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SUBLATTICES CORRESPONDING TO VERY TRUE OPERATORS IN COMMUTATIVE BASIC ALGEBRAS

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Abstract

We introduce the concept of very true operator on a commutative basic algebra in a way analogous to that for fuzzy logics. We are motivated by the fact that commutative basic algebras form an algebraic axiomatization of certain non-associative fuzzy logics. We prove that every such operator is fully determined by a certain relatively complete sublattice provided its idempotency is assumed.[1]

Keywords: commutative basic algebra, very true operator, idempotent operator, relatively complete sublattice.

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The concept of a very true operator was in fact introduced by L.A. Zadeh [6] under the name "linguistic hedge" for a certain class of fuzzy logics. The name "very true" was given by P. Hájek [5]. It was shown by M. Botur and R. Halaš that every commutative basic algebra is an axiomatization of a non-associative fuzzy logic that can be used e.g. in expert systems or another tasks in Artificial

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Inteligence. Hence, the question about an operator reducing the structure of values in these logics was established.

It was proved by M. Botur and the second author that the concept of a very true operator can be extended in these logics and, in a pure axiomatic setting, also in commutative basic algebras. Moreover, both of these concepts are in a one-to-one correspondence, see [1].

On the other hand, several properties of very true operators on commutative basic algebras have not been revealed in [1] and we feel the opportunity to do it now. In particular, we show that the image of a very true operator in a commutative basic algebra is in fact a sublattice of the induced lattice and, if moreover an idempotency of the operator is assumed, every very true operator is fully determined by this sublattice in the way developed here.

At first, we recall the basic concepts. By a *basic algebra* is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ satisfying the axioms

(B1) $x \oplus 0 = x$,

(B2) $\neg\neg x = x$,

(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

where $1 = \neg 0$. If, moreover, $\mathcal{A}$ satisfies the identity

(C) $x \oplus y = y \oplus x$

then it is called a *commutative basic algebra.* In every basic algebra $\mathcal{A}$ we can introduce an *induced order* "$\leq$" as follows

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

It was shown in [3, 4] that $0 \leq x \leq 1$ for all $x \in A$ and $(A; \leq)$ is in fact a lattice where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y).$$

The bounded lattice $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, 0, 1)$ will be called the *induced lattice* of $\mathcal{A}$. It was proved in [3, 4] that for every commutative basic algebra $\mathcal{A}$ the induced lattice $\mathcal{L}(\mathcal{A})$ is distributive (which is not true in a non-commutative case). Recall that a basic algebra is an MV-algebra if and only if it is associative and that every finite commutative basic algebra is an MV-algebra. M. Botur constructed an example of an infinite basic algebra which is not an MV-algebra.

Having an algebra $(A; \to, 1)$ of type $\langle 2, 0\rangle$, one can introduce a very true operator t on $(A; \to, 1)$ in the sense of P. Hájek as a mapping $t : A \to A$ satisfying $t(1) = 1$, $t(x) \leq x$, $t(x \to y) \leq t(x) \to t(y)$, where "$\leq$" is an order on $(A; \to, 1)$ induced by $x \leq y$ iff $x \to y = 1$.

However, in basic algebras we have different operations and the previous concept seems to be rather weak. First, let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Define $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \to y = \neg x \oplus y$. Hence, $\mathcal{A}$ can be alternatively considered in operations $(\to, 0)$ or in operations $(\odot, \neg, 0)$. The converse way is also possible, since

$$\neg x = x \to 0, \quad x \oplus y = (x \to 0) \to y$$

or $x \oplus y = \neg(\neg x \odot \neg y)$.

Therefore, consider the operation $\to$ as defined above, we can introduce the so-called *vt-operator* on $\mathcal{A}$ as a mapping $t : A \to A$ satisfying the following:

(T1) $t(1) = 1$;

(T2) $t(x) \leq x$;

(T3) $t(x \to y) \leq t(x) \to t(y)$;

(T4) $t(x \vee y) \leq t(x) \vee t(y)$.

If, moreover, t satisfies

(T5) $t(t(x)) = t(x)$,

it will be called a *strong vt*-operator. It is elementary to prove the following (see e.g. Lemma 4.1 in [1]).

Lemma 1. *If t is a vt-operator on a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ then*

(a) $t(x) = 1$ *iff* $x = 1$,

(b) $t(0) = 0$,

(c) $x \leq y$ *implies* $t(x) \leq t(y)$.

If a basic algebra $\mathcal{A}$ is commutative, we are able to prove a bit more due to the fact that a commutative basic algebra is in fact a (non-associative) residuated lattice. We can repeat this important result from [1, 2]:

Lemma 2 (Lemma 3.1 in [1])**.** *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra. Then*

(i) $x \odot y \leq z$ *iff* $x \leq y \to z$, (*adjoitness*)

(ii) $x \wedge y = x \odot (x \to y)$, (*divisibility*)

(iii) $(x \to y) \vee (y \to x) = 1$, (*prelinearity*)

(iv) $x \leq y \implies x \to z \geq y \to z,\ z \to x \leq z \to y$.

For the proof of the following statement, see Lemma 4.1 in [1] again.

Lemma 3. *Let t be a vt-operator on a commutative basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$. Then*

(a) $t(x) \odot t(y) \leq t(x \odot y)$,

(b) $t(\neg x) \leq \neg t(x)$.

The following useful results holds in every basic algebra.

Lemma 4. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $a, b, c \in A$. Then*

$$(a \wedge b) \oplus c = (a \oplus c) \wedge (b \oplus c).$$

Proof. Since basic algebras are in a one-to-one correspondence with bounded lattices with sectionally antitone involutions, see e.g. [3, 4], we can compute

$$\begin{aligned} (a \wedge b) \oplus c &= (\neg(a \wedge b) \vee c)^c = (\neg a \vee \neg b \vee c)^c \\ &= ((\neg a \vee c) \vee (\neg b \vee c))^c = (\neg a \vee c)^c \wedge (\neg b \vee c)^c \\ &= (a \oplus c) \wedge (b \oplus c) \end{aligned}$$

(where x^y means an involution of x in the interval $[y, 1]$, i.e., $x^y = \neg x \oplus y$ provided $x \in [y, 1]$). ■

Now, we are able to formulate our new results on vt-operators.

Theorem 5. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra and let $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, 0, 1)$ be its induced lattice. Let t be a vt-operator on $\mathcal{A}$. Then t is a* 0-1-*homomorphism of $\mathcal{L}(\mathcal{A})$ to $\mathcal{L}(\mathcal{A})$.*

Proof. By (T1) and (b) of Lemma 1 we have $t(0) = 0$, $t(1) = 1$.
By (T4) we have $t(x \vee y) \leq t(x) \vee t(y)$. By (c) of Lemma 1, $t(x \vee y) \geq t(x), t(y)$, thus also $t(x \vee y) \geq t(x) \vee t(y)$. Altogether we conclude

$$(*) \qquad t(x \vee y) = t(x) \vee t(y).$$

By (ii) of Lemma 2 we get $x \wedge y = x \odot (x \to y)$, thus, using (a) of Lemma 3, $t(x \wedge y) = t(x \odot (x \to y)) \geq t(x) \odot t(x \to y)$. Applying (i) of Lemma 3 we infer $t(x) \leq t(x \to y) \to t(x \wedge y)$.

Interchanging the roles of x and y in the previous, we obtain analogously $t(y) \leq t(y \to x) \to t(x \wedge y)$. Together, it gets

$$\begin{aligned} t(x) \wedge t(y) &\leq (t(y \to x) \to t(x \wedge y)) \wedge (t(x \to y) \to t(x \wedge y)) \\ &= (\neg t(y \to x) \oplus t(x \wedge y)) \wedge (\neg t(x \to y) \oplus t(x \wedge y)) \\ &= (\neg t(y \to x) \wedge \neg t(x \to y)) \oplus t(x \wedge y) \\ &= \neg(t(y \to x) \vee t(x \to y)) \oplus t(x \wedge y) \end{aligned}$$

due to Lemma 4 and de Morgan laws. Using (iii) of Lemma 2 and $(*)$ we have

$$t(y \to x) \vee t(x \to y) = t((y \to x) \vee (x \to y)) = t(1) = 1.$$

Substituting this in the previous computation, we obtain

$$t(x) \wedge t(y) \leq \neg 1 \oplus t(x \wedge y) = 0 \oplus t(x \wedge y) = t(x \wedge y).$$

However, by (c) of Lemma 1 we infer $t(x) \wedge t(y) \geq t(x \wedge y)$, whence $t(x) \wedge t(y) = t(x \wedge y)$. We have proved that t is 0-1-homomorphism of $\mathcal{L}(\mathcal{A})$ to $\mathcal{L}(\mathcal{A})$. ∎

Corollary 6. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra and t be a vt-operator on $\mathcal{A}$. Then t is a 0-1-homomorphism of $\mathcal{L}(\mathcal{A})$ onto $t(\mathcal{A})$.*

Let $\mathcal{L} = (L; \vee, \wedge)$ be a lattice and $\mathcal{B} = (B; \vee, \wedge)$ its sublattice. We say that $\mathcal{B}$ is a *relatively complete sublattice* of $\mathcal{L}$ if for each $a \in L$ there exists $\sup\{b \in B; b \leq a\}$ computed in $\mathcal{L}$.

We are able now to state our main results which characterize strong *vt*-operators on commutative basic algebras by means of relatively complete sublattices of the included lattices.

Theorem 7. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra and t a strong vt-operator on $\mathcal{A}$. Let $A_0 = t(A)$. Then $\mathcal{A}_0 = (A_0; \vee, \wedge, 0, 1)$ is a relatively complete 0-1-sublattice of $\mathcal{L}(\mathcal{A})$ satisfying the following conditions*

(I) $\quad \sup\{b \in A_0; b \leq x \to y\} \leq \sup\{b \in A_0; b \leq x\} \to \sup\{b \in A_0; b \leq y\}$.

(J) $\quad \sup\{b \in A_0; b \leq x \vee y\} \leq \sup\{b \in A_0; b \leq x\} \vee \sup\{b \in A_0; b \leq y\}$.

Proof. By Theorem 5, $\mathcal{A}_0 = (A_0; \vee, \wedge, 0, 1)$ is a 0-1-sublattice of $\mathcal{L}(\mathcal{A})$ for $A_0 = t(A)$. Assume $a \in A$. Then $t(A) \in \{b \in A_0; b \leq a\}$. If $y \in A_0$ and $y \leq a$ then $t(y) = y$ due to (T5) and, using (c) of Lemma 1, $y = t(y) \leq t(a)$. Hence,

$t(a) = \{b \in A_0; b \leq a\}$ for all $a \in A$ thus $\mathcal{A}_0$ is relatively complete. By (T3), we conclude

$$\begin{aligned}\sup\{b \in A_0; b \leq x \to y\} &= t(x \to y) \leq t(x) \to t(y)\\ &= \sup\{b \in A_0; b \leq x\} \to \sup\{b \in A_0; b \leq y\},\end{aligned}$$

which is the condition (I), and analogously by (T4)

$$\begin{aligned}\sup\{b \in A_0; b \leq x \vee y\} &= t(x \vee y) \leq t(x) \vee t(y)\\ &= \sup\{b \in A_0; b \leq x\} \vee \sup\{b \in A_0; b \leq y\},\end{aligned}$$

which is the inequality (J). ∎

We can prove also the converse to show that relatively complete 0-1-sublattices satisfying (I) and (J) really characterize strong vt-operators on commutative basic algebras.

Theorem 8. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra and let $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, 0, 1)$ be its induced lattice. Let $\mathcal{A}_0 = (A_0; \vee, \wedge, 0, 1)$ be a relatively complete* 0-1-*sublattice of $\mathcal{L}(\mathcal{A})$ satisfying* (I) *and* (J)*. Define a mapping $t : A \to A$ as follows:*

$$t(a) = \sup\{b \in A_0; b \leq a\}.$$

Then t is a strong vt-operator on $\mathcal{A}$.

Proof. Since $\mathcal{A}_0$ is relatively complete, $t(a) = \sup\{b \in A_0; b \leq a\} \in A_0$. It is evident that for any $b \in A_0$ we have $t(b) = b$ thus, in particular, $t(a) = A_0$ and $t(1) = 1$ and $t(t(a)) = t(a)$ proving (T1) and (T5). Immediately from the definition of t we have $t(x) \leq x$ which is (T2) and the condition (I) yields (T3).

Since $\mathcal{A}_0$ is relatively complete, there exists $\sup\{b \in A_0; b \leq a\}$ for all $a \in A$ and, by (J), we conclude

$$\begin{aligned}t(x) \vee t(y) &= \sup\{b \in A_0; b \leq x\} \vee \sup\{b \in A_0; b \leq y\}\\ &\geq \sup\{b \in A_0; b \leq x \vee y\} = t(x \vee y),\end{aligned}$$

which is (T4). ∎

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