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# ON A PERIOD OF ELEMENTS OF PSEUDO-BCI-ALGEBRAS

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#### Abstract

The notions of a period of an element of a pseudo-BCI-algebra and a periodic pseudo-BCI-algebra are defined. Some of their properties and characterizations are given.

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# 1. INTRODUCTION

In 1966 K. Iséki introduced the notion of BCI-algebra (see [10]). BCI-algebras have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. The name of BCI-algebra originates from the combinatories B, C, I in combinatory logic.

The concept of pseudo-BCI-algebra has been introduced in [1] as an extension of BCI-algebras. Pseudo-BCI-algebras are algebraic models of some extension of a noncommutative version of the BCI-logic (see [5] for details). These algebras have also connections with other algebras of logic such as pseudo-BCK-algebras, pseudo-BL-algebras and pseudo-MV-algebras introduced by G. Georgescu and A. Iorgulescu in [6, 7] and [8], respectively. More about those algebras the reader can find in [9].

In this paper we define the notion of a period of an element of a pseudo-BCIalgebra. Some of its properties are also given. Finally, we study the concept of a periodic pseudo-BCI-algebra proving some of its interesting characterization. All necessary material needed in the sequel is presented in Section 2 making our exposition self-contained.

#### 2. Preliminaries

A pseudo-BCI-algebra is a structure  $\mathcal{X} = (X; \leq, \rightarrow, \rightsquigarrow, 1)$ , where  $\leq$  is binary relation on a set  $X, \rightarrow$  and  $\rightsquigarrow$  are binary operations on X and 1 is an element of X such that for all  $x, y, z \in X$ , we have

(a1)  $x \to y \le (y \to z) \rightsquigarrow (x \to z), \quad x \rightsquigarrow y \le (y \rightsquigarrow z) \to (x \rightsquigarrow z),$ 

(a2) 
$$x \le (x \to y) \rightsquigarrow y, \quad x \le (x \rightsquigarrow y) \to y,$$

- (a3)  $x \leq x$ ,
- (a4) if  $x \leq y$  and  $y \leq x$ , then x = y,
- (a5)  $x \leq y$  iff  $x \to y = 1$  iff  $x \rightsquigarrow y = 1$ .

It is obvious that any pseudo-BCI-algebra  $(X; \leq, \rightarrow, \rightsquigarrow, 1)$  can be regarded as a universal algebra  $(X; \rightarrow, \rightsquigarrow, 1)$  of type (2, 2, 0). Note that every pseudo-BCIalgebra satisfying  $x \rightarrow y = x \rightsquigarrow y$  for all  $x, y \in X$  is a BCI-algebra.

Every pseudo-BCI-algebra satisfying  $x \leq 1$  for all  $x \in X$  is a pseudo-BCK-algebra. A pseudo-BCI-algebra which is not a pseudo-BCK-algebra will be called *proper*.

Any pseudo-BCI-algebra  $\mathcal{X} = (X; \leq, \rightarrow, \rightsquigarrow, 1)$  satisfies the following, for all  $x, y, z \in X$ ,

- (b1) if  $1 \le x$ , then x = 1,
- (b2) if  $x \leq y$ , then  $y \to z \leq x \to z$  and  $y \rightsquigarrow z \leq x \rightsquigarrow z$ ,
- (b3) if  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ ,
- (b4)  $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z),$
- (b5)  $x \leq y \rightarrow z$  iff  $y \leq x \rightsquigarrow z$ ,
- (b6)  $x \to y \le (z \to x) \to (z \to y), \quad x \rightsquigarrow y \le (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y),$
- (b7) if  $x \leq y$ , then  $z \to x \leq z \to y$  and  $z \rightsquigarrow x \leq z \rightsquigarrow y$ ,
- (b8)  $1 \rightarrow x = 1 \rightsquigarrow x = x$ ,
- (b9)  $((x \to y) \rightsquigarrow y) \to y = x \to y, \quad ((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y,$
- (b10)  $x \to y \le (y \to x) \rightsquigarrow 1$ ,  $x \rightsquigarrow y \le (y \rightsquigarrow x) \to 1$ ,
- $(\text{b11}) \ (x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \rightsquigarrow 1), \ (x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \to 1),$
- (b12)  $x \to 1 = x \rightsquigarrow 1$ .

If  $(X; \leq, \rightarrow, \rightsquigarrow, 1)$  is a pseudo-BCI-algebra, then, by (a3), (a4), (b3) and (b1),  $(X; \leq)$  is a poset with 1 as a maximal element.

**Example 2.1** ([3]). Let  $X = \{a, b, c, d, e, f, 1\}$  and define binary operations  $\rightarrow$  and  $\rightsquigarrow$  on X by the following tables:

$\rightarrow$	a	b	c	d	e	f	1	$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	d	e	b	c	a	a	a	1	c	b	e	d	a	a
b	c	1	a	e	d	b	b	b	d	1	e	a	c	b	b
c	e	a	1	c	b	d	d	c	b	e	1	c	a	d	d
d	b	e	d	1	a	c	c	d	e	a	d	1	b	c	c
e	d	c	b	a	1	e	e	e	c	d	a	b	1	e	e
f	a	b	c	d	e	1	1	f	a	b	c	d	e	1	1
1	a	b	c	d	e	f	1	1	a	b	c	d	e	f	1

Then  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is a (proper) pseudo-BCI-algebra. Observe that it is not a pseudo-BCK-algebra because  $a \nleq 1$ .

**Example 2.2** ([11]). Let  $Y_1 = (-\infty, 0]$  and let  $\leq$  be the usual order on  $Y_1$ . Define binary operations  $\rightarrow$  and  $\sim$  on  $Y_1$  by

$$x \to y = \begin{cases} 0 & \text{if } x \le y, \\ \frac{2y}{\pi} \arctan(\ln(\frac{y}{x})) & \text{if } y < x, \end{cases}$$
$$x \rightsquigarrow y = \begin{cases} 0 & \text{if } x \le y, \\ ye^{-\tan(\frac{\pi x}{2y})} & \text{if } y < x \end{cases}$$

for all  $x, y \in Y_1$ . Then  $\mathcal{Y}_1 = (Y_1; \leq, \rightarrow, \rightsquigarrow, 0)$  is a pseudo-BCK-algebra, and hence it is a nonproper pseudo-BCI-algebra.

**Example 2.3** ([4]). Let  $Y_2 = \mathbb{R}^2$  and define binary operations  $\rightarrow$  and  $\rightarrow$  and a binary relation  $\leq$  on  $Y_2$  by

$$(x_1, y_1) \to (x_2, y_2) = (x_2 - x_1, (y_2 - y_1)e^{-x_1}), (x_1, y_1) \rightsquigarrow (x_2, y_2) = (x_2 - x_1, y_2 - y_1e^{x_2 - x_1}),$$

$$(x_1, y_1) \le (x_2, y_2) \Leftrightarrow (x_1, y_1) \to (x_2, y_2) = (0, 0) = (x_1, y_1) \rightsquigarrow (x_2, y_2)$$

for all  $(x_1, y_1), (x_2, y_2) \in Y_2$ . Then  $\mathcal{Y}_2 = (Y_2; \leq, \rightarrow, \rightsquigarrow, (0, 0))$  is a proper pseudo-BCI-algebra. Notice that  $\mathcal{Y}_2$  is not a pseudo-BCK-algebra because there exists  $(x, y) = (1, 1) \in Y_2$  such that  $(x, y) \nleq (0, 0)$ .

**Example 2.4** ([4]). Let  $\mathcal{Y}$  be the direct product of pseudo-BCI-algebras  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  from Examples 2.2 and 2.3, respectively. Then  $\mathcal{Y}$  is a proper pseudo-BCI-algebra, where  $Y = (-\infty, 0] \times \mathbb{R}^2$  and binary operations  $\rightarrow$  and  $\rightsquigarrow$  and binary relation  $\leq$  are defined on Y by

$$\begin{aligned} (x_1, y_1, z_1) &\to (x_2, y_2, z_2) = \\ & \begin{cases} (0, y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_1 \le x_2, \\ (\frac{2x_2}{\pi} \arctan(\ln(\frac{x_2}{x_1})), y_2 - y_1, (z_2 - z_1)e^{-y_1}) & \text{if } x_2 < x_1, \end{cases} \end{aligned}$$

$$\begin{aligned} (x_1, y_1, z_1) &\rightsquigarrow (x_2, y_2, z_2) = \\ & \begin{cases} (0, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_1 \leq x_2, \\ (x_2 e^{-\tan(\frac{\pi x_1}{2x_2})}, y_2 - y_1, z_2 - z_1 e^{y_2 - y_1}) & \text{if } x_2 < x_1, \end{cases} \\ (x_1, y_1, z_1) &\leq (x_2, y_2, z_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 = y_2 \text{ and } z_1 = z_2. \end{aligned}$$

Notice that  $\mathcal{Y}$  is not a pseudo-BCK-algebra because there exists  $(x, y, z) = (0, 1, 1) \in Y$  such that  $(x, y, z) \nleq (0, 0, 0)$ .

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Define

$$\begin{aligned} x &\to^0 y = y, \\ x &\to^n y = x \to (x \to^{n-1} y), \end{aligned}$$

where  $x, y \in X$  and n = 1, 2, ... Similarly we define  $x \rightsquigarrow^n y$  for any n = 0, 1, 2, ...

**Proposition 2.5.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following are equivalent for any  $x, y \in X$  and  $n = 0, 1, 2, \ldots$ ,

(i)  $x \to^n y = 1$ ,

(ii) 
$$x \rightsquigarrow^n y = 1$$
.

**Proof.** It follows by (a5) and (b4).

For any pseudo-BCI-algebra  $\mathcal{X}=(X;\rightarrow,\rightsquigarrow,1)$  the set

$$K(X) = \{ x \in X : x \le 1 \}$$

is a subalgebra of  $\mathcal{X}$  (called pseudo-BCK-part of  $\mathcal{X}$ , see [1]). Then  $(K(X); \rightarrow, \rightsquigarrow, 1)$  is a pseudo-BCK-algebra. Note that if  $\mathcal{X}$  is a pseudo-BCK-algebra, then X = K(X).

It is easily seen that for the pseudo-BCI-algebras  $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$  and  $\mathcal{Y}$  from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have  $K(X) = \{f, 1\}, K(Y_1) = Y_1, K(Y_2) = \{(0,0)\}$  and  $K(Y) = \{(x,0,0) : x \leq 0\}.$ 

An element a of a pseudo-BCI-algebra  $\mathcal{X}$  is called a *maximal element* of  $\mathcal{X}$  if for every  $x \in X$  the following holds

if 
$$a \leq x$$
, then  $x = a$ .

We will denote by M(X) the set of all maximal elements of  $\mathcal{X}$ . Obviously,  $1 \in M(X)$ . Notice that  $M(X) \cap K(X) = \{1\}$ . Indeed, if  $a \in M(X) \cap K(X)$ , then  $a \leq 1$  and, by above implication, a = 1. Moreover, observe that 1 is the only maximal element of a pseudo-BCK-algebra. Therefore, for a pseudo-BCK-algebra  $\mathcal{X}, M(X) = \{1\}$ . In [2] there is shown that  $M(X) = \{x \in X : x = (x \to 1) \to 1\}$ . Moreover we have the following simple lemma.

**Lemma 2.6.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $x, y \in X$ . If  $x \leq y$ , then  $x \to 1 = x \rightsquigarrow 1 = y \to 1 = y \rightsquigarrow 1$ .

Observe that for the pseudo-BCI-algebras  $\mathcal{X}$ ,  $\mathcal{Y}_1$ ,  $\mathcal{Y}_2$  and  $\mathcal{Y}$  from Examples 2.1, 2.2, 2.3 and 2.4, respectively, we have  $M(X) = \{a, b, c, d, e, 1\}$ ,  $M(Y_1) = \{0\}$ ,  $M(Y_2) = Y_2$  and  $M(Y) = \{(0, y, z) : y, z \in \mathbb{R}\}$ .

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then  $\mathcal{X}$  is *p*-semisimple if it satisfies for all  $x \in X$ ,

if 
$$x \leq 1$$
, then  $x = 1$ .

Note that if  $\mathcal{X}$  is a p-semisimple pseudo-BCI-algebra, then  $K(X) = \{1\}$ . Hence, if  $\mathcal{X}$  is a p-semisimple pseudo-BCK-algebra, then  $X = \{1\}$ . Moreover, as it is proved in [4], M(X) is a p-semisimple pseudo-BCI-subalgebra of  $\mathcal{X}$  and  $\mathcal{X}$  is p-semisimple if and only if X = M(X).

**Proposition 2.7** ([4]). Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following are equivalent:

- (i)  $\mathcal{X}$  is p-semisimple,
- (ii) for all  $x, y \in X$ ,  $(x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ ,
- (iii) for all  $x \in X$ ,  $x = (x \to 1) \to 1$ .

It is not difficult to see that the pseudo-BCI-algebras  $\mathcal{X}$ ,  $\mathcal{Y}_1$  and  $\mathcal{Y}$  from Examples 2.1, 2.2 and 2.4, respectively, are not p-semisimple, and the pseudo-BCI-algebra  $\mathcal{Y}_2$  from Example 2.3 is a p-semisimple algebra.

**Theorem 2.8.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following are equivalent:

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- (i)  $\mathcal{X}$  is p-semisimple,
- (ii)  $X = \{x \to 1 : x \in X\}.$

**Proof.** (i) $\Rightarrow$ (ii) Take  $y \in X$ . Since  $\mathcal{X}$  is p-semisimple,  $y = (y \to 1) \to 1$ . Putting  $x = y \to 1 \in X$ , we get  $y = x \to 1$ .

(ii) $\Rightarrow$ (i) Take  $a \in X$ . We show that a is a maximal element of  $\mathcal{X}$ , that is, X = M(X). Suppose that  $a = x \to 1$  for some  $x \in X$ . Let  $y \in X$  be such that  $a \leq y$ . Then  $(x \to 1) \to y = 1$  and, by (b4), (b9), (b11) and (b12), we have

$$y \to a = y \to (x \to 1) = y \to (((x \to 1) \to 1) \rightsquigarrow 1)$$
$$= ((x \to 1) \to 1) \rightsquigarrow (y \to 1) = ((x \to 1) \to y) \to 1$$
$$= 1 \to 1 = 1.$$

Hence  $y \leq a$ . So, y = a, that is,  $a \in M(X)$  and  $\mathcal{X}$  is p-semisimple.

For p-semisimple pseudo-BCI-algebras we have the following useful fact.

**Theorem 2.9** [4]. A pseudo-BCI-algebra  $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$  is p-semisimple if and only if  $(X; \cdot, ^{-1}, 1)$  is a group, where  $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ ,  $x^{-1} = x \to 1 = x \rightsquigarrow 1, x \to y = y \cdot x^{-1}$  and  $x \rightsquigarrow y = x^{-1} \cdot y$  for any  $x, y \in X$ .

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. We say that a subset D of X is a *deductive system* of  $\mathcal{X}$  if it satisfies: (i)  $1 \in D$ , (ii) for all  $x, y \in X$ , if  $x \in D$ and  $x \to y \in D$ , then  $y \in D$ . Under this definition,  $\{1\}$  and X are the simplest examples of deductive systems. Note that the condition (ii) can be replaced by (ii') for all  $x, y \in X$ , if  $x \in D$  and  $x \rightsquigarrow y \in D$ , then  $y \in D$ . It can be easily proved that for any  $x, y \in X$ , if  $x \in D$  and  $x \leq y$ , then  $y \in D$ . A deductive system Dof a pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is called *closed* if D is closed under operations  $\rightarrow$  and  $\rightsquigarrow$ , that is, if D is a subalgebra of  $\mathcal{X}$ . It is not difficult to show (see [3]) that a deductive system D of a pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$ is closed if and only if for any  $x \in D$ ,  $x \to 1 = x \rightsquigarrow 1 \in D$ . Obviously, the pseudo-BCK-part K(X) is a closed deductive system of  $\mathcal{X}$ .

**Proposition 2.10** ([2]). Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and M(X) be finite. Then every deductive system of  $\mathcal{X}$  is closed.

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. It is obvious that the intersection of arbitrary number of deductive systems is a deductive system. Hence, for any  $A \subseteq X$  there exists the least deductive system containing A. Denote it by D(A)and call it the deductive system *generated* by A. In particular, if  $A = \{a_1, \ldots, a_n\}$ , then we write  $D(a_1, \ldots, a_n)$  instead of  $D(\{a_1, \ldots, a_n\})$ . It is also obvious that  $D(\emptyset) = \{1\}$ . **Proposition 2.11** ([3]). Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. For any  $a \in X$ ,

$$D(a) = \{1\} \cup \{x \in X : a \to^n x = 1 \text{ for some } n \in \mathbb{N}\}\$$
  
=  $\{1\} \cup \{x \in X : a \rightsquigarrow^n x = 1 \text{ for some } n \in \mathbb{N}\}.$ 

## 3. Period of elements

**Proposition 3.1.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. The following hold for any  $x, y, z \in X$  and  $m, n = 0, 1, 2, \ldots$ ,

- (i)  $x \to^n 1 = x \rightsquigarrow^n 1$ ,
- (ii)  $x \to^n x = x \to^{n-1} 1$ ,  $x \rightsquigarrow^n x = x \rightsquigarrow^{n-1} 1$ ,
- (iii)  $(x \to 1) \to^n 1 = (x \to^n 1) \to 1, \quad (x \rightsquigarrow 1) \rightsquigarrow^n 1 = (x \rightsquigarrow^n 1) \rightsquigarrow 1,$
- (iv)  $x \to (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \to z), \quad x \rightsquigarrow (y \to^n z) = y \to^n (x \rightsquigarrow z),$

(v) 
$$x \to^m (y \rightsquigarrow^n z) = y \rightsquigarrow^n (x \to^m z),$$

$$(\mathrm{vi}) \ x \to^n 1 = ((x \to 1) \to 1) \to^n 1, \quad x \rightsquigarrow^n 1 = ((x \rightsquigarrow 1) \rightsquigarrow 1) \rightsquigarrow^n 1.$$

**Proof.** (i) Follows from (b4) and (b12).

(ii) Obvious.

(iii) We prove first equation by induction. The proof of second equation is analogous. For n = 0 it is obvious. Assume it for n = k:

$$(x \to 1) \to^k 1 = (x \to^k 1) \to 1.$$

We have, by definition, assumption, (i), (b11) and (b12),

$$(x \to 1) \to^{k+1} 1 = (x \to 1) \to ((x \to 1) \to^k 1) = (x \to 1) \to ((x \to^k 1) \to 1)$$
$$= (x \to 1) \to ((x \to^k 1) \to 1) = (x \to (x \to^k 1)) \to 1$$
$$= (x \to^{k+1} 1) \to 1 = (x \to^{k+1} 1) \to 1.$$

So, the equation holds for any  $n = 0, 1, 2, \ldots$ 

(iv) We prove first equation by induction. The proof of second equation is analogous. For n = 0 it is obvious. Assume it for n = k, that is,

$$x \to (y \rightsquigarrow^k z) = y \rightsquigarrow^k (x \to z).$$

We have, by definition, assumption and (b4),

$$\begin{aligned} x \to (y \rightsquigarrow^{k+1} z) &= x \to (y \rightsquigarrow (y \rightsquigarrow^k z)) = y \rightsquigarrow (x \to (y \rightsquigarrow^k z)) \\ &= y \rightsquigarrow (y \rightsquigarrow^k (x \to z)) = y \rightsquigarrow^{k+1} (x \to z). \end{aligned}$$

Hence, the equation holds for any  $n = 0, 1, 2, \ldots$ 

(v) We get it easily by (iv).

(vi) We prove first equation by induction. The proof of second equation is analogous. For n = 0 it is obvious. Assume it for n = k, that is,

$$x \to^k 1 = ((x \to 1) \to 1) \to^k 1$$

We have, by definition, assumption, (i), (iv), (b9) and (b12),

$$((x \to 1) \to 1) \to^{k+1} 1 = ((x \to 1) \to 1) \to (((x \to 1) \to 1) \to^k 1)$$
$$= ((x \to 1) \to 1) \to (x \to^k 1)$$
$$= ((x \to 1) \to 1) \to (x \to^k 1)$$
$$= x \to^k (((x \to 1) \to 1) \to 1)$$
$$= x \to^k (x \to 1)$$
$$= x \to (x \to^k 1)$$
$$= x \to^{k+1} 1.$$

Hence, the equation holds for any  $n = 0, 1, 2, \ldots$ 

Let  $\mathcal{X} = (X; \to, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. For any  $x \in X$ , if there exists the least natural number n such that  $x \to^n 1 = 1$ , then n is called a *period of* xdenoted p(x). If, for any natural number  $n, x \to^n 1 \neq 1$ , then a period of x is called to be infinite and denoted  $p(x) = \infty$ . Obviously, p(1) = 1.

**Proposition 3.2.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then  $p(x) = p(x \rightarrow 1) = p(x \rightsquigarrow 1)$  for all  $x \in X$ .

**Proof.** Obviously,  $p(x \to 1) = p(x \rightsquigarrow 1)$ . For any  $x \in X$ , by Proposition 3.1(iii,v), we have

$$x \to^k 1 = ((x \to 1) \to 1) \to^k 1 = ((x \to 1) \to^k 1) \to 1.$$

Since  $(x \to 1) \to^k 1$  is a maximal element, we have that  $x \to^k 1 = 1$  if and only if  $(x \to 1) \to^k 1 = 1$ . Thus,  $p(x) = p(x \to 1)$ .

**Proposition 3.3.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $x, y \in X$ . If  $x \leq y$ , then p(x) = p(y). **Proof.** Let  $x, y \in X$ . By Lemma 2.6 and Proposition 3.2, if  $x \leq y$ , then  $x \to 1 = y \to 1$  and  $p(x) = p(x \to 1) = p(y \to 1) = p(y)$ .

**Theorem 3.4.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a p-semisimple pseudo-BCI-algebra and  $(X; \cdot, ^{-1}, 1)$  be a group related with  $\mathcal{X}$ . Then p(x) = o(x) for any  $x \in X$ , where o(x) means an order of an element x in a group  $(X; \cdot, ^{-1}, 1)$ .

**Proof.** Let  $x \in X$ . Since  $x \to y = y \cdot x^{-1}$ , it is not difficult to see that  $(x \to 1) \to^k 1 = x^k$  for any  $k = 0, 1, 2, \dots$  Then,

$$(x \to 1) \to^k 1 = 1$$
 iff  $x^k = 1$ 

So,  $p(x \to 1) = o(x)$ . Thus, by Proposition 3.2, p(x) = o(x).

**Corollary 3.5.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a *p*-semisimple pseudo-BCI-algebra. Then the following hold for any  $x, y \in X$ ,

- (i)  $p(x \to y) = p(y \to x), \quad p(x \rightsquigarrow y) = p(y \rightsquigarrow x),$
- (ii)  $p(x \rightarrow y) = p(x \rightsquigarrow y)$ .

Now we prove that identities from Corollary 3.5 hold also for arbitrary pseudo-BCI-algebras.

**Theorem 3.6.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then the following hold for any  $x, y \in X$ ,

- (i)  $p(x \to y) = p(y \to x), \quad p(x \rightsquigarrow y) = p(y \rightsquigarrow x),$
- (ii)  $p(x \rightarrow y) = p(x \rightsquigarrow y)$ .

**Proof.** (i) We show the first equation. The proof of the second one is analogous. Let  $x, y \in X$ . Then  $x \to 1, y \to 1 \in M(X)$ . By Proposition 3.2, (b11), (b12) and Corollary 3.5 we have

$$p(x \to y) = p((x \to y) \to 1) = p((x \to 1) \rightsquigarrow (y \to 1))$$
$$= p((y \to 1) \rightsquigarrow (x \to 1)) = p((y \to x) \to 1)$$
$$= p(y \to x).$$

(ii) Similarly we have

$$p(x \to y) = p((x \to y) \to 1) = p((x \to 1) \rightsquigarrow (y \to 1))$$
$$= p((x \rightsquigarrow 1) \to (y \rightsquigarrow 1)) = p((x \rightsquigarrow y) \to 1)$$
$$= p(x \rightsquigarrow y)$$

for any  $x, y \in X$ .

Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra and  $x \in X$ . It is not difficult to see that

$$p(x) = 1$$
 iff  $x \leq 1$ .

Hence we have the following proposition.

**Proposition 3.7.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then it is a pseudo-BCK-algebra if and only if p(x) = 1 for any  $x \in X$ .

**Corollary 3.8.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then it is proper if and only if there exists  $x \in X$  such that p(x) > 1.

**Corollary 3.9.** Let  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  be a pseudo-BCI-algebra. Then it is psemisimple if and only if p(x) > 1 for any  $x \in X \setminus \{1\}$ .

A pseudo-BCI-algebra  $\mathcal{X} = (X; \rightarrow, \rightsquigarrow, 1)$  is called *periodic* if  $p(x) < \infty$  for any  $x \in X$ . It is immediately seen that every pseudo-BCK-algebra is periodic.

Now give an interesting characterization of periodic pseudo-BCI-algebras.

**Theorem 3.10.** A pseudo-BCI-algebra  $\mathcal{X}$  is periodic if and only if every deductive system of  $\mathcal{X}$  is closed.

**Proof.** Assume that  $\mathcal{X}$  is periodic and D is a deductice system of  $\mathcal{X}$ . Let  $x \in D$ . Then there exists a natural number n such that  $x \to^n 1 = 1$ . Since  $x, x \to^n 1 \in D$ and D is a deductive system, we have  $x \to 1 \in D$ , that is, D is closed.

Conversely, for any  $x \in X$ , a deductive system D(x) is closed. Hence,  $x \to 1 \in D(x)$ . So, there exists a natural number n such that  $x \to^n (x \to 1) = 1$ , that is,  $p(x) < \infty$ . Thus  $\mathcal{X}$  is periodic.

By Proposition 2.10 we have the following.

**Corollary 3.11.** Let  $\mathcal{X}$  be a pseudo-BCI-algebra. If M(X) is finite, then  $\mathcal{X}$  is periodic.

Corollary 3.12. Every finite pseudo-BCI-algebra is periodic.

**Example 3.13.** The pseudo-BCI-algebra  $\mathcal{X}$  from Example 2.1 is periodic because it is finite and the pseudo-BCI-algebra  $\mathcal{Y}$  from Example 2.4 is not periodic because a deductive system  $D = \{(x, y, y) : x \leq 0, y \in \mathbb{R}\}$  is not closed.

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