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JORDAN NUMBERS, STIRLING NUMBERS AND SUMS OF POWERS

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Abstract

In the paper a new combinatorical interpretation of the Jordan numbers is presented. Binomial type formulae connecting both kinds of numbers mentioned in the title are given. The decomposition of the product of polynomial of variable n into the sums of kth powers of consecutive integers from 1 to n is also studied.

Keywords: Bernoulli numbers, binomial coefficients, Jordan numbers, Stirling numbers, Živković numbers.

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1. Introduction

This paper was inspired by the wish to generalize the elementary formulae (formulae (2)–(7) given below) generated by the authors in the course of discussion on D. Knuth's excellent publication [9] (devoted to the analysis of relations between sums of the powers of consecutive positive integers, inspired by Faulhaber's "old" and "deeper" results). It led us first to find the general connection of type (8) (known form) and (9) (probably new form). In the course of discussing these relations a very interesting and original problem has appeared, which consisted in deriving a formula of type (11), meaning a decomposition of the product $n^k \sum_{l=1}^{n} l^r$ into the sum of type $\sum_{s=1}^{k+r} a_s(k,r) \sum_{l=1}^{n} l^s$. Section 3 is devoted to this problem. Section 4 presents the general formula (26) (known formula) giving the decomposition of the Stirling numbers of second kind in the linear combination of binomial coefficients by using the Jordan numbers. Finally, in Section 5 the new combinatoric interpretation of the Jordan numbers is presented.

2. Stirling numbers

It should be reminded that Stirling numbers of the second kind or the partition numbers $S(n,k)$, $k,n \in \mathbb{N}$, $k \le n$, satisfy the triangular recurrence relation [12, 13, 14]:

$$\begin{aligned} &S(n,k) = S(n-1,k-1) + k\,S(n-1,k),\\ &S(k,k) = 1, \qquad S(k,0) = 0. \end{aligned} \tag{1}$$

Moreover, we adopt here Donald Knuth's notation [6, 9]:

$$\sum f(n) := \sum_{l=1}^{n} f(l)$$

for every function $f : \mathbb{N} \to \mathbb{C}$. For example we have

$$\sum n^k := \sum_{l=1}^{n} l^k, \qquad k,n \in \mathbb{N}.$$

We shall start our deliberations by presenting the above-mentioned basic identities which can be easily verified by direct calculations. So, the following identities hold:

$$\sum n = S(n+1,n), \tag{2}$$

$$\begin{aligned} \sum n^3 + \sum n^2 = \sum n^2(n+1) &= \frac{1}{2}(3\,n+1)\binom{n+2}{3}\\ &= \frac{1}{2}\big(3(n-1)+4\big)\binom{n+2}{3}\\ &= 6\binom{n+2}{4} + 2\binom{n+2}{3} = 2\,S(n+2,n), \end{aligned} \tag{3}$$

$$\begin{aligned} 3\sum n^5 + 10\sum n^4 + 9\sum n^3 + 2\sum n^2 &= \sum n^2(n+1)(n+2)(3n+1)\\ &= 24\binom{n+3}{4}\binom{n+1}{2} = 24\,S(n+3,n), \end{aligned} \tag{4}$$

$$S(n+3,n) = 15\binom{n+3}{6} + 10\binom{n+3}{5} + \binom{n+3}{4}, \tag{5}$$

$$\sum n^7 + 7\sum n^6 + 17\sum n^5 + 17\sum n^4 + 6\sum n^3 = \sum n^3(n+1)^2(n+2)(n+3)$$

$$= 48\, S(n+4,n) = \left(15\,n^3 + 30\,n^2 + 5\,n - 2\right)\binom{n+4}{5}, \tag{6}$$

$$S(n+4,n) = 105\binom{n+4}{8} + 105\binom{n+4}{7} + 25\binom{n+4}{6} + \binom{n+4}{5}. \tag{7}$$

Stirling numbers of the second kind are presented in equations (3), (5) and (7) which is not surprising because the following inversion formulae hold.

Proposition 1. *We have*

$$\sum n^r = \sum_{l=0}^{r} S(r,l)\, l!\binom{n+1}{l+1} \tag{8}$$

$$= \sum_{l=1}^{r+1} S(r+1,l)\,(l-1)!\binom{n}{l}. \tag{9}$$

Proof. Formula (8) is drawn from [9], whereas the second formula seems to be new, yet it may be easily derived from the first one:

$$\sum_{l=0}^{r} S(r,l)\, l!\binom{n+1}{l+1} = \sum_{l=0}^{r} S(r,l)\, l!\left(\binom{n}{l+1} + \binom{n}{l}\right)$$

$$= S(r,0) + \sum_{l=0}^{r-1}\binom{n}{l+1}\Big(S(r,l)\, l! + S(r,l+1)\,(l+1)!\Big) + S(r,r)\, r!\binom{n}{r+1}$$

$$= \sum_{l=0}^{r-1}\binom{n}{l+1} l!\Big(S(r,l) + S(r,l+1)\,(l+1)\Big) + S(r+1,r+1)\, r!\binom{n}{r+1}$$

$$= \sum_{l=1}^{r+1}\binom{n}{l}(l-1)!\, S(r+1,l).$$

∎

3. DECOMPOSITION OF PRODUCTS $n^k \sum n^m$

We note that from (9) we obtain

$$(n+1)\sum n^r = \sum_{l=1}^{r+1} S(r+1,l)\,(l-1)!\,(l+1)\binom{n+1}{l+1}$$

$$\overset{\text{by (8)}}{=} \sum n^{r+1} + \sum_{l=1}^{r+1} S(r+1,l)\,(l-1)!\binom{n+1}{l+1}, \tag{10}$$

which, again by (8), suggests a formula of following form

$$n^k \sum n^r = \sum_{s=1}^{k+r} a_s(k,r) \sum n^{k+r+1-s}. \tag{11}$$

Since we have (see [6]):

$$(r+1) \sum n^r = \sum_{l=0}^{r} (-1)^l \binom{r+1}{l} B_l \, n^{r+1-l} \qquad \Longleftrightarrow$$

$$n^{r+1} = (r+1) \sum n^r + (r+1)\, B_1\, n^r - \sum_{k=1}^{\lfloor r/2 \rfloor} \binom{r+1}{2\,k} B_{2k}\, n^{r+1-2k}, \tag{12}$$

where B_l are the Bernoulli's numbers, we obtain indeed formulae of the type (11). From (12) the following formulae can be generated.

Theorem 2. *We have*

$$(r+1)\, n \sum n^r = (r+2) \sum n^{r+1} + \sum_{s=1}^{r} \binom{r+1}{s} B_s \sum n^{r-s+1}, \tag{13}$$

(14)

$$\begin{aligned}
&(r+1)\, n^2 \sum n^r = (r+2)\, n \sum n^{r+1} + \sum_{s=1}^{r} \binom{r+1}{s} B_s\, n \sum n^{r-s+1} \\
&= (r+3) \sum n^{r+2} + \sum_{p=1}^{r+1} \binom{r+2}{p} B_p \sum n^{r-p+2} \\
&+ \sum_{s=1}^{r} \binom{r+1}{s} \frac{B_s}{r-s+2} \Bigg((r-s+3) \sum n^{r-s+2} \\
&+ \sum_{q=1}^{r-s+1} \binom{r-s+2}{q} B_q \sum n^{r-s-q+2} \Bigg) \\
&= (r+3) \sum n^{r+2} + \sum_{p=1}^{r} \frac{2\,r-p+5}{r+2} \binom{r+2}{p} B_p \sum n^{r-p+2} \\
&+ (r+2)\, B_{r+1} \sum n + \sum_{s=1}^{r} \sum_{q=1}^{r-s+1} \binom{r+1}{s} \binom{r-s+2}{q} \frac{B_s\, B_q}{r-s+2} \sum n^{r-s-q+2}.
\end{aligned}$$

Hence, after the appropriate regrouping, we get

(15)

$$(r+1)\,n^2\sum n^r = (r+3)\sum n^{r+2} - (r+2)\sum n^{r+1} + \frac{1}{6}(r+1)(r+3)\sum n^r$$
$$-\frac{1}{6}\binom{r+1}{2}\sum n^{r-1} - \frac{1}{15}\binom{r+1}{4}\sum n^{r-2} + \frac{1}{30}\binom{r+1}{4}\sum n^{r-3}$$
$$+\frac{1}{21}\binom{r+1}{6}\sum n^{r-4} + \ldots$$
$$+\frac{1}{2}\left[(r+5)(r+1)B_r + \sum_{s=1}^{r-1}\binom{r+1}{s}(r-s+1)B_sB_{r-s}\right]\sum n^2$$
$$+\left((r+2)B_{r+1} + \sum_{s=1}^{r}\binom{r+1}{s}B_sB_{r-s+1}\right)\sum n.$$

The last two coefficients can be reduced to the following ones (the classical splitting formulae are applied here [1, 5, 7, 10, 11]):

$$\frac{1}{2}\left[(r+5)(r+1)B_r - \left(2\left\lfloor\frac{r}{2}\right\rfloor+1\right)(r+1)B_{2\lfloor\frac{r}{2}\rfloor}\right]$$

since we have the equality

$$\sum_{s=1}^{r-1}\binom{r+1}{s}(r-s+1)B_sB_{r-s} = -\left(2\left\lfloor\frac{r}{2}\right\rfloor+1\right)(r+1)B_{2\lfloor\frac{r}{2}\rfloor}$$

for every $r = 3, 4, \ldots$ (we have $\left(2\left\lfloor\frac{r}{2}\right\rfloor+1\right)(r+1)B_{2\lfloor\frac{r}{2}\rfloor}$ for $r = 2$), and

$$(r+2)B_{r+1} + \sum_{s=1}^{r}\binom{r+1}{s}B_sB_{r-s+1} = -(r+1)B_r.$$

From these relations, we obtain

$$\sum_{s=1}^{r-1}\binom{r+1}{s}(r-s+1)B_sB_{r-s} = (r+1)\sum_{s=1}^{r-1}\binom{r}{s}B_sB_{r-s}$$

if r is odd and $r \geq 3$.

Proof. We prove only (13). To this aim let us fix $r \in \mathbb{N}$ and put

(16) $$T_{k,r} := (r+1)\,k^r - (k+1)^{r+1} + k^{r+1}.$$

We have

$$
\begin{aligned}
T_{k,r} &= (r+1)\,k^r + k^{r+1} - \sum_{i=0}^{r+1}\binom{r+1}{i}k^i = -\sum_{i=0}^{r-1}\binom{r+1}{i}k^i,\\
\sum_{k=1}^{n} T_{k,r} &= (r+1)\sum n^r + \sum_{k=1}^{n}\left(k^{r+1}-(k+1)^{r+1}\right)\\
&= (r+1)\sum n^r + 1 - (n+1)^{r+1}.
\end{aligned}
\tag{17}
$$

On the other hand

$$
\sum_{k=1}^{n} T_{k,r} = -\sum_{i=0}^{r-1}\binom{r+1}{i}\sum n^i. \tag{18}
$$

We note that (13) holds for $n=1$ since it is equivalent to the following known relation

$$
B_{r+1} = \sum_{s=0}^{r+1}\binom{r+1}{s}B_s.
$$

Let us assume that (13) holds for some $n \in \mathbb{N}$. Then we get

$$
\begin{aligned}
&(r+1)n\sum n^r - (r+2)\sum n^{r+1} - \sum_{i=0}^{r-1}\binom{r+1}{i}\sum n^i\\
&\qquad = \sum_{s=1}^{r}\binom{r+1}{s}B_s\sum n^{r-s+1} - \sum_{i=0}^{r-1}\binom{r+1}{i}\sum n^i\\
&\qquad = \sum_{s=1}^{r}\binom{r+1}{s}B_s\sum (n+1)^{r-s+1}\\
&\qquad - \sum_{s=1}^{r}\binom{r+1}{s}B_s(n+1)^{r-s+1} - \sum_{i=0}^{r-1}\binom{r+1}{i}\sum n^i,
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
&-\sum_{s=1}^{r}\binom{r+1}{s}B_s(n+1)^{r-s+1}\\
&= (n+1)^{r+1} - 2(r+1)B_1(n+1)^r - \sum_{s=0}^{r}(-1)^s\binom{r+1}{s}B_s(n+1)^{r-s+1}\\
&\overset{(12)}{=} (n+1)^{r+1} + (r+1)(n+1)^r - (r+1)\sum (n+1)^r\\
&= (n+1)^{r+1} - (r+1)\sum n^r,
\end{aligned}
$$

which implies

$$(20)\qquad \begin{aligned} &-\sum_{s=1}^{r}\binom{r+1}{s}B_s(n+1)^{r-s+1}-\sum_{i=0}^{r-1}\binom{r+1}{i}\sum n^i \\ &= (n+1)^{r+1}-\sum_{i=0}^{r}\binom{r+1}{i}\sum n^i = 1. \end{aligned}$$

We note that the last identity can be easily deduced after summing the following equalities

$$\begin{aligned} (n+1)^{r+1} &= n^{r+1}+\sum_{i=0}^{r}\binom{r+1}{i}n^i, \\ n^{r+1} &= (n-1)^{r+1}+\sum_{i=0}^{r}\binom{r+1}{i}(n-1)^i, \\ &\dots \\ 2^{r+1} &= 1^{r+1}+\sum_{i=0}^{r}\binom{r+1}{i}1^i. \end{aligned}$$

Moreover, by (17) and (18), we obtain

$$(21)\qquad \begin{aligned} &(r+1)n\sum n^r-(r+2)\sum n^{r+1}-\sum_{i=0}^{r-1}\binom{r+1}{i}\sum n^i \\ &= (r+1)n\sum n^r-(r+2)\sum n^{r+1}+(r+1)\sum n^r+1-(n+1)^{r+1} \\ &= 1+(r+1)(n+1)\sum(n+1)^r-(r+2)\sum(n+1)^{r+1}. \end{aligned}$$

By comparing (19)–(21) we conclude that relation (16) holds also for n replaced by $n+1$ which, by the principle of mathematical induction, ends the proof. ∎

Corollary 3. *From* (15) *the following special formulae can be deduced*

$$\begin{aligned} 2n^2\sum n &= 4\sum n^3-3\sum n^2+\sum n, \\ 3n^2\sum n^2 &= 5\sum n^4-4\sum n^2+\frac{5}{2}\sum n^2-\frac{1}{2}\sum n, \\ 4n^2\sum n^3 &= 6\sum n^5-5\sum n^4+4\sum n^3-\sum n^2. \end{aligned}$$

Some more general formula, than the one in (15), can be also obtained.

Proposition 4. *We have*

$$(22)\qquad (m+1)\,n^k \sum n^m = (m+k+1)\sum n^{m+k} - \frac{k}{2}\,(m+k)\sum n^{m+k-1}$$
$$+\frac{k}{12}\,(m+2\,k-1)\,(m+k-1)\sum n^{m+k-2}$$
$$-\frac{k}{24}\,(k-1)\,(m+k-1)\,(m+k-2)\sum n^{m+k-3}$$
$$+\frac{1}{720}\,(m+k-3)\,k\,(m+2\,k-3)\,\big(3\,k^2+3\,k\,(m-3)-m\,(m+4)+6\big)\sum n^{m+k-4}$$
$$-\frac{1}{360}\binom{m+k-3}{2}\binom{k}{2}\big(2\,k^2+2\,k\,(m-4)-m\,(m+5)+8\big)\sum n^{m+k-5}+\dots$$
$$+\operatorname{coeff}(m,k)\sum n.$$

We have found the general formula only for the first six coefficients from above. The last absent coefficient $\operatorname{coeff}(m,k)$ *can be found by subtracting from the left side of* (22) *the expression staying on the right side of this formula with sums* $\sum n^p$ *where* $p \geq 2$.

For example, we find

$$2n\sum n = 3\sum n^2 - \sum n,$$
$$2n^3\sum n = 5\sum n^4 - 6\sum n^3 + \frac{9}{2}\sum n^2 - \frac{7}{2}\sum n,$$
$$4n\sum n^3 = 5\sum n^4 - 2\sum n^3 + \sum n^2,$$
$$3n^4\sum n^2 = \frac{1}{2}n^5(n+1)(2n+1)$$
$$= 7\sum n^6 - 12\sum n^5 + 15\sum n^4 - 10\sum n^3 + \frac{7}{2}\sum n^2 - \frac{1}{2}\sum n,$$

and the following special one

$$(r+1)n^3\sum n^r = (r+4)\sum n^{r+3} - \frac{3}{2}(r+3)\sum n^{r+2} + \frac{1}{4}(r+2)(r+5)\sum n^{r+1}$$
$$-\frac{1}{4}(r+1)(r+2)\sum n^r - \frac{1}{240}(r-6)r(r+1)(r+3)\sum n^{r-1}$$
$$+\frac{1}{240}(r-2)(r-1)r(r+1)\sum n^{r-2} + \dots$$

4. Jordan numbers

In this section, we present the generalization of relations (3), (5) and (7).

Proposition 5. *We have*

$$S(n+k,n)=\sum_{l=0}^{k-1} a_{l,k}\binom{n+k}{2\,k-l}, \tag{23}$$

where

$$\begin{aligned} a_{0,k+1}&=(2k+1)a_{0,k}, \qquad a_{k,k+1}=a_{k-1,k},\\ a_{l,k+1}&=(k-l+1)\,a_{l-1,k}+(2\,k-l+1)\,a_{l,k}, \end{aligned} \tag{24}$$

for $l=1,2,\ldots,k-1$. The numbers $a_{l,k}$ are called the Jordan numbers [2, 3, 4, 8, 15]. *Authors of the present paper obtained the above relations independently. For the sake of selfcontainedness, the proof of* (23) *will be given now. We proceed by induction.*

Proof. (23) We shall employ the following basic formula

$$\begin{aligned} \sum_{k=m-l}^{n} k\binom{k+l}{m} &= \sum_{k=m-l}^{n}\big((k+l-m)+(m-l)\big)\binom{k+l}{m}\\ &= (m+1)\sum_{k=m-l+1}^{n}\binom{k+l}{m+1}+(m-l)\sum_{k=m-l}^{n}\binom{k+l}{m}\\ &= (m+1)\binom{n+l+1}{m+2}+(m-l)\binom{n+l+1}{m+1}. \end{aligned} \tag{25}$$

Thus, assuming that formula (23) holds for certain $k,n\in\mathbb{N}$, from (25) and (1) the following relation can be derived

$$\begin{aligned} &S(n+k+1,n)=S(n+k+1,n)-S(k+2,0)\\ &=\sum_{l=0}^{n-1}\big(S(l+k+2,l+1)-S(l+k+1,l)\big)=\sum_{l=0}^{n-1}(l+1)\,S(l+k+1,l+1)\\ &=\sum_{l=0}^{n-1}(l+1)\sum_{\tau=0}^{k-1}a_{\tau,k}\binom{l+k+1}{2\,k-\tau}=\sum_{\tau=0}^{k-1}a_{\tau,k}\sum_{l=k-1-\tau}^{n-1}(l+1)\binom{l+k+1}{2\,k-\tau}\\ &=\sum_{\tau=0}^{k-1}a_{\tau,k}\left((2\,k-\tau+1)\binom{n+k+1}{2\,k-\tau+2}+(k-\tau)\binom{n+k+1}{2\,k-\tau+1}\right)\\ &=\ a_{0,k}\,(2\,k+1)\binom{n{+}k{+}1}{2k{+}2}+\sum_{\tau=0}^{k-2}\big(a_{\tau,k}\,(k-\tau)+a_{\tau+1,k}\,(2\,k-\tau)\big)\binom{n{+}k{+}1}{2k{-}\tau{+}1}\\ &+a_{k-1,k}\binom{n+k+1}{k+2}=\sum_{l=0}^{k}a_{l,k+1}\binom{n+k+1}{2\,k-l+2}. \end{aligned}$$

■

Remark 6. It is possible to prove the following formulae:

$$a_{0,k} = (2\,k-1)!!, \qquad a_{1,k} = \frac{k-1}{3}\, a_{0,k},$$
$$a_{2,k} = \frac{1}{12}\binom{2\,k-2}{3}(2\,k-3)!!, \qquad a_{3,k} = S2(k, 2\,k+3),$$
$$a_{k-2,k} = 2^{k+1} - k - 3 \qquad a_{k-1,k} = 1,$$

where $S2(n,k)$ denotes the 2-associated Stirling number of the second kind (see [12] and sequences A000478 and A000247 in [14]).

In Table 1 the triangle of coefficients $a_{l,k}$, $l = 0, 1, \ldots, k-1$ is presented.

Table 1. Triangle of coefficients $a_{l,k}$

l,k	1	2	3	4	5	6	7	8
0	1	3	15	105	945	10395	135135	2027025
1		1	10	105	1260	17325	270270	4729725
2			1	25	490	9450	190575	4099095
3				1	56	1918	56980	1636635
4					1	119	6825	302995
5						1	246	22935
6							1	501
7								1

Remark 7. Another formula connecting Stirling numbers of the second kind with binomial coefficients is also known

$$S(n, n-k) = \sum_r \left\langle\!\!\left\langle {k \atop r} \right\rangle\!\!\right\rangle \binom{n+k-1-r}{2k},$$

where $\left\langle\!\!\left\langle {k \atop r} \right\rangle\!\!\right\rangle$ are the Eulerian numbers of the second kind (see [6]).

5. Combinatoric interpretation of the Jordan and Živković numbers

As it can be inferred from [12, pp. 76–77], the numbers $a_{n-2k-1,n-k}$, for $k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$, enumerate the permutations of n elements with k cycles, none of which is a unit cycle. We note that in [12] the numbers

$$b(n,k) := a_{n-2k-1,n-k}$$

are called the associated Stirling number of the second kind. Moreover, we have the following recurrence relation

$$b(n+1,k) = k\,b(n,k) + n\,b(n-1,k-1).$$

Conversely, in [15] the Živković numbers $G(k,i)$ are defined by the following equality

$$s(n,n-k) = \sum_{i=0}^{k} (-1)^i\, G(k,i) \binom{n+i-1}{k+i},$$

where $G(k,i) = 0$ if $k \geq 1$ and $i > k$ or $i < 1$ ($G(0,i) = \delta_{0,i}$), and $s(n,k)$ denotes the Stirling number of the first kind.

Numbers $G(k,i)$ are determined by the initial condition $G(1,i) = \delta_{1,i}$ and by the recurrence relation

$$G(k+1,i) = i\,G(k,i) + (k+i)\,G(k,i-1). \tag{26}$$

It is easy to check that

$$G(r-l,r) = a_{l,r} \iff G(k,k+l) = a_{l,k+l} \quad (r-l := k). \tag{27}$$

Hence, as it is verifiable on the grounds of relation (26), $a_{l,k+l}$ denotes the number of ways of placing $2k+l$ labeled balls into k indistinguishable boxes with at least two balls in each box.

The values of numbers $G(k,i)$ are compiled in Table 2 (we note that $G(k,i)$, $k = i, i+1, \ldots$ make up i-th diagonal – Table 1 – for each $i \in \mathbb{N}$; the diagonal is from the main line upwards).

Table 2. Values of numbers $G(k,i)$

$G(1,k)$	$G(2,k)$	$G(3,k)$	$G(4,k)$	$G(5,k)$	$G(6,k)$
1					
1	3				
1	10	15			
1	25	105	105		
1	56	490	1260	945	
1	119	1918	9450	17325	10395

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