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ON THE INTERSECTION GRAPHS OF IDEALS OF DIRECT PRODUCT OF RINGS

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Abstract

In this paper we first calculate the number of vertices and edges of the intersection graph of ideals of direct product of rings and fields. Then we study Eulerianity and Hamiltonicity in the intersection graph of ideals of direct product of commutative rings.

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1. Introduction

For graph theory terminology in general we follow [6]. Specifically, let $G = (V, E)$ be an (undirected) graph with vertex set V and edge set E. The *size* of G is $|E|$, the number of edges of G. A *Hamiltonian graph* is a graph with a spanning cycle, called a *Hamiltonian cycle*. A graph is *Eulerian* if it has a closed trail containing all edges. Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ is the one-dimensional skeleton of the nerves of F , i.e., $G(F)$ is the graph whose vertices are S_i, $i \in I$, and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$. It is shown that every simple graph (without loops and multiple edges) is an intersection graph [5].

The study of algebraic structures using properties of graphs has become an exciting research topic in the last few decades, leading to many fascinating results and questions. It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. Several mathematicians studied such graphs on various algebraic structures. Recently Chakrabarty et al. [2] studied intersection graphs of ideals of rings. The *intersection graph of ideals of a ring* R, denoted by $G(R)$, is the undirected simple graph whose vertices are in a one-to-one correspondence with all nontrivial (nonzero, proper) ideals of R and two distinct vertices are joined by an edge if and only if the corresponding ideals of R have a nonzero intersection. For references on graphs related to the ring structures see for example [2, 3, 4, 7].

In this paper, we first calculate the number of vertices and the number of edges of the intersection graph of ideals of rings and fields. Then we study Eulerianity and Hamiltonicity in the intersection graph of ideals of direct product of commutative rings. For a ring R we denote the number of edges of $G(R)$ by $e(G(R))$. For ring theory terminology in general we follow [1].

2. Order and size

Theorem 1. *Let R_1 and R_2 be two rings with identity. If R_i has t_i ideals for $i = 1, 2$, then $|V(G(R_1 \times R_2))| = t_1t_2 - 2$ and*

$$(1) \qquad e(G(R_1 \times R_2)) = \binom{t_1t_2 - 2}{2} - 2(x + a)(y + b) - x - y + ab,$$

where $x = \binom{t_1 - 2}{2} - e(G(R_1))$, $y = \binom{t_2 - 2}{2} - e(G(R_2))$, $a = t_1 - 1$, $b = t_2 - 1$.

Proof. Let $R = R_1 \times R_2$. We show that $e(\overline{G(R)}) = 2(x + a)(y + b) + x + y - ab$, where $\overline{G(R)}$ is the graph complement of $G(R)$ and hence the result follows.

Let I be a nontrivial ideal of R. Then $I = I_1 \times I_2 \notin \{\{(0_{R_1}, 0_{R_2})\}, R_1 \times R_2\}$. Let A be the set of ideals I of R such that $I = I_1 \times I_2$, where $I_1 \notin \{\{0_{R_1}\}, R_1\}$ and $I_2 \notin \{\{0_{R_2}\}, R_2\}$. Let B be the set of all other nontrivial ideals of R. Then any ideal of R in B is one of the following forms:

$$\{0_{R_1}\} \times R_2, R_1 \times \{0_{R_2}\}, \{0_{R_1}\} \times I_2, I_1 \times \{0_{R_2}\}, R_1 \times I_2, I_1 \times R_2$$

where I_1 and I_2 are nontrivial ideals of R_1 and R_2, respectively.

Let $I = I_1 \times I_2$ and $J = I_3 \times I_4$ be two nontrivial ideals of R such that $I \cap J = \{(0_{R_1}, 0_{R_2})\}$. Then $I_1 \cap I_3 = \{0_{R_1}\}$ and $I_2 \cap I_4 = \{0_{R_2}\}$. So the number of edges of $\overline{G(R)}$ with both end points corresponding to ideals in A is given by

$$2\, e(\overline{G(R_1)})\, e(\overline{G(R_2)}) \;=\; 2xy,$$

where x and y are defined as above.

Now the vertices corresponding to the ideals $\{0_{R_1}\} \times R_2, R_1 \times \{0_{R_2}\}, R_1 \times I_2, I_1 \times R_2$ (for all nontrivial ideals I_1 and I_2 of R_1 and R_2 respectively) are adjacent to any vertex corresponding to an ideal in A in the graph $G(R)$. Thus there are no such edges in $\overline{G(R)}$.

Next we note that for each pair of nontrivial ideals I_2, I_4 of R_2 with $I_2 \cap I_4 = \{0_{R_2}\}$ and for all nontrivial ideals I_1, I_3 of R_1, we have

$$\{0_{R_1}\} \times I_2 \cap I_3 \times I_4 = \{(0_{R_1}, 0_{R_2})\} = I_1 \times I_2 \cap \{0_{R_1}\} \times I_4.$$

So the number of edges of $\overline{G(R)}$ with one end point corresponding to an ideal in A and the other end point corresponding to an ideal in B is given by

$$2(t_1 - 2)e(\overline{G(R_2)}) + 2(t_2 - 2)e(\overline{G(R_1)}) = 2(a-1)y + 2(b-1)x,$$

where a and b are defined as above.

Finally we calculate the number of edges of $\overline{G(R)}$ with both end points corresponding to ideals in B. We denote the degree of the vertex corresponding to an ideal T in a graph X by $d_X(T)$.

Let M be the induced subgraph of $\overline{G(R)}$ generated by the set of vertices corresponding to ideals in B. Then the following computations are straightforward:

Fact 1. $d_M(\{0_{R_1}\} \times R_2) = 1 + (t_1 - 2) = t_1 - 1.$

Note that since the vertex corresponding to $\{0_{R_1}\} \times R_2$ is adjacent to the vertex corresponding to $R_1 \times \{0_{R_2}\}$ and all the vertices corresponding to $I_1 \times \{0_{R_2}\}$ (for all nontrivial ideals I_1 of R_1) in M.

Fact 2. $d_M(R_1 \times \{0_{R_2}\}) = t_2 - 1.$

Fact 3. $d_M(\{0_{R_1}\} \times I_2) = 1 + (t_1 - 2) + d_{\overline{G(R_2)}}(I_2) + d_{\overline{G(R_2)}}(I_2) = t_1 - 1 + 2d_{\overline{G(R_2)}}(I_2)$.

Note that since the vertex corresponding to $\{0_{R_1}\} \times I_2$ is adjacent to the vertex corresponding to $R_1 \times \{0_{R_2}\}$, all the vertices corresponding to $I_1 \times \{0_{R_2}\}$ (for all nontrivial ideals I_1 of R_1) and all the vertices corresponding to ideals of the forms $R_1 \times I_4$ and $\{0_{R_1}\} \times I_4$ (for all nontrivial ideals I_4 of R_2 such that $I_2 \cap I_4 = \{0_{R_2}\}$) in M.

Fact 4. $d_M(I_1 \times \{0_{R_2}\}) = t_2 - 1 + 2d_{\overline{G(R_1)}}(I_1)$.

Fact 5. $d_M(R_1 \times I_2) = d_{\overline{G(R_2)}}(I_2)$.

Fact 6. $d_M(I_1 \times R_2) = d_{\overline{G(R_1)}}(I_1)$.

Thus the total degree of vertices of M is

$$(t_1 + t_2 - 2) + (t_2 - 2)(t_1 - 1) + (t_1 - 2)(t_2 - 1) + 3(2x + 2y) = 2(ab + 3x + 3y),$$

as the sum of degrees of all nontrivial ideals of R_1 (respectively, R_2) in the graph $\overline{G(R_1)}$ (respectively, $\overline{G(R_2)}$) is twice the number of edges of $\overline{G(R_1)}$ (respectively, $\overline{G(R_2)}$). Therefore the number of edges of $\overline{G(R)}$ with both end points corresponding to ideals in B is given by

$$ab + 3x + 3y.$$

Hence the total number of edges of $\overline{G(R)}$ is

$$\begin{aligned} 2xy + 2(a-1)y + 2(b-1)x + ab + 3x + 3y &= 2xy + 2ay + 2bx + ab + x + y \\ &= 2(x+a)(y+b) + x + y - ab \end{aligned}$$

as required. ■

Corollary 2. *Let $F_1, F_2, \ldots, F_n$ be $n \geqslant 2$ fields and $F = F_1 \times F_2 \times \cdots \times F_n$. Then $|V(G(F))| = 2^n - 2$ and*

$$e(G(F)) = \frac{1}{2}(2^{2n} - 3 \cdot 2^n - 3^n + 5).$$

Proof. Since the number of ideals of F is 2^n, we have $|V(G(F))| = 2^n - 2$. We prove the result by induction on n. For $n = 2$, there are only 2 nontrivial ideals of F, namely, $F_1 \times \{0_{F_2}\}$ and $\{0_{F_1}\} \times F_2$. The vertices of $G(F)$ corresponding to these ideals are non-adjacent and so $e(G(F)) = 0 = \frac{1}{2}(2^4 - 3 \cdot 2^2 - 3^2 + 5)$.

Thus the result is true for $n = 2$. Suppose the result is true for $n = k - 1$. Let $R_1 = F_1 \times F_2 \times \cdots \times F_{k-1}$ and $R_2 = F_k$. Then $F = R_1 \times R_2$. Now by inductive hypothesis, $e(G(R_1)) = \frac{1}{2}(2^{2(k-1)} - 3 \cdot 2^{(k-1)} - 3^{(k-1)} + 5)$. Then straightforward calculations show that $x = \frac{1}{2}(3^{k-1} - 2^k + 1)$, $y = 0$, $a = 2^{k-1} - 1$, $b = 1$ and $e(G(F)) = \frac{1}{2}(2^{2k} - 3 \cdot 2^k - 3^k + 5)$, by (1), as required. ∎

3. Eulerianity and Hamiltonicity

Let $R = R_1 \times R_2$, where R_i is a commutative ring with identity, and with t_i ideals for $i = 1, 2$. By definition $G(R_i)$ has $t_i - 2$ vertices, for $i = 1, 2$. Let the ideals of R_1 be $0 = I_0, I_1, \ldots, I_{t_1-2}, I_{t_1-1} = R_1$, and let the ideals of R_2 be $0 = J_0, J_1, \ldots, J_{t_2-2}, J_{t_2-1} = R_2$. It is clear that $I \trianglelefteq R_1 \times R_2$ if and only if $I = I_1 \times I_2$, where $I_1 = \{x_1 : (x_1, x_2) \in I\}$, and $I_2 = \{x_2 : (x_1, x_2) \in I\}$.

For an ideal I of R in order to avoid repeating "the vertex in $G(R)$ corresponded with I" we henceforth simply use "the vertex I". It is well known that a connected graph is Eulerian if and only if its vertices all have even degree. We now calculate the degree of a vertex in $G(R)$. Let $I_i \times J_j \trianglelefteq R$.

If $i = 0, j \notin \{0, t_2 - 1\}$, then $deg_{G(R)}(I_i \times J_j) = t_1 - 1 + t_1 - 1 + t_1 deg_{R_2}(J_j)$. If $i = 0, j = t_2 - 1$, then $deg_{G(R)}(I_i \times J_j) = t_1(t_2 - 1) - 2$. If $i \notin \{0, t_1 - 1\}$, $j = 0$, then $deg_{G(R)}(I_i \times J_j) = t_2 - 1 + t_2 - 1 + t_2 deg_{R_1}(I_i)$. If $i = t_1 - 1$, $j = 0$, then $deg_{G(R)}(I_i \times J_j) = t_2(t_1 - 1) - 2$. If $i = t_1 - 1, j \notin \{0, t_2 - 1\}$, then $deg_{G(R)}(I_i \times J_j) = t_2(t_1 - 1) - 1 + deg_{G(R_2)}(J_j) + 1$. If $i \notin \{0, t_1 - 1\}, j = t_2 - 1$, then $deg_{G(R)}(I_i \times J_j) = t_1(t_2 - 1) - 1 + deg_{G(R_1)}(I_i) + 1$. If $\{i, j\} \cap \{0, t_2 - 1, t_1 - 1\} = \emptyset$, then

$$\begin{aligned} deg_{G(R)}(I_i \times J_j) &= t_2 - 1 + t_1 - 1 + t_1 - 2 + t_2 - 2 \\ &+ (t_2 - 2) deg_{G(R_1)}(I_i) + (t_1 - 2) deg_{G(R_2)}(J_j) \\ &- deg_{G(R_1)}(I_i) deg_{G(R_2)}(J_j). \end{aligned}$$

So the degrees of vertices of $G(R)$ are listed in the following forms:

- (1): $2(t_1 - 1) + t_1 deg_{R_2}(J_j)$, and $2(t_2 - 1) + t_2 deg_{R_1}(I_i)$.
- (2): $t_1(t_2 - 1) - 2$, and $t_2(t_1 - 1) - 2$.
- (3): $t_2(t_1 - 1) + deg_{G(R_2)}(J_j)$, and $t_1(t_2 - 1) + deg_{G(R_1)}(I_i)$.
- (4): $2(t_1 - 1) + 2(t_2 - 1) - 2 + (t_2 - 2) deg_{G(R_1)}(I_i) + (t_1 - 2) deg_{G(R_2)}(J_j) -$ $deg_{G(R_1)}(I_i) deg_{G(R_2)}(J_j)$.

Theorem 3. *Let $G(R_i)$ be connected for $i = 1, 2$. If $G(R_1 \times R_2)$ is Eulerian, then both $G(R_1)$ and $G(R_2)$ are Eulerian.*

Proof. Let $G(R_1 \times R_2)$ be Eulerian. Then the degree of any vertex of $G(R)$ is even. From Form (2) we have that both $t_1(t_2-1)$ and $t_2(t_1-1)$ are even. Then by Form (3), $deg_{G(R_1)}(I_i)$ and $deg_{G(R_2)}(J_j)$ are even. Then both $G(R_1)$ and $G(R_2)$ are Eulerian. ■

The converse of Theorem 3 is not true in general.

Theorem 4. *Let R_i be a ring with t_i ideals such that $G(R_i)$ be Eulerian for $i = 1, 2$. Then $G(R_1 \times R_2)$ is Eulerian if and only if $t_1 + t_2$ is even.*

Proof. Let $G(R_1)$ and $G(R_2)$ be Eulerian. Then $deg_{G(R_1)}(I_i)$ and $deg_{G(R_2)}(J_j)$ are even. So Forms (1) and (4) are even. If $t_1 + t_2$ is even, then either both t_1 and t_2 are even, or both are odd. In each case Forms (2) and (3) are even. Consequently, $G(R)$ is Eulerian. For the converse suppose that $G(R)$ is Eulerian. Then Forms (2) and (3) are even. Since $deg_{G(R_1)}(I_i)$ and $deg_{G(R_2)}(J_j)$ are even, both $t_1(t_2 - 1)$ and $t_2(t_1 - 1)$ are even. This implies that $t_1 + t_2$ is even. ■

We next study Hamiltonicity of the intersection graph of ideals of direct product of commutative rings. If C is a cycle in a graph with vertex set $V(C) = \{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_1 v_n\}$, then we refer to this cycle as $C : v_1 - v_2 - \cdots - v_n - v_1$. We begin with the following.

Proposition 5. *If $G(R_1)$ or $G(R_2)$ is Hamiltonian, then $G(R)$ is Hamiltonian.*

Proof. Let $G(R_2)$ be Hamiltonian. It is obvious that $t_2 \geq 5$. Without loss of generality assume that the Hamiltonian cycle of $G(R_2)$ is $J_1 - J_2 - \cdots - J_{t_2-2} - J_1$. Consider the following cycles in $G(R)$:

$$C_0 : 0 \times J_1 - 0 \times J_2 - \cdots - 0 \times J_{t_2-2} - 0 \times R_2 - 0 \times J_1,$$

$$C_{t_1-1} : R_1 \times 0 - R_1 \times J_1 - R_1 \times J_2 - \cdots - R_1 \times J_{t_2-2} - R_1 \times 0,$$

and for $1 \leq i \leq t_1 - 2$,

$$C_i : I_i \times 0 - I_i \times J_1 - \cdots - I_i \times J_{t_2-2} - I_i \times R_2 - I_i \times 0.$$

From the above cycles we obtain a Hamiltonian cycle as follows. Let $0 \leq r \leq t_1 - 2$.

If r is even, then we remove the edges $(I_r \times J_1)(I_r \times J_2)$, $(I_{r+1} \times J_1)(I_{r+1} \times J_2)$, and add the edges $(I_r \times J_1)(I_{r+1} \times J_1)$, $(I_r \times J_2)(I_{r+1} \times J_2)$.

If r is odd, then we remove the edges $(I_r \times J_2)(I_r \times J_3)$, $(I_{r+1} \times J_2)(I_{r+1} \times J_3)$, and add the edges $(I_r \times J_2)(I_{r+1} \times J_2)$, $(I_r \times J_3)(I_{r+1} \times J_3)$.

Thus $G(R)$ is Hamiltonian. ■

Note that the converse of Proposition 5 is not true in general. As an example let F_i is a field for $i = 1, 2, 3$. Then $F_1 \times 0 \times 0 - F_1 \times F_2 \times 0 - 0 \times F_2 \times 0 - 0 \times F_2 \times F_3 - 0 \times 0 \times F_3 - F_1 \times 0 \times F_3 - F_1 \times 0 \times 0$ is a Hamiltonian cycle in $G(F_1 \times F_2 \times F_3)$. But $G(S)$ is not Hamiltonian for $S \in \{F_i, F_i \times F_j : i, j = 1, 2, 3, i \neq j\}$.

Lemma 6. *If* $\min\{t_1, t_2\} = 3$*, then* $G(R)$ *is Hamiltonian.*

Proof. Let $\min\{t_1, t_2\} = 3$. Without loss of generality assume that $t_2 = \min\{t_1, t_2\}$. If $t_1 = 3$, then $I_1 \times 0 - I_1 \times J_1 - 0 \times J_1 - 0 \times R_2 - I_1 \times R_2 - R_1 \times J_1 - R_1 \times 0 - I_1 \times 0$ is a Hamiltonian cycle in $G(R)$. So we suppose that $t_1 \geq 4$. Then,

$$\begin{array}{rll}
0 \times J_1 - & 0 \times R_2 - & \\
I_1 \times J_1 - & I_1 \times 0 - & I_1 \times R_2 - \\
I_2 \times J_1 - & I_2 \times 0 - & I_2 \times R_2 - \\
I_3 \times J_1 - & I_3 \times 0 - & I_3 \times R_2 - \\
\cdots & & \\
I_i \times J_1 - & I_i \times 0 - & I_i \times R_2 - \\
\cdots & & \\
I_{t_1-3} \times J_1 - & I_{t_1-3} \times 0 - & I_{t_1-3} \times R_2 - \\
I_{t_1-2} \times J_1 - & R_1 \times 0 - & I_{t_1-2} \times 0 - I_{t_1-2} \times R_2 - R_1 \times J_1 - 0 \times J_1
\end{array}$$

is a Hamiltonian cycle. ∎

Proposition 7. *If* $\min\{t_1, t_2\} \geq 4$*, then* $G(R)$ *is Hamiltonian.*

Proof. Let $4 \leq t_2 \leq t_1$. For $i = 1, 2, \ldots$ or $\ldots, t_2 - 3$ let C_i be the following cycle.
$I_i \times 0 - I_i \times J_1 - \cdots - I_i \times J_i - 0 \times J_i - I_{i-1} \times R_2 - I_i \times J_{i+1} - \cdots - I_i \times J_{t_2-2} - I_i \times 0$.
Let C_{t_2-2} be the following cycle.
$I_{t_2-2} \times 0 - I_{t_2-2} \times J_1 - \cdots - I_{t_2-2} \times J_{t_2-2} - 0 \times J_{t_2-2} - I_{t_2-3} \times R_2 - I_{t_2-2} \times R_2 - I_{t_2-2} \times 0$.
For $t_2 - 1 \leq i \leq t_1 - 2$, let C_i be the cycle
$I_i \times 0 - I_i \times J_1 - \cdots - I_i \times J_{t_2-2} - I_i \times R_2 - I_i \times 0$.
Let C_{t_1-1} be the cycle $R_1 \times 0 - R_1 \times J_1 - \cdots - R_1 \times J_{t_2-2} - R_1 \times 0$.

We produce a Hamiltonian cycle from above cycles as follows. We remove the edges $(0 \times J_1)(0 \times R_2)$ and $(I_1 \times R_2)(I_2 \times J_3)$, and add the edges $(0 \times J_1)(I_1 \times R_2)$ and $(0 \times R_2)(I_2 \times J_3)$ to obtain a cycle C'_2 from C_1 and C_2. We remove the edges $(0 \times J_2)(I_1 \times R_2)$ and $(I_2 \times R_2)(I_3 \times J_3)$ and add the edges $(0 \times J_2)(I_2 \times R_2)$ and $(I_1 \times R_2)(I_3 \times J_3)$ to obtain a cycle C'_3 from C'_2 and C_3. In general from C'_i and C_{i+1} we obtain a cycle C'_{i+1} by removing the edges $(0 \times J_i)(I_{i-1} \times R_2)$ and $(I_i \times R_2)(I_{i+1} \times J_{i+2})$, and add the edges $(0 \times J_i)(I_i \times R_2)$ and $(I_{i-1} \times R_2)(I_{i+1} \times J_{i+2})$. We proceed this process to obtain the cycle C'_{t_2-2}.

Now we remove the edges $(I_{t_2-2} \times J_1)(I_{t_2-2} \times J_2)$ and $(I_{t_2-1} \times J_1)(I_{t_2-1} \times J_2)$ from C'_{t_2-2} and C_{t_2-1} and add the edges $(I_{t_2-2} \times J_1)(I_{t_2-1} \times J_1)$ and $(I_{t_2-2} \times J_2)(I_{t_2-1} \times J_2)$, to obtain a cycle C'_{t_2-1}. We remove $(I_{t_2-1} \times J_2)(I_{t_2-1} \times J_3)$ and $(I_{t_2} \times J_2)(I_{t_2} \times J_3)$ from C'_{t_2-1} and C_{t_2} and add the edges $(I_{t_2-1} \times J_2)(I_{t_2} \times J_2)$ and $(I_{t_2-1} \times J_3)(I_{t_2} \times J_3)$, obtain a cycle C'_{t_2}.

We remove $(I_{t_2} \times J_1)(I_{t_2} \times J_2)$ and $(I_{t_2+1} \times J_1)(I_{t_2+1} \times J_2)$ from C'_{t_2} and C_{t_2+1} and add the edges $(I_{t_2} \times J_1)(I_{t_2+1} \times J_1)$ and $(I_{t_2} \times J_2)(I_{t_2+1} \times J_2)$, obtain a cycle C'_{t_2+1}.

We proceed this process to obtain C'_{t_1-2}. Lastly, we remove $(I_{t_1-2} \times 0)(I_{t_1-2} \times J_1)$ and $(R_2 \times 0)(R_2 \times J_1)$ from C'_{t_1-2}, and C_{t_1-1}, and add $(I_{t_1-2} \times 0)(R_2 \times 0)$ and $(I_{t_1-2} \times J_1)(R_2 \times J_1)$ to obtain C'_{t_1-2}. Then C'_{t_1-2} is a Hamiltonian cycle. ■

Corollary 8. *If* $\min\{t_1, t_2\} \geq 3$, *then* $G(R)$ *is Hamiltonian.*

Lemma 9. *If* R_i *is a commutative ring with identity for* $i = 1, 2, 3$, *then* $G(R_1 \times R_2 \times R_3)$ *is Hamiltonian.*

Proof. Let the number of ideals of R_i be equal to t_i for $i = 1, 2, 3$. Since the graph of direct product of three fields is Hamiltonian, we assume that $t_i \geq 3$ for some i. Then Corollary 8 implies that $G(R_1 \times R_2 \times R_3)$ is Hamiltonian. ■

By Lemma 9, we have the following corollary.

Corollary 10. *For two commutative rings* R_1 *and* R_2 *with identity, if* R_1 *or* R_2 *is direct product of two rings, then* $G(R)$ *is Hamiltonian.*

In the rest of the paper we consider $t_1 = 2$. By Corollary 8 it remains to study Hamiltonicity in $G(R)$ when R_1 is a field (i.e., $t_1 = 2$). The following has a straightforward proof.

Observation 11. *If* $t_2 \in \{2, 3\}$, *then* $G(R)$ *is not Hamiltonian.*

So henceforth we suppose that $t_2 \geq 4$. Also by Corollary 10, we assume R_2 is not direct product of two rings. Since R_2 is an Artinian ring, henceforth $(R_2.m)$ is local. (We recall that a ring is *local* if it has only one maximal ideal. We also refer (R, M) as a local ring R with the unique maximal ideal M).

Lemma 12. *If* R_2 *has a unique minimal ideal, then* $G(R)$ *is Hamiltonian.*

Proof. First suppose that $t_2 = 4$. Let J_1 be the unique minimal ideal of R_2. Let J_2 be the other proper non-trivial ideal. Since R_2 is not direct product of two rings, J_2 is the unique maximal ideal of R_2. Now $0 \times J_1 - 0 \times R_2 - 0 \times J_2 - R_1 \times J_2 - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ is a Hamiltonian cycle. We next suppose that $t_2 \geq 5$. Notice that $G(R_2)$ is a complete graph and so is Hamiltonian. Then the result follows by Proposition 5. ■

Since for any $n \geq 2$, the ring $\mathbb{Z}_{p^n}$ has a unique minimal ideal, a consequence of Corollaries 8, 10, Observation 11, and Lemma 12 we obtain the following.

Proposition 13. *For $k \geq 2$, $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$ is not Hamiltonian if and only if $k = 2$ and $n_1 + n_2 \leq 3$.*

Since for a prime p, $\mathbb{Z}_{p^n}$ is not Hamiltonian if and only if $n \leq 3$, as a consequence of Proposition 13 we obtain the following.

Corollary 14 (Theorem 5.2 of [2])**.** *All the graphs of $\mathbb{Z}_n$ are Hamiltonian except when n is one of the following forms: p^2, p^3, pq, p^2q, where p, q are distinct primes.*

Lemma 15. *If F is an infinite field and V is a n-dimensional vector space over F, then V has infinite subspaces.*

Lemma 16. *If F is a finite field of order q and V is a n-dimensional vector space over F, then V has $\frac{q^n-1}{q-1} = q^{n-1} + q^{n-2} + \cdots + 1$, 1-dimensional subspaces.*

Corollary 17. *Let (R_2, M) is a local ring such that $\frac{R_2}{M}$ is finite. If $I_1, \ldots, I_t$ are minimal ideals of R_2 for $t \geq 2$ such that for any i, $I_i \cong \frac{R_2}{M}$, and $I_1 + \cdots + I_t = I_1 \bigoplus I_2 \bigoplus \cdots \bigoplus I_t = J$, then J contains $|\frac{R_2}{M}|^{t-1} + |\frac{R_2}{M}|^{t-2} + \cdots + 1$ minimal ideals of R_2.*

Proof. Notice that R_2 is a Noetherian ring and $MI_i = 0$ for $i = 1, 2, \ldots, t$. Thus J is an $\frac{R_2}{M}$-module (vector space) which a subset T is an $\frac{R_2}{M}$-submodule of J if and only if T is an R_2-submodule. ∎

Theorem 18. *For $t_2 = 4, 5, \ldots, 9$, $G(R)$ is Hamiltonian.*

Proof. For $t_2 = 4$ the result follows from the proof of Lemma 12.
If R_2 has a unique minimal ideal then by Lemma 12, $G(R_2)$ is complete and so is Hamiltonian. Then by Proposition 5, $G(R)$ is Hamiltonian. Thus we assume that R_2 has at least two minimal ideals.

If $t_2 = 5$, then by Corollary 17, R_2 has at least six ideals, a contradiction. Thus $t_2 \neq 5$.

• $t_2 = 6$. Let J_1, J_2 be two minimal ideals of R_2. By Corollary 17, $J_1 + J_2$ contains at least three minimal ideals of R_2. Without loss of generality assume that $J_1 + J_2$ contains three minimal ideals J_1, J_2 and J_3. Since $t_2 = 6$, all ideals of R_2 are $0, J_1, J_2, J_3, J_1 + J_2 = M, R$. Now, $0 \times J_1 - 0 \times R_2 - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times (J_1 + J_2) - R_1 \times (J_1 + J_2) - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ is a Hamiltonian cycle.

• $t_2 = 7$. Let J_1, J_2 be two distinct minimal ideals of R_2. By Corollary 17, $J_1 + J_2$ contains at least three minimal ideals of R_2. Without loss of generality assume that $J_1 + J_2$ contains three minimal ideals J_1, J_2 and J_3. Let J be an ideal of R_2 different from J_1, J_2, J_3 and $J_1 + J_2$. If $J \cap (J_1 + J_2) = 0$, then R_2 has more than 7 ideals, a contradiction. So $J \cap (J_1 + J_2) \neq 0$. Now $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times J - R_1 \times J - R_1 \times (J_1 + J_2) - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$, is a Hamiltonian cycle.

• $t_2 = 8$. Let J_1, J_2 be two distinct minimal ideals of R_2. Since R_2 is local, $J_1 \cong J_2$ ($\cong \frac{R_2}{M}$). Then $J_1 + J_2 = J_1 \bigoplus J_2$. If R_2 has another minimal ideal K such that $J_1 + J_2 + K = J_1 \bigoplus J_2 \bigoplus K$, then by Corollary 17, $J_1 + J_2 + K$ contains at least 7 minimal ideals contradicting that $t_2 = 8$. We deduce that $J_1 + J_2$ contains all minimal ideals of R_2. Further, $|\frac{R_2}{M}| \leq 4$. We discuss the possiblities for $|\frac{R_2}{M}|$.

If $|\frac{R_2}{M}| = 2$, then R_2 has three minimal ideals J_1, J_2, J_3. Also note that $M \neq J_1 + J_2$, since $t_2 = 8$. Let J be a proper non-trivial ideal different from $J_1, J_2, J_3, J_1 + J_2, M$. Since J is not minimal we may assume that $J_1 \subseteq J$. Now $0 \times J_1 - 0 \times (J_1 + J_2) - R_1 \times (J_1 + J_2) - R_1 \times J_2 - 0 \times J_2 - 0 \times M - R_1 \times M - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times J - R_1 \times J - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ is a Hamiltonian cycle.

If $|\frac{R_2}{M}| = 3$, then R_2 has four minimal ideals J_1, J_2, J_3, J_4. Since $t_2 = 8$, we have $M \neq J_1 + J_2$ and so $0, J_1, J_2, J_3, J_4, J_1 + J_2, M, R_2$ are all ideals of R_2. Now $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times M - 0 \times J_4 - R_1 \times J_4 - R_1 \times (J_1 + J_2) - R_1 \times M - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ is a Hamiltonian cycle.

If $|\frac{R_2}{M}| = 4$, then R_2 has five minimal ideals J_1, J_2, J_3, J_4, J_5. So $0, J_1, J_2, J_3, J_4, J_5, J_1 + J_2, R_2$ are all ideals of R_2. Now $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times (J_1 + J_2) - 0 \times J_5 - R_1 \times J_5 - R_1 \times J_4 - 0 \times J_4 - 0 \times R_2 - 0 \times J_3 - R_1 \times J_3 - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ is a Hamiltonian cycle.

• $t_2 = 9$. Let J_1, J_2 be two distinct minimal ideals of R_2. Since R is local, $J_1 \cong J_2$ ($\cong \frac{R_2}{M}$). Let $|\frac{R_2}{M}| = q$. Then by Corollary 17 $q \in \{2, 3, 4, 5, 7\}$, since $\frac{R_2}{M}$ is a field. Consequently by Corollary 17 R_2 has $q + 1$ minimal ideals. It is obvious that $q \neq 7$. If $q = 5$, then the ideals of R_2 are $0, J_1, J_2, J_3, J_4, J_5, J_6, J_1 + J_2, R_2$, where J_i is minimal and $J_i \subseteq J_1 + J_2$ for $i = 1, 2, 3, 4, 5, 6$. Then $R_1 \times 0 - R_1 \times J_2 - 0 \times J_2 - 0 \times R_2 - 0 \times J_1 - R_1 \times J_1 - R_1 \times J_3 - 0 \times J_3 - 0 \times (J_1 + J_2) - 0 \times J_4 - R_1 \times J_4 - R_1 \times J_5 - 0 \times J_5 - R_1 \times (J_1 + J_2) - 0 \times J_6 - R_1 \times J_6 - R_1 \times 0$ is a Hamiltonian cycle. If $q = 4$, then the ideals of R_2 are $0, J_1, J_2, J_3, J_4, J_5, J_1 + J_2, M, R_2$, where J_i is minimal for $i = 1, 2, 3, 4, 5$. Then $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times J_4 - R_1 \times J_4 - R_1 \times J_5 - 0 \times J_5 - 0 \times M - R_1 \times M - R_1 \times (J_1 + J_2) - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$, is a Hamiltonian cycle. If $q = 3$, $J_1 + J_2$ contains four minimal ideals J_1, J_2, J_3, J_4. Let J be another proper non-trivial ideal. This time $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - 0 \times M - 0 \times J_3 - 0 \times R_2 - 0 \times J - R_1 \times J - R_1 \times M - R_1 \times 0 - R_1 \times (J_1 + J_2) - 0 \times J_4 - R_1 \times J_4 - R_1 \times J_3 - R_1 \times J_2 -$

$R_1 \times J_1 - 0 \times J_1$ is a Hamiltonian cycle. If $q = 2$, then by Corollary 17, $J_1 + J_2$ contains three minimal ideals of J_1, J_2, J_3 of R_2. Also R_2 has no minimal ideal. Suppose that the ideals of R_2 are $0, J_1, J_2, J_1 + j_2, J_3, K_1, K_2, K_3, R_2$. Without loss of generality assume that $J_1 \subseteq K_1$. Note that $K_i \cap (J_1 + J_2) \neq 0$. Then $0\times J_1 - 0\times K_1 - 0\times (J_1+J_2) - 0\times J_3 - 0\times R_2 - 0\times J_2 - R_1\times J_2 - R_1\times K_2 - 0\times K_2 - 0\times K_3 - R_1\times K_3 - R_1\times 0 - R_1\times K_1 - R_1\times (J_1+J_2) - R_1\times J_3 - R_1\times J_1 - 0\times J_1$ is a Hamiltonian cycle. ■

We close with the following problem.

Problem 19. Is $G(R)$ Hamiltonian for $t_2 \geq 10$?

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