# ON A PERIOD OF ELEMENTS OF PSEUDO-BCI-ALGEBRAS 

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#### Abstract

The notions of a period of an element of a pseudo-BCI-algebra and a periodic pseudo-BCI-algebra are defined. Some of their properties and characterizations are given.


Keywords: pseudo-BCI-algebra, period.
2010 Mathematics Subject Classification: 03G25, 06F35.

## 1. Introduction

In 1966 K . Iséki introduced the notion of BCI-algebra (see [10]). BCI-algebras have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. The name of BCIalgebra originates from the combinatories B, C, I in combinatory logic.

The concept of pseudo-BCI-algebra has been introduced in [1] as an extension of BCI-algebras. Pseudo-BCI-algebras are algebraic models of some extension of a noncommutative version of the BCI-logic (see [5] for details). These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras introduced by G. Georgescu and A. Iorgulescu in $[6,7]$ and [8], respectively. More about those algebras the reader can find in [9].

In this paper we define the notion of a period of an element of a pseudo-BCIalgebra. Some of its properties are also given. Finally, we study the concept of a periodic pseudo-BCI-algebra proving some of its interesting characterization. All necessary material needed in the sequel is presented in Section 2 making our exposition self-contained.

## 2. Preliminaries

A pseudo-BCI-algebra is a structure $\mathcal{X}=(X ; \leq, \rightarrow, \rightsquigarrow, 1)$, where $\leq$ is binary relation on a set $X, \rightarrow$ and $\rightsquigarrow$ are binary operations on $X$ and 1 is an element of $X$ such that for all $x, y, z \in X$, we have
(a1) $x \rightarrow y \leq(y \rightarrow z) \rightsquigarrow(x \rightarrow z), \quad x \rightsquigarrow y \leq(y \rightsquigarrow z) \rightarrow(x \rightsquigarrow z)$,
(a2) $x \leq(x \rightarrow y) \rightsquigarrow y, \quad x \leq(x \rightsquigarrow y) \rightarrow y$,
(a3) $x \leq x$,
(a4) if $x \leq y$ and $y \leq x$, then $x=y$,
(a5) $x \leq y$ iff $x \rightarrow y=1$ iff $x \rightsquigarrow y=1$.
It is obvious that any pseudo-BCI-algebra $(X ; \leq, \rightarrow, \rightsquigarrow, 1)$ can be regarded as a universal algebra $(X ; \rightarrow, \rightsquigarrow, 1)$ of type $(2,2,0)$. Note that every pseudo-BCIalgebra satisfying $x \rightarrow y=x \rightsquigarrow y$ for all $x, y \in X$ is a BCI-algebra.

Every pseudo-BCI-algebra satisfying $x \leq 1$ for all $x \in X$ is a pseudo-BCKalgebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called proper.

Any pseudo-BCI-algebra $\mathcal{X}=(X ; \leq, \rightarrow, \rightsquigarrow, 1)$ satisfies the following, for all $x, y, z \in X$,
(b1) if $1 \leq x$, then $x=1$,
(b2) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,
(b3) if $x \leq y$ and $y \leq z$, then $x \leq z$,
$(\mathrm{b} 4) x \rightarrow(y \rightsquigarrow z)=y \rightsquigarrow(x \rightarrow z)$,
(b5) $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$,
(b6) $x \rightarrow y \leq(z \rightarrow x) \rightarrow(z \rightarrow y), \quad x \rightsquigarrow y \leq(z \rightsquigarrow x) \rightsquigarrow(z \rightsquigarrow y)$,
(b7) if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,
(b8) $1 \rightarrow x=1 \rightsquigarrow x=x$,
$(\mathrm{b} 9) \quad((x \rightarrow y) \rightsquigarrow y) \rightarrow y=x \rightarrow y, \quad((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y=x \rightsquigarrow y$,
(b10) $x \rightarrow y \leq(y \rightarrow x) \rightsquigarrow 1, \quad x \rightsquigarrow y \leq(y \rightsquigarrow x) \rightarrow 1$,
$(\mathrm{b} 11)(x \rightarrow y) \rightarrow 1=(x \rightarrow 1) \rightsquigarrow(y \rightsquigarrow 1), \quad(x \rightsquigarrow y) \rightsquigarrow 1=(x \rightsquigarrow 1) \rightarrow(y \rightarrow 1)$,
(b12) $x \rightarrow 1=x \rightsquigarrow 1$.

If $(X ; \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1), ( $X ; \leq$ ) is a poset with 1 as a maximal element.

Example $2.1([3])$. Let $X=\{a, b, c, d, e, f, 1\}$ and define binary operations $\rightarrow$ and $\rightsquigarrow$ on $X$ by the following tables:

| $\rightarrow$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a$ | 1 | $d$ | $e$ | $b$ | $c$ | $a$ | $a$ |
| $b$ | $c$ | 1 | $a$ | $e$ | $d$ | $b$ | $b$ |
| $c$ | $e$ | $a$ | 1 | $c$ | $b$ | $d$ | $d$ |
| $d$ | $b$ | $e$ | $d$ | 1 | $a$ | $c$ | $c$ |
| $e$ | $d$ | $c$ | $b$ | $a$ | 1 | $e$ | $e$ |
| $f$ | $a$ | $b$ | $c$ | $d$ | $e$ | 1 | 1 |
| 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |


| $\rightsquigarrow$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a$ | 1 | $c$ | $b$ | $e$ | $d$ | $a$ | $a$ |
| $b$ | $d$ | 1 | $e$ | $a$ | $c$ | $b$ | $b$ |
| $c$ | $b$ | $e$ | 1 | $c$ | $a$ | $d$ | $d$ |
| $d$ | $e$ | $a$ | $d$ | 1 | $b$ | $c$ | $c$ |
| $e$ | $c$ | $d$ | $a$ | $b$ | 1 | $e$ | $e$ |
| $f$ | $a$ | $b$ | $c$ | $d$ | $e$ | 1 | 1 |
| 1 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

Then $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because $a \not \leq 1$.

Example 2.2 ([11]). Let $Y_{1}=(-\infty, 0]$ and let $\leq$ be the usual order on $Y_{1}$. Define binary operations $\rightarrow$ and $\rightsquigarrow$ on $Y_{1}$ by

$$
\begin{aligned}
& x \rightarrow y= \begin{cases}0 & \text { if } x \leq y, \\
\frac{2 y}{\pi} \arctan \left(\ln \left(\frac{y}{x}\right)\right) & \text { if } y<x,\end{cases} \\
& x \rightsquigarrow y=\left\{\begin{array}{lll}
0 & \text { if } x \leq y \\
y e^{-\tan \left(\frac{\pi x}{2 y}\right)} & \text { if } & y<x
\end{array}\right.
\end{aligned}
$$

for all $x, y \in Y_{1}$. Then $\mathcal{Y}_{1}=\left(Y_{1} ; \leq, \rightarrow, \rightsquigarrow, 0\right)$ is a pseudo-BCK-algebra, and hence it is a nonproper pseudo-BCI-algebra.

Example 2.3 ([4]). Let $Y_{2}=\mathbb{R}^{2}$ and define binary operations $\rightarrow$ and $\rightsquigarrow$ and a binary relation $\leq$ on $Y_{2}$ by

$$
\begin{aligned}
\left(x_{1}, y_{1}\right) & \rightarrow\left(x_{2}, y_{2}\right)=\left(x_{2}-x_{1},\left(y_{2}-y_{1}\right) e^{-x_{1}}\right) \\
\left(x_{1}, y_{1}\right) & \rightsquigarrow\left(x_{2}, y_{2}\right)=\left(x_{2}-x_{1}, y_{2}-y_{1} e^{x_{2}-x_{1}}\right) \\
\left(x_{1}, y_{1}\right) \leq\left(x_{2}, y_{2}\right) \Leftrightarrow\left(x_{1}, y_{1}\right) & \rightarrow\left(x_{2}, y_{2}\right)=(0,0)=\left(x_{1}, y_{1}\right) \rightsquigarrow\left(x_{2}, y_{2}\right)
\end{aligned}
$$

for all $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right) \in Y_{2}$. Then $\mathcal{Y}_{2}=\left(Y_{2} ; \leq, \rightarrow, \rightsquigarrow,(0,0)\right)$ is a proper pseudo-BCI-algebra. Notice that $\mathcal{Y}_{2}$ is not a pseudo-BCK-algebra because there exists $(x, y)=(1,1) \in Y_{2}$ such that $(x, y) \not \leq(0,0)$.

Example 2.4 ([4]). Let $\mathcal{Y}$ be the direct product of pseudo-BCI-algebras $\mathcal{Y}_{1}$ and $\mathcal{Y}_{2}$ from Examples 2.2 and 2.3, respectively. Then $\mathcal{Y}$ is a proper pseudo-BCIalgebra, where $Y=(-\infty, 0] \times \mathbb{R}^{2}$ and binary operations $\rightarrow$ and $\rightsquigarrow$ and binary relation $\leq$ are defined on $Y$ by

$$
\begin{aligned}
& \left(x_{1}, y_{1}, z_{1}\right) \rightarrow\left(x_{2}, y_{2}, z_{2}\right)= \\
& \begin{cases}\left(0, y_{2}-y_{1},\left(z_{2}-z_{1}\right) e^{-y_{1}}\right) & \text { if } x_{1} \leq x_{2}, \\
\left(\frac{2 x_{2}}{\pi} \arctan \left(\ln \left(\frac{x_{2}}{x_{1}}\right)\right), y_{2}-y_{1},\left(z_{2}-z_{1}\right) e^{-y_{1}}\right) & \text { if } x_{2}<x_{1},\end{cases} \\
& \left(x_{1}, y_{1}, z_{1}\right) \rightsquigarrow\left(x_{2}, y_{2}, z_{2}\right)= \\
& \begin{cases}\left(0, y_{2}-y_{1}, z_{2}-z_{1} e^{y_{2}-y_{1}}\right) & \text { if } x_{1} \leq x_{2}, \\
\left(x_{2} e^{-\tan \left(\frac{\pi x_{1}}{2 x_{2}}\right)}, y_{2}-y_{1}, z_{2}-z_{1} e^{y_{2}-y_{1}}\right) & \text { if } x_{2}<x_{1},\end{cases} \\
& \left(x_{1}, y_{1}, z_{1}\right) \leq\left(x_{2}, y_{2}, z_{2}\right) \Leftrightarrow x_{1} \leq x_{2} \text { and } y_{1}=y_{2} \text { and } z_{1}=z_{2} .
\end{aligned}
$$

Notice that $\mathcal{Y}$ is not a pseudo-BCK-algebra because there exists $(x, y, z)=$ $(0,1,1) \in Y$ such that $(x, y, z) \nsubseteq(0,0,0)$.

Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Define

$$
\begin{aligned}
& x \rightarrow^{0} y=y, \\
& x \rightarrow^{n} y=x \rightarrow\left(x \rightarrow^{n-1} y\right),
\end{aligned}
$$

where $x, y \in X$ and $n=1,2, \ldots$. Similarly we define $x \rightsquigarrow^{n} y$ for any $n=$ $0,1,2, \ldots$.

Proposition 2.5. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent for any $x, y \in X$ and $n=0,1,2, \ldots$,
(i) $x \rightarrow^{n} y=1$,
(ii) $x \rightsquigarrow^{n} y=1$.

Proof. It follows by (a5) and (b4).
For any pseudo-BCI-algebra $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ the set

$$
K(X)=\{x \in X: x \leq 1\}
$$

is a subalgebra of $\mathcal{X}$ (called pseudo-BCK-part of $\mathcal{X}$, see [1]). Then $(K(X) ; \rightarrow$, $\rightsquigarrow, 1$ ) is a pseudo-BCK-algebra. Note that if $\mathcal{X}$ is a pseudo-BCK-algebra, then $X=K(X)$.

It is easily seen that for the pseudo-BCI-algebras $\mathcal{X}, \mathcal{Y}_{1}, \mathcal{Y}_{2}$ and $\mathcal{Y}$ from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have $K(X)=\{f, 1\}, K\left(Y_{1}\right)=Y_{1}, K\left(Y_{2}\right)=$ $\{(0,0)\}$ and $K(Y)=\{(x, 0,0): x \leq 0\}$.

An element $a$ of a pseudo-BCI-algebra $\mathcal{X}$ is called a maximal element of $\mathcal{X}$ if for every $x \in X$ the following holds

$$
\text { if } a \leq x \text {, then } x=a \text {. }
$$

We will denote by $M(X)$ the set of all maximal elements of $\mathcal{X}$. Obviously, $1 \in M(X)$. Notice that $M(X) \cap K(X)=\{1\}$. Indeed, if $a \in M(X) \cap K(X)$, then $a \leq 1$ and, by above implication, $a=1$. Moreover, observe that 1 is the only maximal element of a pseudo-BCK-algebra. Therefore, for a pseudo-BCK-algebra $\mathcal{X}, M(X)=\{1\}$. In [2] there is shown that $M(X)=\{x \in X: x=(x \rightarrow 1) \rightarrow 1\}$. Moreover we have the following simple lemma.

Lemma 2.6. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $x, y \in X$. If $x \leq y$, then $x \rightarrow 1=x \rightsquigarrow 1=y \rightarrow 1=y \rightsquigarrow 1$.

Observe that for the pseudo-BCI-algebras $\mathcal{X}, \mathcal{Y}_{1}, \mathcal{Y}_{2}$ and $\mathcal{Y}$ from Examples 2.1, $2.2,2.3$ and 2.4, respectively, we have $M(X)=\{a, b, c, d, e, 1\}, M\left(Y_{1}\right)=\{0\}$, $M\left(Y_{2}\right)=Y_{2}$ and $M(Y)=\{(0, y, z): y, z \in \mathbb{R}\}$.

Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then $\mathcal{X}$ is $p$-semisimple if it satisfies for all $x \in X$,

$$
\text { if } x \leq 1 \text {, then } x=1 \text {. }
$$

Note that if $\mathcal{X}$ is a p-semisimple pseudo-BCI-algebra, then $K(X)=\{1\}$. Hence, if $\mathcal{X}$ is a p-semisimple pseudo-BCK-algebra, then $X=\{1\}$. Moreover, as it is proved in [4], $M(X)$ is a p-semisimple pseudo-BCI-subalgebra of $\mathcal{X}$ and $\mathcal{X}$ is p-semisimple if and only if $X=M(X)$.

Proposition 2.7 ([4]). Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent:
(i) $\mathcal{X}$ is $p$-semisimple,
(ii) for all $x, y \in X,(x \rightarrow 1) \rightsquigarrow y=(y \rightsquigarrow 1) \rightarrow x$,
(iii) for all $x \in X, x=(x \rightarrow 1) \rightarrow 1$.

It is not difficult to see that the pseudo-BCI-algebras $\mathcal{X}, \mathcal{Y}_{1}$ and $\mathcal{Y}$ from Examples 2.1, 2.2 and 2.4 , respectively, are not p-semisimple, and the pseudo-BCI-algebra $\mathcal{Y}_{2}$ from Example 2.3 is a p-semisimple algebra.

Theorem 2.8. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following are equivalent:
(i) $\mathcal{X}$ is p-semisimple,
(ii) $X=\{x \rightarrow 1: x \in X\}$.

Proof. (i) $\Rightarrow$ (ii) Take $y \in X$. Since $\mathcal{X}$ is p-semisimple, $y=(y \rightarrow 1) \rightarrow 1$. Putting $x=y \rightarrow 1 \in X$, we get $y=x \rightarrow 1$.
(ii) $\Rightarrow$ (i) Take $a \in X$. We show that $a$ is a maximal element of $\mathcal{X}$, that is, $X=M(X)$. Suppose that $a=x \rightarrow 1$ for some $x \in X$. Let $y \in X$ be such that $a \leq y$. Then $(x \rightarrow 1) \rightarrow y=1$ and, by (b4), (b9), (b11) and (b12), we have

$$
\begin{aligned}
y \rightarrow a & =y \rightarrow(x \rightarrow 1)=y \rightarrow(((x \rightarrow 1) \rightarrow 1) \rightsquigarrow 1) \\
& =((x \rightarrow 1) \rightarrow 1) \rightsquigarrow(y \rightarrow 1)=((x \rightarrow 1) \rightarrow y) \rightarrow 1 \\
& =1 \rightarrow 1=1 .
\end{aligned}
$$

Hence $y \leq a$. So, $y=a$, that is, $a \in M(X)$ and $\mathcal{X}$ is p-semisimple.
For p-semisimple pseudo-BCI-algebras we have the following useful fact.
Theorem 2.9 [4]. A pseudo-BCI-algebra $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ is p-semisimple if and only if $\left(X ; \cdot,^{-1}, 1\right)$ is a group, where $x \cdot y=(x \rightarrow 1) \rightsquigarrow y=(y \rightsquigarrow 1) \rightarrow x$, $x^{-1}=x \rightarrow 1=x \rightsquigarrow 1, x \rightarrow y=y \cdot x^{-1}$ and $x \rightsquigarrow y=x^{-1} \cdot y$ for any $x, y \in X$.

Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. We say that a subset $D$ of $X$ is a deductive system of $\mathcal{X}$ if it satisfies: (i) $1 \in D$, (ii) for all $x, y \in X$, if $x \in D$ and $x \rightarrow y \in D$, then $y \in D$. Under this definition, $\{1\}$ and $X$ are the simplest examples of deductive systems. Note that the condition (ii) can be replaced by (ii') for all $x, y \in X$, if $x \in D$ and $x \rightsquigarrow y \in D$, then $y \in D$. It can be easily proved that for any $x, y \in X$, if $x \in D$ and $x \leq y$, then $y \in D$. A deductive system $D$ of a pseudo-BCI-algebra $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ is called closed if $D$ is closed under operations $\rightarrow$ and $\rightsquigarrow$, that is, if $D$ is a subalgebra of $\mathcal{X}$. It is not difficult to show (see [3]) that a deductive system $D$ of a pseudo-BCI-algebra $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ is closed if and only if for any $x \in D, x \rightarrow 1=x \rightsquigarrow 1 \in D$. Obviously, the pseudo-BCK-part $K(X)$ is a closed deductive system of $\mathcal{X}$.

Proposition 2.10 ([2]). Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $M(X)$ be finite. Then every deductive system of $\mathcal{X}$ is closed.

Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. It is obvious that the intersection of arbitrary number of deductive systems is a deductive system. Hence, for any $A \subseteq X$ there exists the least deductive system containing $A$. Denote it by $D(A)$ and call it the deductive system generated by $A$. In particular, if $A=\left\{a_{1}, \ldots, a_{n}\right\}$, then we write $D\left(a_{1}, \ldots, a_{n}\right)$ instead of $D\left(\left\{a_{1}, \ldots, a_{n}\right\}\right)$. It is also obvious that $D(\emptyset)=\{1\}$.

Proposition 2.11 ([3]). Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. For any $a \in X$,

$$
\begin{aligned}
D(a) & =\{1\} \cup\left\{x \in X: a \rightarrow^{n} x=1 \text { for some } n \in \mathbb{N}\right\} \\
& =\{1\} \cup\left\{x \in X: a \rightsquigarrow^{n} x=1 \text { for some } n \in \mathbb{N}\right\} .
\end{aligned}
$$

## 3. Period of elements

Proposition 3.1. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. The following hold for any $x, y, z \in X$ and $m, n=0,1,2, \ldots$,
(i) $x \rightarrow^{n} 1=x \rightsquigarrow^{n} 1$,
(ii) $x \rightarrow^{n} x=x \rightarrow^{n-1} 1, \quad x \rightsquigarrow^{n} x=x \rightsquigarrow^{n-1} 1$,
(iii) $(x \rightarrow 1) \rightarrow^{n} 1=\left(x \rightarrow^{n} 1\right) \rightarrow 1, \quad(x \rightsquigarrow 1) \rightsquigarrow^{n} 1=\left(x \rightsquigarrow^{n} 1\right) \rightsquigarrow 1$,
(iv) $x \rightarrow\left(y \rightsquigarrow^{n} z\right)=y \rightsquigarrow^{n}(x \rightarrow z), \quad x \rightsquigarrow\left(y \rightarrow^{n} z\right)=y \rightarrow^{n}(x \rightsquigarrow z)$,
(v) $x \rightarrow^{m}\left(y \rightsquigarrow^{n} z\right)=y \rightsquigarrow^{n}\left(x \rightarrow^{m} z\right)$,
(vi) $x \rightarrow^{n} 1=((x \rightarrow 1) \rightarrow 1) \rightarrow^{n} 1, \quad x \rightsquigarrow^{n} 1=((x \rightsquigarrow 1) \rightsquigarrow 1) \rightsquigarrow^{n} 1$.

Proof. (i) Follows from (b4) and (b12).
(ii) Obvious.
(iii) We prove first equation by induction. The proof of second equation is analogous. For $n=0$ it is obvious. Assume it for $n=k$ :

$$
(x \rightarrow 1) \rightarrow^{k} 1=\left(x \rightarrow^{k} 1\right) \rightarrow 1
$$

We have, by definition, assumption, (i), (b11) and (b12),

$$
\begin{aligned}
(x \rightarrow 1) \rightarrow^{k+1} 1 & =(x \rightarrow 1) \rightarrow\left((x \rightarrow 1) \rightarrow^{k} 1\right)=(x \rightarrow 1) \rightarrow\left(\left(x \rightarrow^{k} 1\right) \rightarrow 1\right) \\
& =(x \rightsquigarrow 1) \rightarrow\left(\left(x \rightsquigarrow^{k} 1\right) \rightsquigarrow 1\right)=\left(x \rightsquigarrow\left(x \rightsquigarrow^{k} 1\right)\right) \rightsquigarrow 1 \\
& =\left(x \rightsquigarrow^{k+1} 1\right) \rightsquigarrow 1=\left(x \rightarrow^{k+1} 1\right) \rightarrow 1 .
\end{aligned}
$$

So, the equation holds for any $n=0,1,2, \ldots$
(iv) We prove first equation by induction. The proof of second equation is analogous. For $n=0$ it is obvious. Assume it for $n=k$, that is,

$$
x \rightarrow\left(y \rightsquigarrow^{k} z\right)=y \rightsquigarrow^{k}(x \rightarrow z) .
$$

We have, by definition, assumption and (b4),

$$
\begin{aligned}
x \rightarrow\left(y \rightsquigarrow^{k+1} z\right) & =x \rightarrow\left(y \rightsquigarrow\left(y \rightsquigarrow^{k} z\right)\right)=y \rightsquigarrow\left(x \rightarrow\left(y \rightsquigarrow^{k} z\right)\right) \\
& =y \rightsquigarrow\left(y \rightsquigarrow^{k}(x \rightarrow z)\right)=y \rightsquigarrow^{k+1}(x \rightarrow z) .
\end{aligned}
$$

Hence, the equation holds for any $n=0,1,2, \ldots$.
(v) We get it easily by (iv).
(vi) We prove first equation by induction. The proof of second equation is analogous. For $n=0$ it is obvious. Assume it for $n=k$, that is,

$$
x \rightarrow^{k} 1=((x \rightarrow 1) \rightarrow 1) \rightarrow^{k} 1
$$

We have, by definition, assumption, (i), (iv), (b9) and (b12),

$$
\begin{aligned}
((x \rightarrow 1) \rightarrow 1) \rightarrow^{k+1} 1 & =((x \rightarrow 1) \rightarrow 1) \rightarrow\left(((x \rightarrow 1) \rightarrow 1) \rightarrow^{k} 1\right) \\
& =((x \rightarrow 1) \rightarrow 1) \rightarrow\left(x \rightarrow^{k} 1\right) \\
& =((x \rightarrow 1) \rightarrow 1) \rightarrow\left(x \rightsquigarrow^{k} 1\right) \\
& =x \rightsquigarrow^{k}(((x \rightarrow 1) \rightarrow 1) \rightarrow 1) \\
& =x \rightsquigarrow^{k}(x \rightarrow 1) \\
& =x \rightarrow\left(x \rightsquigarrow^{k} 1\right) \\
& =x \rightarrow\left(x \rightarrow^{k} 1\right) \\
& =x \rightarrow^{k+1} 1 .
\end{aligned}
$$

Hence, the equation holds for any $n=0,1,2, \ldots$.
Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. For any $x \in X$, if there exists the least natural number $n$ such that $x \rightarrow^{n} 1=1$, then $n$ is called a period of $x$ denoted $p(x)$. If, for any natural number $n, x \rightarrow^{n} 1 \neq 1$, then a period of $x$ is called to be infinite and denoted $p(x)=\infty$. Obviously, $p(1)=1$.

Proposition 3.2. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then $p(x)=$ $p(x \rightarrow 1)=p(x \rightsquigarrow 1)$ for all $x \in X$.

Proof. Obviously, $p(x \rightarrow 1)=p(x \rightsquigarrow 1)$. For any $x \in X$, by Proposition 3.1(iii,v), we have

$$
x \rightarrow^{k} 1=((x \rightarrow 1) \rightarrow 1) \rightarrow^{k} 1=\left((x \rightarrow 1) \rightarrow^{k} 1\right) \rightarrow 1
$$

Since $(x \rightarrow 1) \rightarrow^{k} 1$ is a maximal element, we have that $x \rightarrow^{k} 1=1$ if and only if $(x \rightarrow 1) \rightarrow^{k} 1=1$. Thus, $p(x)=p(x \rightarrow 1)$.

Proposition 3.3. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $x, y \in X$. If $x \leq y$, then $p(x)=p(y)$.

Proof. Let $x, y \in X$. By Lemma 2.6 and Proposition 3.2, if $x \leq y$, then $x \rightarrow$ $1=y \rightarrow 1$ and $p(x)=p(x \rightarrow 1)=p(y \rightarrow 1)=p(y)$.

Theorem 3.4. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra and $\left(X ; \cdot,^{-1}, 1\right)$ be a group related with $\mathcal{X}$. Then $p(x)=o(x)$ for any $x \in X$, where $o(x)$ means an order of an element $x$ in a group $\left(X ; \cdot,{ }^{-1}, 1\right)$.
Proof. Let $x \in X$. Since $x \rightarrow y=y \cdot x^{-1}$, it is not difficult to see that $(x \rightarrow$ 1) $\rightarrow^{k} 1=x^{k}$ for any $k=0,1,2, \ldots$ Then,

$$
(x \rightarrow 1) \rightarrow^{k} 1=1 \quad \text { iff } \quad x^{k}=1
$$

So, $p(x \rightarrow 1)=o(x)$. Thus, by Proposition 3.2, $p(x)=o(x)$.
Corollary 3.5. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a p-semisimple pseudo-BCI-algebra. Then the following hold for any $x, y \in X$,
(i) $p(x \rightarrow y)=p(y \rightarrow x), \quad p(x \rightsquigarrow y)=p(y \rightsquigarrow x)$,
(ii) $p(x \rightarrow y)=p(x \rightsquigarrow y)$.

Now we prove that identities from Corollary 3.5 hold also for arbitrary pseudo-BCI-algebras.

Theorem 3.6. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then the following hold for any $x, y \in X$,
(i) $p(x \rightarrow y)=p(y \rightarrow x), \quad p(x \rightsquigarrow y)=p(y \rightsquigarrow x)$,
(ii) $p(x \rightarrow y)=p(x \rightsquigarrow y)$.

Proof. (i) We show the first equation. The proof of the second one is analogous. Let $x, y \in X$. Then $x \rightarrow 1, y \rightarrow 1 \in M(X)$. By Proposition 3.2, (b11), (b12) and Corollary 3.5 we have

$$
\begin{aligned}
p(x \rightarrow y) & =p((x \rightarrow y) \rightarrow 1)=p((x \rightarrow 1) \rightsquigarrow(y \rightarrow 1)) \\
& =p((y \rightarrow 1) \rightsquigarrow(x \rightarrow 1))=p((y \rightarrow x) \rightarrow 1) \\
& =p(y \rightarrow x) .
\end{aligned}
$$

(ii) Similarly we have

$$
\begin{aligned}
p(x \rightarrow y) & =p((x \rightarrow y) \rightarrow 1)=p((x \rightarrow 1) \rightsquigarrow(y \rightarrow 1)) \\
& =p((x \rightsquigarrow 1) \rightarrow(y \rightsquigarrow 1))=p((x \rightsquigarrow y) \rightarrow 1) \\
& =p(x \rightsquigarrow y)
\end{aligned}
$$

for any $x, y \in X$.

Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra and $x \in X$. It is not difficult to see that

$$
p(x)=1 \quad \text { iff } \quad x \leq 1 .
$$

Hence we have the following proposition.
Proposition 3.7. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then it is a pseudo-BCK-algebra if and only if $p(x)=1$ for any $x \in X$.
Corollary 3.8. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then it is proper if and only if there exists $x \in X$ such that $p(x)>1$.

Corollary 3.9. Let $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCI-algebra. Then it is $p$ semisimple if and only if $p(x)>1$ for any $x \in X \backslash\{1\}$.
A pseudo-BCI-algebra $\mathcal{X}=(X ; \rightarrow, \rightsquigarrow, 1)$ is called periodic if $p(x)<\infty$ for any $x \in X$. It is immediately seen that every pseudo-BCK-algebra is periodic.

Now give an interesting characterization of periodic pseudo-BCI-algebras.
Theorem 3.10. A pseudo-BCI-algebra $\mathcal{X}$ is periodic if and only if every deductive system of $\mathcal{X}$ is closed.

Proof. Assume that $\mathcal{X}$ is periodic and $D$ is a deductice system of $\mathcal{X}$. Let $x \in D$. Then there exists a natural number $n$ such that $x \rightarrow^{n} 1=1$. Since $x, x \rightarrow^{n} 1 \in D$ and $D$ is a deductive system, we have $x \rightarrow 1 \in D$, that is, $D$ is closed.

Conversely, for any $x \in X$, a deductive system $D(x)$ is closed. Hence, $x \rightarrow$ $1 \in D(x)$. So, there exists a natural number $n$ such that $x \rightarrow^{n}(x \rightarrow 1)=1$, that is, $p(x)<\infty$. Thus $\mathcal{X}$ is periodic.

By Proposition 2.10 we have the following.
Corollary 3.11. Let $\mathcal{X}$ be a pseudo-BCI-algebra. If $M(X)$ is finite, then $\mathcal{X}$ is periodic.

Corollary 3.12. Every finite pseudo-BCI-algebra is periodic.
Example 3.13. The pseudo-BCI-algebra $\mathcal{X}$ from Example 2.1 is periodic because it is finite and the pseudo-BCI-algebra $\mathcal{Y}$ from Example 2.4 is not periodic because a deductive system $D=\{(x, y, y): x \leq 0, y \in \mathbb{R}\}$ is not closed.

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