly we have

$$\begin{aligned} p(x \to y) &= p((x \to y) \to 1) = p((x \to 1) \rightsquigarrow (y \to 1)) \\ &= p((x \rightsquigarrow 1) \to (y \rightsquigarrow 1)) = p((x \rightsquigarrow y) \to 1) \\ &= p(x \rightsquigarrow y) \end{aligned}$$

for any $x, y \in X$. ■

Let $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $x \in X$. It is not difficult to see that

$$p(x) = 1 \text{ iff } x \leq 1.$$

Hence we have the following proposition.

Proposition 3.7. *Let $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then it is a pseudo-BCK-algebra if and only if $p(x) = 1$ for any $x \in X$.*

Corollary 3.8. *Let $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then it is proper if and only if there exists $x \in X$ such that $p(x) > 1$.*

Corollary 3.9. *Let $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then it is p-semisimple if and only if $p(x) > 1$ for any $x \in X \backslash \{1\}$.*

A pseudo-BCI-algebra $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$ is called *periodic* if $p(x) < \infty$ for any $x \in X$. It is immediately seen that every pseudo-BCK-algebra is periodic.

Now give an interesting characterization of periodic pseudo-BCI-algebras.

Theorem 3.10. *A pseudo-BCI-algebra $\mathcal{X}$ is periodic if and only if every deductive system of $\mathcal{X}$ is closed.*

Proof. Assume that $\mathcal{X}$ is periodic and D is a deductice system of $\mathcal{X}$. Let $x \in D$. Then there exists a natural number n such that $x \rightarrow^n 1 = 1$. Since $x, x \rightarrow^n 1 \in D$ and D is a deductive system, we have $x \rightarrow 1 \in D$, that is, D is closed.

Conversely, for any $x \in X$, a deductive system $D(x)$ is closed. Hence, $x \rightarrow 1 \in D(x)$. So, there exists a natural number n such that $x \rightarrow^n (x \rightarrow 1) = 1$, that is, $p(x) < \infty$. Thus $\mathcal{X}$ is periodic. ■

By Proposition 2.10 we have the following.

Corollary 3.11. *Let $\mathcal{X}$ be a pseudo-BCI-algebra. If $M(X)$ is finite, then $\mathcal{X}$ is periodic.*

Corollary 3.12. *Every finite pseudo-BCI-algebra is periodic.*

Example 3.13. The pseudo-BCI-algebra $\mathcal{X}$ from Example 2.1 is periodic because it is finite and the pseudo-BCI-algebra $\mathcal{Y}$ from Example 2.4 is not periodic because a deductive system $D = \{(x, y, y) : x \leq 0, y \in \mathbb{R}\}$ is not closed.

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