# COMPLICATED BE-ALGEBRAS AND CHARACTERIZATIONS OF IDEALS 

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#### Abstract

In this paper, using the notion of upper sets, we introduced the notions of complicated BE-Algebras and gave some related properties on complicated, self-distributive and commutative BE-algebras. In a self-distributive and complicated BE-algebra, characterizations of ideals are obtained.


Keywords: BE-algebras, complicated BE-algebras, ideals in BE-algebras. 2010 Mathematics Subject Classification: 03G25, 06F35.

## 1. Introduction

Y. Imai and K. Isėki introduced two classes of abstract algebras called BCKalgebras and BCI-algebras [8, 10]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In [5, 6], Q.P. Hu and X . Li introduced a wide class of abstract algebras called BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of BCH-algebras. J. Neggers and H.S. Kim ([16]) introduced the notion of a d-algebra which is a generalization of BCKalgebras, and also they introduced the notion of B-algebras ([17, 18]). Y.B. Jun, E.H. Roh and H.S. Kim ([11]) introduced a new notion called BH-algebra which
is another generalization of $\mathrm{BCH} / \mathrm{BCI} / \mathrm{BCK}$-algebras. A. Walendziak obtained another equivalent axioms for B-algebras ([20]). C.B. Kim and H.S. Kim ([13]) introduced the notion of BM-algebra which is a specialization of B -algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0 -commutative B -algebra. In [14], H.S. Kim and Y.H. Kim introduced the notion of BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. In [2] and [3], S.S. Ahn and K.S. So introduced the notion of ideals in BE-algebras, and proved several characterizations of such ideals. Also they generalized the notion of upper sets in BE-algebras and discussed some properties of the characterizations of generalized upper sets related to the structure of ideals in transitive and self distributive BE-algebras. In [4], S.S. Ahn, Y.H. Kim and J.M. Ko are introduced the notion of terminal section of BE-algebras and provided the characterization of a commutative BE-algebras.
B.M. Schein [19] considered systems of the form ( $\phi ; \circ, \backslash$ ), where $\phi$ is a set of functions closed under the composition "०" of functions (and hence ( $\phi ; \circ$ ) is a function semigroup) and the set theoretic subtraction " $\backslash$ " (and hence $(\phi ; \backslash)$ is a subtraction algebra in the sence of [1]). B. Zelinka [22] discussed a problem proposed by B.M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y.B. Jun et al. [12] introduced the complicated subtraction algebras and investigated several properties on it.

In this paper, using the notion of upper sets, we introduced the notions of complicated BE-Algebras and gave some related properties on complicated, selfdistrubutive and commutative BE-algebras. In a self-distributive and complicated BE-algebra, characterizations of ideals are obtained.

## 2. Preliminaries

Definition 2.1 [14]. An algebra $(X ; *, 1)$ of type $(2,0)$ is called a BE-algebra if, for all $a, b, c$ in $X$, the following identities hold:
(BE1) $a * a=1$,
(BE2) $a * 1=1$,
(BE3) $1 * a=a$,
(BE4) $a *(b * c)=b *(a * c)$.
In a BE-algebra $X$, the relation " $\leq$ " is defined by $a \leq b$ if and only if $a * b=1$.
Proposition 2.2 [14]. If $(X ; *, 1)$ is a BE-algebra, then
(i) $a *(b * a)=1$,
(ii) $a *((a * b) * b)=1$
for any $a, b \in X$.
Example 2.1. [14] Let $X=\{1, a, b, c, d, 0\}$ be a set with the following table:

| $*$ | 1 | $a$ | $b$ | $c$ | $d$ | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | 0 |
| $a$ | 1 | 1 | $a$ | $c$ | $c$ | $d$ |
| $b$ | 1 | 1 | 1 | $c$ | $c$ | $c$ |
| $c$ | 1 | $a$ | $b$ | 1 | $a$ | $b$ |
| $d$ | 1 | 1 | $a$ | 1 | 1 | $a$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |.

Then $(X ; *, 1)$ is a BE-algebra.
Definition 2.3 [14]. A BE-algebra $(X ; *, 1)$ is said to be self-distributive if $a *$ $(b * c)=(a * b) *(a * c)$ for all $a, b, c \in X$.

Example 2.2 [14]. Let $X=\{1, a, b, c, d\}$ be a set with the following table:

| $*$ | 1 | $a$ | $b$ | $c$ | $d$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | $a$ | $b$ | $c$ | $d$ |
| $a$ | 1 | 1 | $b$ | $c$ | $d$ |
| $b$ | 1 | $a$ | 1 | $c$ | $c$ |
| $c$ | 1 | 1 | $b$ | 1 | $b$ |
| $d$ | 1 | 1 | 1 | 1 | 1 |.

Then $(X ; *, 1)$ is a self-distributive BE-algebra.
Proposition $2.4([2,4])$. Let $(X ; *, 1)$ be a self-distributive BE-algebra. If $a \leq b$, then, for all $a, b, c$ in $X$, the following hold:
(i) $c * a \leq c * b$,
(ii) $b * c \leq a * c$,
(iii) $a * b \leq(b * c) *(a * c)$.

Definition 2.5 [21]. Let $X$ be a BE-algebra. We say that $X$ is commutative if (C) $(a * b) * b=(b * a) * a$
for all $a, b \in X$.
Proposition 2.6 [21]. If $(X ; *, 1)$ is a commutative BE-algebra, then for all $a, b \in X$,

$$
a * b=1 \text { and } b * a=1 \text { imply } a=b \text {. }
$$

Definition 2.7 [2]. Let $X$ be a BE-algebra. A nonempty subset $I$ of $X$ is called an ideal of $X$ if
(I1) $\forall x \in X$ and $\forall a \in I$ imply $x * a \in I$,
(I2) $\forall x \in X$ and $\forall a, b \in I$ imply $(a *(b * x)) * x \in I$.
Corollary 2.8 [2]. Let $I$ be an ideal of $X$. If $a \in I$ and $a \leq x$, then $x \in I$.
Corollary 2.9 [2]. Let $X$ be a self-distributive BE-algebra. A nonempty subset $I$ of $X$ is an ideal of $X$ if and only if it satisfies the following conditions
(I3) $1 \in I$,
(I4) $x *(y * z) \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

## 3. Complicated BE-algebras

Definition 3.1. Let $(X ; *, 1)$ be a BE-algebra and $a, b \in X$. The set

$$
A(a, b)=\{x \in X: a *(b * x)=1\}
$$

is called an upper set of $a$ and $b$. It is easy to see that $1, a, b \in A(a, b)$.
Proposition 3.2. Let $(X ; *, 1)$ be a BE-algebra. Then $A(a, b)=A(b, a)$ for all $a, b \in X$.
Proof. It is clear by (BE4).
Example 3.1. Let $X=\{1, a, b, c\}$ be a set with the following table:

| $*$ | 1 | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $b$ | $c$ |
| $b$ | 1 | 1 | 1 | $c$ |
| $c$ | 1 | 1 | 1 | 1 |.

It is clear that $X$ is a BE-algebra and $A(1,1)=\{1\}, A(1, a)=A(a, a)=$ $\{1, a\}, A(1, b)=A(a, b)=A(b, b)=\{1, a, b\}$ and $A(1, c)=A(a, c)=A(b, c)=$ $A(c, c)=X$.

Example 3.2. Let $X=\{1, a, b, c\}$ be a set with the following table:

| $*$ | 1 | $a$ | $b$ | $c$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $a$ | $b$ | $c$ |
| $a$ | 1 | 1 | $b$ | $c$ |
| $b$ | 1 | $a$ | 1 | $c$ |
| $c$ | 1 | 1 | 1 | 1 |.

It is clear that $X$ is a BE-algebra and $A(1,1)=\{1\}, A(1, a)=A(a, a)=$ $\{1, a\}, A(1, b)=A(b, b)=\{1, b\}, A(a, b)=\{1, a, b\}$ and $A(1, c)=A(a, c)=$ $A(b, c)=A(c, c)=X$.

Definition 3.3. A BE-algebra $(X ; *, 1)$ is called a complicated BE-algebra (c-BE-algebra, shortly) if for all $a, b \in X$, the set $A(a, b)$ has the smallest element. The smallest element of $A(a, b)$ is denoted by $a(s b$.

Example 3.3. The BE-algebra $X$ in Example 3.1 is a c-BE-algebra since $1(51=$ $1,1(\mathrm{~S} a=a, a(\mathrm{~S}) a=a, 1(\mathrm{~S} b=a(5) b=b(\mathrm{~S} b=b$ and $1(5) c=a(5) c=b(\mathrm{~S} c=c(\mathrm{~S} c=c$. But the BE-algebra in Example 3.2 is not a c-BE-algebra since $A(a, b)=\{1, a, b\}$ has no the smallest element.

Proposition 3.4. Let $(X ; *, 1)$ be a $c$-BE-algebra. Then, for all $a, b \in X$,
(i) $a(\mathrm{~S} b \leq a$ and $a(\mathrm{~S} b \leq b$,
(ii) $a(\mathbb{S} 1=a$,
(iii) $a(\mathfrak{S} b=b$ S $a$,
(iv) $a(S)(a * b) \leq b$.

Proof. (i) and (ii) are easily seen by the definition of the c-BE algebra.
(iii) is clear since $A(a, b)=A(b, a)$.
(iv) From Proposition 2.1 (i), since $a *((a * b) * b)=1$, we have $b \in A(a, a * b)$ and hence $a(S)(a * b) \leq b$.

Proposition 3.5. Let $(X ; *, 1)$ be a self-distributive BE-algebra. If, for all $a, b, c$ $\in X, a \leq b$ and $b \leq c$ then $a \leq c$.

Proof. Since $a * c=1 *(a * c)=(a * b) *(a * c)=a *(b * c)=a * 1=1$, we have $a \leq c$.

Proposition 3.6. Let $(X ; *, 1)$ be a self-distributive $c$-BE-algebra. Then, for all $a, b, c \in X$,
(i) $a \leq b$ implies $a(5) c \leq b(5) c$,
(ii) $(a * b)(\mathbb{S}(b * c) \leq a * c$.

Proof. (i) Let $a \leq b$. Since $X$ is self-distributive, by Proposition 2.4 (ii), we have $b *(b(\mathbb{S} c) \leq a *(b(\mathbb{S}) c)$. Also since $b(\mathbb{S} c \in A(b, c)$, we have $c \leq b *(b(S c)$. Then by Proposition 3.5, we get $c \leq a *(b(S) c)$. Hence we obtain $b(S) c \in A(a, c)$ and $a(\mathrm{~S}) c \leq b(\mathrm{~S}) c$.
(ii) By Proposition 2.4. (iii), we have $a * b \leq(b * c) *(a * c)$. Hence we see that $a * c \in A(a * b, b * c)$ and $(a * b)(S)(b * c) \leq a * c$.

Theorem 3.7. Let $(X ; *, 1)$ be a self-distributive and commutative c-BE-algebra. Then $(X ; S)$ is a commutative monoid.

Proof. By Proposition 3.4 (ii) and (iii), we need only to show that ( $X$; (S) is associative. Say $(a \subseteq b) \subseteq c=u$. Then, since $u \in A(a(S b, c)$ and $A(a(5 b, c)=$ $A(c, a(S) b)$, we know that

$$
\begin{equation*}
a(\mathrm{~S} b \leq c * u \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
c \leq(a \subseteq b) * u \tag{3.2}
\end{equation*}
$$

Hence using the equation (3.1), we have, by Proposition 2.4 (i) and (BE4),

$$
\begin{equation*}
b *(a(S) b) \leq b *(c * u)=c *(b * u) \tag{3.3}
\end{equation*}
$$

Since $a \leq b *(a S b)$, using the equation (3.3) and Proposition 3.5, we obtain

$$
\begin{equation*}
a \leq c *(b * u) \tag{3.4}
\end{equation*}
$$

From the equation (3.4), we have $b * u \in A(a, c)$ and $a(S) c \leq b * u$. So we see that $u \in A(a(S) c, b)$, that is,

$$
\begin{equation*}
(a \subseteq c) \subseteq b \leq(a(S) b) \subseteq c=u \tag{3.5}
\end{equation*}
$$

Since the equation (3.5) is true for all $a, b, c \in X$, the following inequality is true:

$$
\begin{equation*}
(a(S) b) \subseteq c \leq(a \subseteq c) \subseteq \tag{3.6}
\end{equation*}
$$

Hence by Proposition 2.6, using the equation (3.5) and (3.6), we get

$$
\begin{equation*}
(a(\mathrm{~S} b)(\mathbb{S}) c=(a(\mathbb{S} c) \subseteq \tag{3.7}
\end{equation*}
$$

Then we obtain $(a(S b)(S) c=(b \subseteq a)(S c=(b(s) \subseteq) a=a \subseteq(b \subseteq c)$.
Proposition 3.8. If $(X ; *, 1)$ is a self-distributive and commutative $c$-BE-algebra and $X \neq\{1\}$, then $(X$; (S) has no group structure.

Proof. Let $1 \neq a \in X$. Hence we have $a \leq 1$. If there exists an element $b \in X$ such that $a(S) b=b(S) a=1$, then since $1=a(S) b \leq a \leq 1$, we have $a=1$.This is a contradiction.

Proposition 3.9. Let $(X ; *, 1)$ be a self-distributive and commutative $c-B E$ algebra. Then $a \leq b$ implies $a(S b=a$.

Proof. (i) Let $a \leq b$. Hence we have $a * b=1$. Then we get

$$
\begin{aligned}
a *(a \Im b) & =1 *(a *(a(b)) \\
& =(a * b) *(a *(a \Im b)) \\
& =a *(b *(a \Im b)), \quad \text { by self-distributivity property } \\
& =1
\end{aligned}
$$

since $a(s b \in A(a, b)$. Hence $a \leq b *(a(S b)$. Then we have $a \leq a(b)$. Also we know that $a(s b \leq a$. Hence we obtain $a(S b=a$ by Proposition 2.6.

Now, in a c-BE-algebra, define the set

$$
\begin{equation*}
B(a, b)=\{x \in X: x(a \leq b\} \tag{3.8}
\end{equation*}
$$

Theorem 3.10. Let $(X ; *, 1)$ is a self-distributive $c-B E$-algebra. Then the set $B(a, b)$ in equation (3.8) has the greatest element and it is $a * b$.

Proof. Since $a * b \leq a * b$, we have $b \in A(a * b, a)$. Hence we get $(a * b)$ (S) $a \leq b$. So, it is seen that $a * b \in B(a, b)$. If $c \in B(a, b)$, we write $c(a \leq b$. By Proposition 2.4 (i), we have $a *(c(\mathbb{S} a) \leq a * b$. Since $c(\mathbb{S}) a \in A(c, a)$, we have $c \leq a *(c(\mathbb{S}) a)$. Then we obtain $c \leq a * b$, by Proposition 3.5. Hence $a * b$ is the greatest element of $B(a, b)$.

Proposition 3.11. Let $(X ; *, 1)$ be a self-distributive and commutative $c-B E-$ algebra. Then
(i) $a(S b \leq a * b \leq(a(S) c) *(c(S b)$,
(ii) $(a * b)$ (S $a=a$ (Sb $b$,
(iii) $(a(\mathfrak{S} b) * c=a *(b * c)$,
(iv) $a *(b(S) c)=(a * b)(\mathbb{S}(a * c)$,
(v) $a(5) b$ is the greatest lower bound of the set $\{a, b\}$.

Proof. (i) Using Proposition 3.4 (iv) and Proposition 3.6 (i), we have $c(\mathbb{S}(a(S)(a *$ $b)) \leq c(\mathbb{S} b$. We get $(c(\mathbb{S} a)(\mathbb{S}(a * b)) \leq c(S b$ or by Proposition 3.4 (iii), ( $a *$ $b)(\mathbb{S}(c(S) a) \leq c(S b$. Hence since $a * b \in B(c(S a, c(S b)$, we obtain $a * b \leq(a(s) c) *$ $(c(\mathfrak{S}) b)$. Also it is known that $a(\mathfrak{S} b \leq b \leq a * b$. By Proposition 3.5, we get $a(\mathbb{S} b \leq a * b \leq(a(S) c) *(c(S) b)$.
(ii) Since $a * b \in B(a, b)$, we have $(a * b) \subseteq a \leq b$. Using Proposition 3.4 (i), commutativity and associativity of the operation (S), we get $(a * b)$ (S $(a(S) a) \leq a(S b$. By Proposition 3.9, we see that $a(\mathbb{S}) a=a$ since $a \leq a$. Hence $(a * b)(\mathbb{S}) a \leq a(S) b$.

Secondly, since $b \leq a * b$, by commutativity of the operation (S) and Proposition 3.6 (i), we have $a(\mathfrak{S} b \leq(a * b)(\subseteq) a$. So we obtain $(a * b)(\mathbb{S} a=a(S b b$ by Proposition 2.6 .
(iii) $a(\Im b \in A(a, b)$ implies $a \leq b *(a(s)$. Also from Proposition 2.4 (iii), we have $b *(a @ b) \leq((a @ b) * c) *(b * c)$. So we get $a \leq((a \Im b) * c) *(b * c)$ by Proposition 3.5. Then we have $(a(S)) * c \leq a *(b * c))$. Secondly, using Proposition 2.2 (ii), Proposition 2.4 (iii) and (BE4), since

$$
\begin{aligned}
b & \leq(b * c) * c \\
& \leq(a *(b * c)) *(a * c) \\
& =a *((a *(b * c)) * c),
\end{aligned}
$$

we have $b \leq a *((a *(b * c)) * c)$ or $a \leq b *((a *(b * c)) * c)$. Then we obtain $a(b \leq(a *(b * c)) * c$ or $a *(b * c) \leq(a(S b) * c$. Consequently we see that $a *(b * c)=(a(\mathfrak{S} b) * c$.
(iv) By (i), we have $a * c \leq(a(s) *(b(\subseteq) c)$ or $a(S b \leq(a * c) *(b \Im c)$. Then we get $a *(a(b) \leq a *((a * c) *(b(c))$ by Proposition 2.4 (i). We can write $a * b \leq$ $(a(S) a) *(a(S b) \leq(a * c) *(a *(b(S)))$ by (i). Hence since $a *(b(S) c) \in A(a * b, a * c)$, we have

$$
\begin{equation*}
(a * b)((a * c) \leq a *(b \Im c) . \tag{3.9}
\end{equation*}
$$

Secondly, since $b(\subseteq) c \leq b$, we have $a *(b(c) \leq a * b$ by Proposition 2.4 (i). Hence we get

$$
\begin{equation*}
(a *(b \Im c))(S(a * c) \leq(a * b) \text { S( }(a * c) \tag{3.10}
\end{equation*}
$$

Also since $b(S) c \leq c$, we have $a *(b(\mathbb{S} c) \leq a * c$ and so we get $(a *(b(S) c))(S)(a *$ $(b(S c)) \leq(a *(b \subseteq c))(S(a * c)$, that is

$$
\begin{equation*}
a *(b(\Im) c) \leq(a *(b(S) c))(S)(a * c) . \tag{3.11}
\end{equation*}
$$

Hence from the equation (3.10) and (3.11) and by Proposition 3.5, we obtain

$$
\begin{equation*}
a *(b(\mathrm{~S}) c) \leq(a *(b(\mathrm{~S} c))(\mathbb{S}(a * c) . \tag{3.12}
\end{equation*}
$$

The equations (3.9) and (3.12) show that $a *(b(S c)=(a *(b(S)))(S)(a * c)$ by Proposition 2.6.
(v) Since $a(\mathfrak{S} b \leq a$ and $a(\mathfrak{S} b \leq b, a(\mathfrak{S} b$ is the lower bound of the set $\{a, b\}$. Let $c$ be another lower bound of the set $\{a, b\}$. Then we know that $c * a=1$ and $c * b=1$. So since $c *(a(\mathrm{~S}) b)=(c * a)(\mathbb{S}(c * b)=1(\mathrm{~S} 1=1$, we have $c \leq a(\mathrm{~S} b$.

Remark 3.1. Let $(X ; *, 1)$ be a BE-algebra. In [21], the binary operation " + " on $X$ was defined as the following: for any $a, b \in X$,

$$
a+b=(a * b) * b .
$$

Also the author proved that if $(X ; *, 1)$ is a commutative BE-algebra, then $(X ;+)$ is a semilattice. By Proposition 3.11 (v), we proved that a self-distributive and commutative c-BE-algebra $X$ is a semilattice under the operation "(S)". In a selfdistributive and commutative c-BE-algebra, since $a \leq a+b$ by Proposition 2.2 (ii) and using Proposition 3.9, we see that $a((a+b)=a$. Also, since $a \leq b$ implies $a+b=b$ and since $a(s b \leq a$, we have $(a(s)+a=a$. Therefore any selfdistributive and commutative c - BE -algebra is a lattice with respect to operations "(5)" and "+".

Now we provide characterizations of ideals in a self-distributive c-BE-algebra.
Corollary 3.12 [2]. Let $(X ; *, 1)$ be a self-distributive BE-algebra. A nonempty subset $I$ of $X$ is an ideal of $X$ if and only if $A(u, v) \subseteq I$ for all $u, v \in I$.

Theorem 3.13. Let $(X ; *, 1)$ be a self-distributive c-BE-algebra. A nonempty subset $I$ of $X$ is an ideal of $X$ if and only if it satisfies the following conditions:
(i) $\forall a \in I, \forall x \in X, a \leq x \Longrightarrow x \in I$,
(ii) $\forall a, b \in I, \exists c \in I, c \leq a$ and $c \leq b$.

Proof. Let $I$ be an ideal of $X$. (i) follows from the Corollary 2.8. Let $a, b \in I$. From Corollary 3.12, we have $A(a, b) \subseteq I$. Then we get $a(\mathbb{S} b \in I$. If we take $a \mathrm{~S} b=c$, then we have $c \leq a$ and $c \leq b$ by Proposition 3.4 (i) which proves (ii). Conversely, let $I$ be a non-empty subset of $X$ satisfying (i) and (ii). Since for $a \in I, a \leq 1$ by (BE2), we have $1 \in I$ by (i). For any $a, b, c \in X$, let $b \in I$ and $a *(b * c) \in I$. By (ii), there exists $d \in I$ such that $d \leq b$ and $d \leq a *(b * c)$. Then using (BE3), (BE4), and self-distributivity, we have

$$
1=d *(a *(b * c))=d *(b *(a * c))=(d * b) *(d *(a * c))=d *(a * c) .
$$

Hence, we get $d \leq a * c$. By (i), it is obtained $a * c \in I$. So $I$ is an ideal of $X$ by Corollary 2.9.

Theorem 3.14. Let $(X ; *, 1)$ be a self-distributive c-BE-algebra. A non-empty subset $I$ of $X$ is an ideal of $X$ if and only if it satisfies the following conditions:
(i) $\forall a \in I, \forall x \in X, a \leq x \Longrightarrow x \in I$,
(ii) $\forall a, b \in I, a(b b \in I$.

Proof. The necessity is given in the proof of Theorem 3.13. Conversely, since for $a \in I, a \leq 1$ by (BE2), we have $1 \in I$ by (i). Let $I$ be a non-empty subset of $X$ satisfying (i) and (ii). We know that $x * y \in B(x, y)$ in a self-distributive c-BE-algebra. So $(x * y)$ © $x \leq y$ and hence

$$
\begin{equation*}
x(S)(x * y) \leq y \tag{3.13}
\end{equation*}
$$

Now let $y \in I$ and $x *(y * z) \in I$. By (ii) and (BE4), we get $y(S)(x *(y * z))=$ $y(S(y *(x * z)) \in I$. From the equation 3.13 , it is clear that $y \subseteq(y *(x * z)) \leq x * z$. Hence it is obtained $x * z \in I$ by (i). Consequently, $I$ is an ideal of $X$ by Corollary 2.9.

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