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# ON BALANCED ORDER RELATIONS AND THE NORMAL HULL OF COMPLETELY SIMPLE SEMIRINGS

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**Abstract**

In [1] the authors proved that a semiring $S$ is a completely simple semiring if and only if $S$ is isomorphic to a Rees matrix semiring over a skew-ring $R$ with sandwich matrix $P$ and index sets $I$ and $\Lambda$ which are bands under multiplication. In this paper we characterize all the balanced order relations on completely simple semirings. Also we study the normal hull of a completely simple semiring.

**Keywords:** skew-ring, Rees matrix semiring, balanced order relation, essential extension, normal extension, normal ideal, normal hull.

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## 1. Introduction

Recall that a semiring $(S, +, \cdot)$ is a type (2, 2)-algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by ring like distributivity, that is, $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in S$. A semiring $(S, +, \cdot)$ is called a skew-ring if its additive reduct $(S, +)$ is a group. In a skew-ring $(R, +, \cdot)$, a normal subgroup $K$ of $(R, +)$ is called a skew-ideal of $R$ if $a \in K$ implies $ca, ac \in K$ for all $c \in R$.

A semigroup $(S, \cdot)$ is said to be a regular semigroup if for each element $a \in S$, there exists an element $x \in S$ such that $axa = a$. A semigroup $(S, \cdot)$ is called a completely regular semigroup if for each element $a \in S$, there exists an element $x \in S$ such that $axa = a$ and $ax = xa$. We call a semiring $(S, +, \cdot)$ additively regular if for every element $a \in S$ there exists an element $x \in S$ such that

$a + x + a = a$. If $(S, +, \cdot)$ is a semiring, we denote Green's relations on the semigroup $(S, +)$ by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$, $\mathscr{D}^+$ and $\mathscr{H}^+$.

A semiring $S$ is called a completely regular semiring [2] if for every element $a \in S$ there exists an element $x \in S$ such that the following conditions are satisfied:

$$\text{(i) } a = a + x + a, \ \ \text{(ii) } a + x = x + a \ \text{ and (iii) } a(a + x) = a + x.$$

In fact, conditions (i) and (ii) state that $a \in S$ is a completely regular element in the additive reduct $(S, +)$ of the semiring $(S, +, \cdot)$. Condition (iii) is an extra condition which makes the element $a$ in $(S, +, \cdot)$ to be completely regular. A completely regular semiring $(S, +, \cdot)$ is called completely simple if $\mathscr{J}^+ = S \times S$. Clearly, from definition it follows that every completely simple semiring $(S, +, \cdot)$ is a semiring with completely simple additive semigroup $(S, +)$. Construction of a semiring $(S, +, \cdot)$ whose additive reduct $(S, +)$ is a completely simple semigroup has been studied by M.P. Grillet [3]. We know that a semigroup $(S, \cdot)$ is completely simple if and only if $S$ is isomorphic to a Rees matrix semigroup over a group with sandwich matrix $P$ and index sets $I$ and $\Lambda$. In [1] we extended this important result from semigroups to semirings.

**Theorem 1.1** [1]**.** *Let $R$ be skew-ring, $(I, \cdot)$ and $(\Lambda, \cdot)$ are bands such that $I \cap \Lambda = \{o\}$. Let $P = (p_{\lambda,i})$ be a matrix over $R, i \in I, \lambda \in \Lambda$ and assume*

(1.1) $p_{\lambda,o} = p_{o,i} = 0$;

(1.2) $p_{\lambda\mu,kj} = p_{\lambda\mu,ij} - p_{\nu\mu,ij} + p_{\nu\mu,kj}$;

(1.3) $p_{\mu\lambda,jk} = p_{\mu\lambda,ji} - p_{\mu\nu,ji} + p_{\mu\nu,jk}$;

(1.4) $ap_{\lambda,i} = p_{\lambda,i}a = 0$;

(1.5) $ab + p_{o\mu,io} = p_{o\mu,io} + ab$;

(1.6) $ab + p_{\lambda o,oj} = p_{\lambda o,oj} + ab$ *for all* $i, j, k \in I; \lambda, \mu, \nu \in \Lambda$ *and* $a, b \in R$.

*Let $\mathscr{M}$ consist of the elements of $I \times R \times \Lambda$ and define operations $'+'$ and $'\cdot'$ on $\mathscr{M}$ by*

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu) \tag{1.7}$$

$$(i, a, \lambda) \cdot (j, b, \mu) = (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu). \tag{1.8}$$

*Then $(\mathscr{M}, +, \cdot)$ is a completely simple semiring. Conversely every completely simple semiring is isomorphic to such a semiring.*

The semiring constructed above is denoted by $\mathscr{M}(I, R, \Lambda; P)$ and is called Rees matrix semiring.

**Corollary 1.2** [1]**.** *Let $\mathscr{M}(I, R, \Lambda; P)$ be a Rees matrix semiring. Then $ab+p_{\lambda,i} = p_{\lambda,i} + ab$ for all $i \in I, \lambda \in \Lambda$ and $a, b \in R$.*

In Section 2, we characterize all balanced order relations on a completely simple semiring. For a completely simple semiring, normal extension and essential extension are defined and their properties are studied in Section 3. The notion of normal hull $\Phi(S)$ of a completely simple semiring $S$ is defined in Section 4. Finally, we show that $\Phi(S)$ admits a natural Rees matrix semigroup representation if $S$ is given a Rees matrix semiring representation.

## 2. BALANCED ORDER RELATIONS ON REES MATRIX SEMIRINGS

Let $(G, +)$ be a group with identity 0, $I$ and $\Lambda$ be index sets such that $I \cap \Lambda = \{o\}$ and $P = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I}$ be a matrix over $G$ such that $p_{\lambda,o} = p_{o,i} = 0$ for all $\lambda \in \Lambda$ and $i \in I$. On $S = I \times R \times \Lambda$ define addition by

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu) \text{ for all } (i, a, \lambda), (j, b, \mu) \in S.$$

Then straightforward calculations show that $(S, +)$ is a semigroup and is denoted by $S = \mathscr{M}(I, G, \Lambda; P)$.

**Definition 2.1.** A relation $\varrho$ on a semigroup $(S, \cdot)$ is said to be stable if for $a, b \in S$; $a\,\varrho\,b$ implies $ac\,\varrho\,bc, ca\,\varrho\,cb$ for all $c \in S$; i.e., $\rho$ is stable under left and right multiplication. An order relation on a semigroup $(S, \cdot)$ [a semiring $(S, +, \cdot)$] is a partial order relation on $S$ which is stable in $(S, \cdot)$ [as well as in $(S, +, \cdot)$].

**Lemma 2.2.** *Let $S = \mathscr{M}(I, G, \Lambda; P)$ be a semigroup and $\varrho$ be a stable relation on $S$. Then*

(i) *$(i, a, \lambda)\,\varrho\,(j, b, \lambda)$ for some $\lambda \in \Lambda$ implies $(i, a, \mu)\,\varrho\,(j, b, \mu)$ for all $\mu \in \Lambda$.*

(ii) *$(i, a, \lambda)\,\varrho\,(i, b, \mu)$ for some $i \in I$ implies $(j, a, \lambda)\,\varrho\,(j, b, \mu)$ for all $j \in I$.*

***Proof.*** (i) Since $\varrho$ is a stable relation on $S$ and $(i, a, \lambda)\,\varrho\,(j, b, \lambda)$, then we have $(i, a, \lambda) + (o, 0, \mu)\,\varrho\,(j, b, \lambda) + (o, 0, \mu)$, i.e., $(i, a, \mu)\,\varrho\,(j, b, \mu)$ for all $\mu \in \Lambda$.

(ii) This part follows in a similar way as (i) of this theorem. ■

**Corollary 2.3.** *Let $S = \mathscr{M}(I, G, \Lambda; P)$ be a semigroup and $\varrho$ be a stable relation on $S$. Then $(i, a, \lambda)\,\varrho\,(j, b, \lambda)$ if and only if $(o, a, o)\,\varrho\,(o, b, o)$ if and only if $(o, a - b, o)\,\varrho\,(o, 0, o)$.*

**Remark 2.4.** Let $S = \mathscr{M}(I, G, \Lambda; P)$ be a semigroup and $\varrho$ be a stable relation on $S$. Then $U_\varrho = \{a \in G : (o, a, o)\, \varrho\, (o, 0, o)\}$ satisfies

$$\begin{aligned} U_\varrho &= \{a \in G : (i, a, \lambda)\, \varrho\, (i, 0, \lambda) \text{ for some } i \in I \text{ and for some } \lambda \in \Lambda\} \\ &= \{a \in G : (i, a, \lambda)\, \varrho\, (i, 0, \lambda) \text{ for all } i \in I \text{ and for all } \lambda \in \Lambda\}. \end{aligned}$$

Now special properties of $\varrho$ result in special properties of $U_\varrho$:

(i) If $\varrho$ is reflexive, then $0 \in U_\varrho \neq \emptyset$ and conversely.

(ii) If $\varrho$ is reflexive and transitive, then $(U_\varrho, +)$ is a submonoid of $G$.

(iii) If $\varrho$ is an equivalence relation (and hence a congruence), then $(U_\varrho, +)$ is a normal subgroup of $(G, +)$.

(iv) If $\varrho$ is a partial order, then $U_\varrho \cap (-U_\varrho) = \{0\}$, i.e., $U_\varrho$ is a positive cone of $\varrho$.

**Remark 2.5.** Let $\varrho$ be a transitive and stable relation on $S = \mathscr{M}(I, G, \Lambda; P)$ and $(i, a, \lambda)\, \varrho\, (j, b, \mu)$. Then we get

$$(i, -p_{\nu,j}, \nu) + (i, a, \lambda) + (k, -p_{\mu,k}, \lambda)\, \varrho\, (i, -p_{\nu,j}, \nu) + (j, b, \mu) + (k, -p_{\mu,k}, \lambda),$$

i.e., $$(i, -p_{\nu,j} + p_{\nu,i} + a + p_{\lambda,k} - p_{\mu,k} - b, \lambda)\, \varrho\, (i, 0, \lambda)$$

and hence $-p_{\nu,j} + p_{\nu,i} + a + p_{\lambda,k} - p_{\mu,k} - b \in U_\varrho$ for all $k \in I$ and $\nu \in \Lambda$.

**Definition 2.6.** Let $\varrho$ be a stable relation on $S = \mathscr{M}(I, G, \Lambda; P)$. Define the relation $\varrho_I$ on $I$ by $i\, \varrho_I\, j$ if and only if $(i, a, \lambda)\, \varrho\, (j, b, \lambda)$ for some $a, b \in G$ and $\lambda \in \Lambda$ and likewise the relation $\varrho_\Lambda$ on $\Lambda$ by $\lambda\, \varrho_\Lambda\, \mu$ if and only if $(i, a, \lambda)\, \varrho\, (i, b, \mu)$ for some $a, b \in G$ and $i \in I$.

The relation $\varrho$ is called balanced if $i\, \varrho_I\, j$ implies $(i, -p_{\lambda,i}, \lambda)\, \varrho\, (j, -p_{\lambda,j}, \lambda)$ for all $\lambda \in \Lambda$ and if $\lambda\, \varrho_\Lambda\, \mu$ implies $(i, -p_{\lambda,i}, \lambda)\, \varrho\, (i, -p_{\mu,i}, \mu)$ for all $i \in I$.

**Remark 2.7.** Let $\varrho$ be a stable relation on $S = \mathscr{M}(I, G, \Lambda; P)$. Then it is interesting to note that if $\varrho$ is reflexive, then $\varrho_I$ and $\varrho_\Lambda$ are reflexive. Also, if $\varrho$ is balanced and antisymmetric, then $i\, \varrho_I\, j$ and $j\, \varrho_I\, i$ imply in particular $(i, 0, o)\, \varrho\, (j, 0, o)$ and $(j, 0, o)\, \varrho\, (i, 0, o)$; hence $(i, 0, o) = (j, 0, o)$ and $\varrho_I$ is antisymmetric, too. In the similar way it follows that $\varrho_\Lambda$ is antisymmetric. Again, if $\varrho$ is balanced and transitive, then $i\, \varrho_I\, j$ and $j\, \varrho_I\, k$ imply $(i, 0, o)\, \varrho\, (j, 0, o)$ and $(j, 0, o)\, \varrho\, (k, 0, o)$. Hence $(i, 0, o)\, \varrho\, (k, 0, o)$, which shows that $i\, \varrho_I\, k$, i.e., $\varrho_I$ is transitive. A similar argument shows that $\varrho_\Lambda$ is transitive. Moreover, from Remark 2.5, it follows that $i\, \varrho_I\, j$ implies $-p_{\nu,j} + p_{\nu,i} - p_{\lambda,i} + p_{\lambda,j} \in U_\varrho$ for all $\lambda, \nu \in \Lambda$. Likewise $\lambda\, \varrho_\Lambda\, \mu$ implies $-p_{\lambda,i} + p_{\lambda,j} - p_{\mu,j} + p_{\mu,i} \in U_\varrho$ for all $i, j \in I$.

Now, we characterize all balanced order relations on a Rees matrix semiring $\mathscr{M}(I, R, \Lambda; P)$ in terms of the parameters $I, R, \Lambda$ and $P$.

**Definition 2.8.** Let $U$ be a positive cone of an order relation on $(R, +)$ of a skew-ring $R$ satisfying $ac, ca \in U$ for all $a \in U$ and for all $c \in R$, $\xi$ an order relation on $I$ and $\eta$ an order relation on $\Lambda$. The triple $(\xi, U, \eta)$ is said to be an admissible triple of orders of $I, R$ and $\Lambda$ respectively if $(i, j) \in \xi$ implies $-p_{\mu,i} + p_{\mu,j} - p_{\lambda,j} + p_{\lambda,i} \in U$ for all $\lambda, \mu \in \Lambda$ and $(\lambda, \mu) \in \eta$ implies $-p_{\mu,i} + p_{\mu,j} - p_{\lambda,j} + p_{\lambda,i} \in U$ for all $i, j \in I$.

First we state the following Theorem.

**Theorem 2.9** [5]**.** *Let $S = \mathscr{M}(I, G, \Lambda; P)$ be a completely simple semigroup and let $\pi(\varrho) = (\varrho_I, U_\varrho, \varrho_\Lambda)$ for any balanced order relation $\varrho$ on $S$. Then $\pi$ is an order preserving bijection from the set of all balanced orders on $S$ onto the set of all admissible triples of orders on $I, G$ and $\Lambda$, respectively.*

We now extend Theorem 2.9 from completely simple semigroup to completely simple semiring. In fact this is the main theorem in this section.

**Theorem 2.10.** *Let $S = \mathscr{M}(I, R, \Lambda; P)$ be a completely simple semiring. Then there is an order preserving one-one correspondence between the set of all balanced order relations on $S$ onto the set of all admissible triples of orders of $I, R$ and $\Lambda$ respectively.*

***Proof.*** Let $\mathscr{B}(S)$ denote the set of all balanced order relations on a completely simple semiring $S = \mathscr{M}(I, R, \Lambda; P)$ and $\mathscr{A}\mathscr{T}(S)$ denote the set of all admissible triples of orders of $I, R$ and $\Lambda$, respectively. Then $\mathscr{B}(S)$ and $\mathscr{A}\mathscr{T}(S)$ are both lattices with respect to set inclusion.

We define $\rho : \mathscr{B}(S) \longrightarrow \mathscr{A}\mathscr{T}(S)$ by $\rho(\varrho) = (\varrho_I, U_\varrho, \varrho_\Lambda)$ for all $\varrho \in \mathscr{B}(S)$.

Since $(S, +)$ is a completely simple semigroup, we find from Theorem 2.9, that $\rho$ is order preserving and injective.

To complete the proof it suffices to show that $\rho$ is onto. For this let, $(\xi, U, \eta)$ be an admissible triple of orders of $I, R$ and $\Lambda$ respectively. We define a relation $\varrho$ on $S$ by $(i, a, \lambda)\, \varrho\, (j, b, \mu)$ if and only if $(i, j) \in \xi, (\lambda, \mu) \in \eta$ and $-p_{\nu,i} + p_{\nu,j} + b + p_{\mu,k} - p_{\lambda,k} - a \in U$ for some $\nu \in \Lambda$ and $k \in I$. Then by Theorem 2.9, stated above $\varrho$ is a balanced order relation on $(S, +)$. Moreover, by Theorem 2.9, we have $\varrho_I = \xi$, $\varrho_\Lambda = \eta$ and $U_\varrho = U$.

We now show that $\varrho$ is stable under multiplication. For this let $(i, a, \lambda)$, $(j, b, \mu), (k, c, \nu) \in S$ and $(i, a, \lambda)\varrho(j, b, \mu)$. Then $(i, j) \in \xi, (\lambda, \mu) \in \eta$ and $-p_{\nu',i} + p_{\nu',j} + b + p_{\mu,k'} - p_{\lambda,k'} - a \in U$ for some $\nu' \in \Lambda$ and $k' \in I$. Now, $(i, j) \in \xi$ implies $(ik, jk) \in \xi$. Similarly, $(\lambda\nu, \mu\nu) \in \eta$.

Again, $-p_{\nu',i} + p_{\nu',j} + b + p_{\mu,k'} - p_{\lambda,k'} - a \in U$ implies

$$(-p_{\nu,i} + p_{\nu,j} + b + p_{\mu,k} - p_{\lambda,k} - a)c \in U,$$

i.e., $bc - ac \in U$ [by (1.4) of Theorem 1.1]. This leads to,

$$-p_{\lambda\nu,ik} + p_{\lambda\nu,jk} + (-p_{\mu\nu,jk} + bc) + p_{\mu\nu,jk} - p_{\lambda\nu,jk} - (-p_{\lambda\nu,ik} + ac) \in U.$$

Hence $(ik, -p_{\lambda\nu,ik} + ac, \lambda\nu)\, \varrho\, (jk, -p_{\mu\nu,jk} + bc, \mu\nu)$. This implies

$$(i, a, \lambda)(k, c, \nu)\, \varrho\, (j, b, \mu)(k, c, \nu).$$

Similarly, we can show that $(k, c, \nu)(i, a, \lambda)\, \varrho\, (k, c, \nu)(j, b, \mu)$ and the proof is completed. ∎

## 3. Normal extensions

For the remaining part of this paper, let $E^+(S)$ denote the set of all additive idempotents of the semiring $S$ and by $[E^+(S)]$ we mean the subsemiring of $S$ generated by $E^+(S)$.

**Definition 3.1.** An ideal $K$ of a completely simple semiring $S$ is said to be a normal ideal of $S$, and $S$ is said to be a normal extension of $K$, if (i) $K$ is a completely simple semiring and (ii) $x' + K + x \subseteq K$ for all $x \in S$.

Here it is interesting to mention that the requirement that $x' + K + x \subseteq K$ for all $x \in S$, ensures that $K$ has non-null intersection with every $\mathscr{H}^+$-class of $S$ and so is full, i.e., $E^+(S) \subseteq K$.

**Notation 3.2.** For $S = \mathscr{M}(I, R, \Lambda; P)$, we will write

$$\begin{aligned} \overline{P} &= \text{subskew-ring of } R \text{ generated by the entries of } P, \\ \mathscr{K} &= \text{set of normal ideals of } S, \\ \mathscr{N} &= \text{set of skew-ideals of } R \text{ containing } \overline{P} \end{aligned}$$

where $\mathscr{K}$ and $\mathscr{N}$ are both lattices with respect to set inclusion.

Similar to Rees matrix semigroup, one can easily show the following lemma.

**Lemma 3.3.** *If $S = \mathscr{M}(I, R, \Lambda; P)$, then $[E^+(S)] = \mathscr{M}(I, \overline{P}, \Lambda; P)$.*

**Definition 3.4.** A completely simple semiring $S$ is said to be an essential extension of a normal ideal $K$ if the restriction to $K$ of any non-trivial congruences on $S$ is non-trivial.

**Theorem 3.5.** *Let $S = \mathscr{M}(I, R, \Lambda; P)$.*

(i) *The mappings*

$$K \longrightarrow K^* = \{a \in R : (o, a, o) \in K\},$$
$$N \longrightarrow N^* = \{(i, a, \lambda) \in S : a \in N\},$$

*are mutually inverse lattice isomorphisms of $\mathscr{K}$ and $\mathscr{N}$.*

(ii) *$S$ is an essential extension of $K \in \mathscr{K}$ if and only if $R$ is an essential extension of $K^*$.*

***Proof.*** (i) By Theorem 2.7 (i) [4], we see that $(K^*, +)$ is a normal subgroup of $R$ containing $\overline{P}$. To show $K^*$ is a skew-ideal of $R$, let $a \in K^*$ and $r \in R$. Then $(o, a, o) \in K$. Now, $(o, r, o) \in S$. Since $K$ is an ideal of $S$, so it follows that $(o, a, o)(o, r, o) \in K$, i.e., $(o, ar, o) \in K$. This implies $ar \in K^*$. Similarly, we can show that $ra \in K^*$. Hence $K^*$ is a skew-ideal of $R$ containing $\overline{P}$.

Again, if $N$ is a skew-ideal of $R$ then by Theorem 2.7 (i) [4], $N^*$ is a normal subsemigroup of $(S, +)$. To show $N^*$ is a normal ideal of $S$, let $(i, a, \lambda) \in N^*$ and $(j, r, \mu) \in S$. Then $(i, a, \lambda)(j, r, \mu) = (ij, -p_{\lambda\mu, ij} + ar, \lambda\mu) \in N^*$, since $-p_{\lambda\mu, ij} + ar \in N$. Similarly, we can show that $(j, r, \mu)(i, a, \lambda) \in N^*$. Consequently, $N^*$ is a normal ideal of $S$.

From Theorem 2.7 (i) [4], it follows that $(K^*)^* = K$ and $(N^*)^* = N$. Hence the theorem.

(ii) This part follows from Theorem 2.7 (ii) [4]. ∎

From Theorem 3.5 (i) we can conclude that any normal ideal is completely determined by its intersection with $H_e^+$, $e = (o, 0, o)$. Since the concept of a normal ideal is quite independent of any particular representation as a Rees matrix semiring, we have

**Corollary 3.6.** *Let $K$ be a normal ideal of a completely simple semiring $S$. Then $K$ is determined by its intersection with any $\mathscr{H}^+$-class of $S$.*

## 4. The normal hull

In this section we define the normal hull of a completely simple semiring. In this regard, we point out that if $S$ is a completely simple semiring and $e \in E^+(S)$, then $e + S + e$ is a subskew-ring of $S$ and conversely for any subskew-ring $R$ of $S$ there exists $f \in E^+(S)$ such that $R = f + S + f$.

**Lemma 4.1.** *Let $S$ be a completely simple semiring and $a, b \in S$. Then $(a+b)' = (a+b)^0 + b' + (b+a)^0 + a' + (a+b)^0$.*

***Proof.*** This follows from Lemma 3.2 [4]. ■

In the next lemma we define two mappings which will be useful in the remaining part of this paper.

**Lemma 4.2.** *Let $S$ be a completely simple semiring and $e, f \in E^+(S)$. Then the mapping*

$$\phi_{e,f} : x \to (f+e)^0 + x + f \qquad (x \in H_e^+)$$

*is an isomorphism of $H_e^+$ onto $H_f^+$ with inverse*

$$\phi_{e,f}^{-1} : z \to e + z + (f+e)^0 \qquad (z \in H_f^+). \tag{4.1}$$

***Proof.*** Clearly, $\phi_{e,f}$ is a mapping from $H_e^+$ to $H_f^+$. By Theorem 3.3 [4], we at once have the mapping in (4.1) is the inverse of $\phi_{e,f}$ and $\phi_{e,f}$ is an isomorphism from $(H_e^+, +)$ onto $(H_f^+, +)$.

To show $\phi_{e,f}$ is a homomorphism under multiplication, let $x, y \in H_e^+$. Now,

$$\begin{aligned}
&(x\phi_{e,f})(y\phi_{e,f})\\
&= \Big((f+e)^0 + x + f\Big)\Big((f+e)^0 + y + f\Big)\\
&= \Big((f+e)^0 + x\Big)\Big((f+e)^0 + y\Big) + \Big((f+e)^0 + x\Big)f + f\Big((f+e)^0 + y\Big) + f^2\\
&= (f+e)^0 + (f+e)^0e + e(f+e)^0 + xy + f + ef + ef + f + fe + fe + f\\
&= (f+e)^0 + fe + e + ef + e + xy + f + ef + f + fe + f\\
&= (f+e)^0 + xy + f\\
&= (xy)\phi_{e,f}.
\end{aligned}$$

Consequently, the mapping $\phi_{e,f}$ is an isomorphism. ■

**Notation 4.3.** Let $S = \mathscr{M}(I, R, \Lambda; P)$ be a completely simple semiring. Then for all $i \in I$ and $\lambda \in \Lambda$, we define $\theta_{p_{\lambda,i}} : R \longrightarrow R$ by $x\theta_{p_{\lambda,i}} = -p_{\lambda,i} + x + p_{\lambda,i}$, for all $x \in R$. Then it is easy to verify that $\theta_{p_{\lambda,i}}$ is an automorphism of $R$. For any skew-ring $R$, we will denote by $\mathscr{A}(R)$, the automorphism group of $R$.

**Lemma 4.4.** *Let $S$ be a completely simple semiring. Then for any three elements $e, f, g \in E^+(S)$, $\phi_{e,g}\theta_{p_{(g+e)^0,(f+g)^0}}\phi_{g,f} = \phi_{e,f}$. Moreover, if $e\mathscr{L}^+ g$ or $g\mathscr{R}^+ f$, then $\phi_{e,g}\phi_{g,f} = \phi_{e,f}$.*

***Proof.*** Follows from Lemma 3.5 [4]. ■

**Notation 4.5.** For any completely simple semiring $S$, let

$$\Phi(S) = \bigcup \left\{ \mathscr{A}(H_e^+) : e \in E^+(S) \right\}.$$

We define a binary operation $\star$ on $\Phi(S)$ by

$$\alpha \star \beta = \phi_{(e+f)^0,\, e} \alpha \; \phi_{e,f} \; \beta \phi_{f,\, (e+f)^0} \tag{4.2}$$

for all $\alpha \in \mathscr{A}(H_e^+)$, $\beta \in \mathscr{A}(H_f^+)$. Here we note that $\alpha \star \beta \in \mathscr{A}(H_{(e+f)^0}^+)$ and thus $\Phi(S)$ forms a semigroup.

We now prove that $\Phi(S)$ is a completely simple semigroup with respect to the operation defined in (4.2) by establishing an isomorphism with a Rees matrix semigroup of the following form.

**Definition 4.6.** For a completely simple semiring $S = \mathscr{M}(I, R, \Lambda; P)$, the completely simple semigroup $\mathscr{M}(I, \mathscr{A}(R), \Lambda; P^*)$ where $p_{\lambda,i}^* = \theta_{p_{\lambda,i}}$ is defined to be the automorphism semigroup of the semiring $S$ and it is denoted by $\mathscr{A}(S)$.

**Lemma 4.7.** *Let $S$ be a completely simple semiring. Then the mapping*

$$\psi : \alpha \to ((e+g)^0, \phi_{g,e}\, \alpha\, \phi_{e,g}, (g+e)^0) \;\; \Big(\alpha \in \mathscr{A}(H_e^+) \subseteq \Phi(S);\, e, g \in E^+(S)\Big),$$

*is an isomorphism of $\Phi(S)$ onto $\mathscr{A}(S)$.*

***Proof.*** The proof follows from Theorem 3.10 [4]. ∎

**Definition 4.8.** For any completely simple semiring $S$, the (completely simple) semigroup $\Phi(S)$ is called the normal hull of $S$.

**Corollary 4.9.** *Let $S = \mathscr{M}(I, R, \Lambda; P)$, $S' = \mathscr{M}(I', R', \Lambda'; P')$ be two isomorphic completely simple semirings. Then $\mathscr{A}(S) \cong \mathscr{A}(S')$.*

***Proof.*** Since the definition of $\Phi(S)$ is independent of the matrix representation of $S$, $\Phi(S)$ and $\Phi(S')$ are isomorphic. Hence, by Lemma 4.7, $\mathscr{A}(S) \cong \Phi(S) \cong \Phi(S') \cong \mathscr{A}(S')$. ∎


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## References

[1] M.K. Sen, S.K. Maity and H.J. Weinert, *Completely simple semirings*, Bull. Cal. Math. Soc **97** (2005) 163–172.

[2] M.K. Sen, S.K. Maity and K.P. Shum, *On completely regular semirings*, Bull. Cal. Math. Soc **98** (2006) 319–328.

[3] M.P. Grillet, *Semirings with a completely simple additive semigroup*, J. Austral. Math. Soc.(Series A) **20** (1975) 257–267. doi:10.1017/S1446788700020607

[4] M. Petrich and N. Reilly, *The normal hull of a completely simple semigroup*, J. Algebra **81** (1983) 232–257. doi:10.1016/0021-8693(83)90218-1

[5] S.M. Goberstein, *Balanced order relations on completely simple semigroups*, Semigroup Forum **30** (1984) 121–124. doi:10.1007/BF02573443

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