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INTERVALS OF CERTAIN CLASSES OF Z-MATRICES

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Abstract

Let A and B be M-matrices satisfying $A \leq B$ and $J = [A, B]$ be the set of all matrices C such that $A \leq C \leq B$, where the order is component wise. It is rather well known that if A is an M-matrix and B is an invertible M-matrix and $A \leq B$, then $aA + bB$ is an invertible M-matrix for all $a, b > 0$. In this article, we present an elementary proof of a stronger version of this result and study corresponding results for certain other classes as well.

Keywords: interval matrix, M-matrix, N-matrix, N_0-matrix, nonnegativity.

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1. Introduction and Preliminaries

Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the reals. $T \in \mathbb{R}^{m\times n}$ is said to be nonnegative denoted $T \geq 0$, if each entry of T is nonnegative. $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if all the off-diagonal entries of A are nonpositive. Let $\mathbb{Z}$ denote the set of all Z-matrices. It follows that a Z-matrix A can be written as $A = sI - B$, where $s \geq 0$ and $B \geq 0$.

Let A be Z-matrix with a decomposition as above. Then

(a) A is called an M-matrix, if $s \geq \rho(B)$, where $\rho(.)$ denotes the spectral radius. Let A be an M-matrix. Then A is invertible if $s > \rho(B)$ and singular if $s = \rho(B)$. It is a well known result that if $s > \rho(B)$, then $A^{-1} \geq 0$ [1].

(b) A is called an N-matrix, if $\rho_{n-1}(B) < s < \rho(B)$, where $\rho_{n-1}(.)$ denotes the maximum of the spectral radii of all the principal submatrices of B of order $n-1$ [4].

(c) A is called an N_0-matrix, if $\rho_{n-1}(B) \leq s < \rho(B)$ [4].

Let $\mathbb{M}, \mathbb{M}_{inv}, \mathbb{M}_{sing}, \mathbb{N}$ and $\mathbb{N}_0$ denote the set of all M-matrices, invertible M-matrices, singular M-matrices, N-matrices and N_0-matrices, respectively.

For $A, B \in \mathbb{R}^{n\times n}$ with $A \leq B$, define $J = [A, B] = \{C \in \mathbb{R}^{n\times n} : c_{ij} = t_{ij}a_{ij} + (1-t_{ij})b_{ij},\ t_{ij} \in [0,1]$ for all $i, j \in \{1, \ldots, n\}\}$ and $int(J) = \{C \in \mathbb{R}^{n\times n} : c_{ij} = t_{ij}a_{ij} + (1-t_{ij})b_{ij},\ t_{ij} \in (0,1)$ for all $i, j \in \{1, \ldots, n\}\}$.

It is well known that certain classes of Z-matrices (for example, M-matrices and N_0-matrices) are closed under positive scalar multiplication, but are not closed under addition. In [5], Ky Fan showed that if A and B are nonsingular M-matrices with $A \leq B$, then $A + B$ is also a nonsingular M-matrix. In [8], Smith and Hu proved that if A is an M-matrix and B is a nonsingular M-matrix with $A \leq B$, then $aA+bB$ is a nonsingular M-matrix for all $a, b > 0$. Their proof was based on the existence of a certain semi-positive vector and the principle of mathematical induction. In this paper we extend this result and give a new linear algebraic proof using elementary arguments. More generally, the objective of the present work is to address the following problem: Let K_1, K_2 denote any of the classes $\mathbb{M}_{inv}, \mathbb{M}_{sing}, \mathbb{N}, \mathbb{N}_0$. Suppose that $A \in K_1$ and $B \in K_2$ with $A \leq B$. Does it follow that $J \subseteq K_1$ or K_2? If the answer is in the affirmative, we demonstrate that with a proof. If the inclusion is not true, in general, we illustrate this fact by means of an example and then consider the inclusion $int(J) \subseteq K_1$ or K_2.

The subsets of Z-matrices considered in this article arise in many problems of optimization. Let us only mention that N-matrices have been studied by many authors in connection with the linear complementarity problem, for instance [7]. One of the most widely considered classes of Z-matrices is the subclass $\mathbb{M}_{inv}$. These matrices arise not only with reference to linear complementarity problems ([2], for a survey on many of these matrix classes in the context of the linear complementarity problem) but also in other classical areas such as finite difference methods in elliptic partial differential equations. Our work reported here is expected to have applications in perturbation considerations in the nature of solutions of linear complementarity problems defined in terms of these matrix classes.

The paper is organized as follows. In the rest of this introductory section, we collect certain preliminary results that will used in the sequel. In the next section, we prove the main results. In Theorem 2.3, we show that if A is a singular M-matrix and B is an invertible M-matrix, then any matrix in $int(J)$ is an invertible M-matrix. Theorem 2.4 shows that if A and B are both singular M-matrices then any matrix in J must also be a singular M-matrix. If A is an

N_0-matrix and B is a singular M-matrix, then any matrix in $int(J)$ must also be an N_0-matrix. This is proved in Theorem 2.5. Theorem 2.6 presents a result for N_0-matrices, analogous to Theorem 2.4. Theorem 2.7 shows that if $A \in \mathbb{N}_0, B$ is an invertible M-matrix and if $C \in int(J)$ then C is either an invertible M-matrix or a singular M-matrix or an N_0-matrix depending on the sign of its determinant. In Theorem 2.8, a similar result is proved when $A \in \mathbb{N}$ and B is an invertible M-matrix. Theorem 2.10 shows that if $A \in \mathbb{N}_0$ and $B \in \mathbb{N}$, then $int(J) \subseteq \mathbb{N}$. The concluding result shows that if $A \in \mathbb{N}$, $B \in \mathbb{N}_0$ and $A \leq B$, then $B \in \mathbb{N}$.

Let us recall that a permutation matrix is a square matrix in which each row and each column has one entry unity, all others being zero. It follows that, $A \in \mathbb{M}$ if and only if $QAQ^t \in \mathbb{M}$, for any permutation matrix Q [1].

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be reducible if there exist an $n \times n$ permutation matrix Q such that $QAQ^t = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, where A_{11} is an $r \times r$ sub matrix and A_{22} is an $(n-r) \times (n-r)$ sub matrix with $1 \leq r < n$. If no such permutation matrix Q exists, then A is said to be irreducible.

The following block representation for a reducible matrix will be crucially used in the first main result.

Theorem 1.1 (Page 51, [9]). *Let $A \in \mathbb{R}^{n\times n}$ be reducible. Then there exists a permutation matrix Q such that*

$$QAQ^t = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{pmatrix},$$

where each square submatrix $A_{ii}, 1 \leq i \leq m$, is either irreducible or a 1×1 zero matrix and the eigenvalues of A are precisely the eigenvalues of the square submatrices A_{ii}.

The following two results collect important properties of the spectral radius.

Theorem 1.2 (Theorem 2.20, [9]). *Let $A, B \in \mathbb{R}^{n\times n}$ with $0 \leq A \leq B$. Then $\rho(A) \leq \rho(B)$.*

Theorem 1.3 (Theorem 2.1, [9]). *Let $A \geq 0$ be an irreducible matrix. Then $\rho(A)$ strictly increases when any entry of A increases.*

Finally, let us state a result for N_0-matrices.

Theorem 1.4 (Lemma 2.1, [4]). *Let $A \in \mathbb{R}^{n\times n}$. Then $A \in \mathbb{N}_0$ if and only if all principal submatrices of A belong to $\mathbb{M}$ and A has negative determinant.*

2. Main Results

We begin with the following fundamental result.

Lemma 2.1. Let A be an invertible M-matrix and $A = tI - C$, with $t > \rho(C)$. Then for any $s \geq t$, we have $A = sI - D$ with $D \geq 0$ and $s > \rho(D)$.

Proof. Let $A = tI - C$ with $t > \rho(C)$. Then $A = sI - D = sI - ((s-t)I + C)$ and $\rho(D) = \rho((s-t)I + C) = s - t + \rho(C) < s$. Hence the proof. ■

Next, we prove a simple well known result concerning invertible M-matrices.

Theorem 2.1 (Fact 4, page 9–19, [3])**.** *Let $A \in \mathbb{M}_{inv}$, $B \in \mathbb{Z}$ and $A \leq B$. Then $B \in \mathbb{M}_{inv}$.*

Proof. Let $A \in \mathbb{M}_{inv}$, $B \in \mathbb{Z}$ and $A \leq B$. Set $s = max\ b_{ii} + 1$. Then $A = sI - D, B = sI - E$ for some $D \geq E \geq 0$ and $s > \rho(D)$. Also $\rho(E) \leq \rho(D)$, so that $\rho(E) < s$. Thus $B \in \mathbb{M}_{inv}$. ■

Before proceeding to the main result, let us show that a rather well known result of [6] can be obtained as a corollary to Theorem 2.1. Let us reiterate the fact that if $A \leq C \leq B$, where A, B are Z-matrices, then C is also a Z-matrix.

Theorem 2.2 (Part of Theorem 3.6.5, [6])**.** *Let $J = [A, B]$. Then $J \subseteq \mathbb{M}_{inv}$ if and only if $A, B \in \mathbb{M}_{inv}$.*

Proof. If $J \subseteq \mathbb{M}_{inv}$, then (obviously), $A, B \in \mathbb{M}_{inv}$. Conversely, suppose that $A, B \in \mathbb{M}_{inv}$ and $C \in J$. Now, $A \leq C \leq B$ with $A \in \mathbb{M}_{inv}$ and $C \in \mathbb{Z}$. Again, by Theorem 2.1, it follows that $C \in \mathbb{M}_{inv}$. So $J \subseteq \mathbb{M}_{inv}$. ■

Let us recall the result mentioned earlier. If $a, b > 0$ and if A, B are invertible M-matrices, then $aA + bB$ is an invertible M-matrix. In the next result, we show that there are many more invertible M-matrices of which $aA + bB$ is just one type. Our approach is much simpler than the proof of [8]. This is our main result.

Theorem 2.3. *Let $A, B \in \mathbb{Z}$ and $A \leq B$. If $A \in \mathbb{M}_{sing}$ and $B \in \mathbb{M}_{inv}$, then $int(J) \subseteq \mathbb{M}_{inv}$.*

Proof. Let $C \in int(J)$. Then $c_{ij} = t_{ij}a_{ij} + (1 - t_{ij})b_{ij}$ with $t_{ij} \in (0, 1)$. Since $a_{ij} \leq c_{ij} \leq b_{ij} \leq 0$, so $a_{ij} = 0$ if and only if $c_{ij} = 0$, for $i \neq j$. By Lemma 2.1, there exists an s such that $A = sI - D, B = sI - E, C = sI - F$ for some $D \geq F \geq E \geq 0$ and $s = \rho(D), s > \rho(E)$ (Such a common s could be chosen by Lemma 2.1).

Let A be irreducible. Then D is irreducible. So $\rho(F) < \rho(D) = s$ and hence $C \in \mathbb{M}_{inv}$, as was required to prove.

Suppose that A is reducible. Then there exists a permutation matrix Q such that $QAQ^t = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{pmatrix}$, where each A_{ii} is either irreducible or it is a 1×1 zero matrix. Now, applying the same permutation to B and C, we obtain $QBQ^t = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ 0 & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{mm} \end{pmatrix}$ and $QCQ^t = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ 0 & C_{22} & \cdots & C_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{mm} \end{pmatrix}$. Also, if A_{ii} is irreducible, then C_{ii} is irreducible, for each i. Let $QAQ^t = sI - L, QBQ^t = sI - M, QCQ^t = sI - N$ for some $L \geq N \geq M \geq 0$ and $s = \rho(L), s > \rho(M)$. Then $L = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1m} \\ 0 & L_{22} & \cdots & L_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{mm} \end{pmatrix}, M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ 0 & M_{22} & \cdots & M_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{mm} \end{pmatrix}$ and $N = \begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1m} \\ 0 & N_{22} & \cdots & N_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{mm} \end{pmatrix}$. Also, $\sigma(L) = \bigcup \sigma(L_{ii})$ and $\sigma(N) = \bigcup \sigma(N_{ii})$, where $\sigma(.)$ denotes the spectrum of the matrix. Suppose that A_{ii} is an irreducible M-matrix for some i. Then, arguing as above, we have that $C_{ii} \in \mathbb{M}_{inv}$. Since $QCQ^T = sI - N$, it now follows that $sI - N_{ii}$ is invertible. Already, $s \geq 0$ and $N_{ii} \geq 0$. Hence $s > \rho(N_{ii})$. This argument can be applied for all i such that A_{ii} is irreducible. Since $\rho(N) = max\ \rho(N_{ii})$, it follows that QCQ^t is an invertible M-matrix. Thus $C \in \mathbb{M}_{inv}$. ■

Corollary 2.1 (Theorem 3.5, [8])**.** Let $A, B \in \mathbb{Z}$ and $A \leq B$. If $A \in \mathbb{M}_{sing}$ and $B \in \mathbb{M}_{inv}$, then $aA + bB \in \mathbb{M}_{inv}$, for all $a, b > 0$.

Proof. Let $\lambda \in (0, 1)$. We then have $\lambda A + (1 - \lambda)B \in int(J)$. By Theorem 2.3, $aA + bB = (a + b)(\lambda A + (1 - \lambda)B) \in \mathbb{M}_{inv}$, with $\lambda = \frac{a}{a+b}$. ■

Remark 2.1. The following example shows that the conclusion in Theorem 2.3 is stronger then the conclusion in Corollary 2.1. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and

$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $A \in \mathbb{M}_{sing}, B \in \mathbb{M}_{inv}$. Let $C = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{3} & \frac{3}{2} \end{pmatrix}$, then $C \in \mathbb{M}_{inv}$ and $C \in int(J)$. It is easy to verify that C is not of the form $aA + bB$ for some $a, b > 0$. Let $F = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then $F \in J$ but $F \notin int(J)$. So the result is not true for the interval J, in general.

In the following theorem, we give a condition on the matrices A and B so that all the matrices in the interval J are M-matrices. This generalizes Theorem 3.6, [8].

Theorem 2.4. *Let $A, B \in \mathbb{M}_{sing}$ and $A \leq B$. Then $J \subseteq \mathbb{M}_{sing}$.*

Proof. Let $C \in J$ and $s = max\ b_{ii}+1$. Then $A = sI - D, B = sI - E, C = sI - F$ for some $D \geq F \geq E \geq 0$ and $s = \rho(D) = \rho(E)$. Also $\rho(E) \leq \rho(F) \leq \rho(D)$, so that $\rho(F) = s$. Thus $C \in \mathbb{M}_{sing}$. ■

Corollary 2.2 (Theorem 3.6, [8])**.** Let $A, B \in \mathbb{M}_{sing}$ and $A \leq B$. Then $aA+bB \in \mathbb{M}_{sing}$ for all $a, b > 0$.

In the following theorem, we give a condition on the matrices A and B so that all the matrices in the set $int(J)$ are N_0-matrices. This generalizes Theorem 3.7, [8].

Theorem 2.5. *Let $A \in \mathbb{N}_0, B \in \mathbb{M}_{sing}$ and $A \leq B$. Then $int(J) \subseteq \mathbb{N}_0$.*

Proof. Let $C \in int(J)$ and $s = max\ b_{ii} + 1$. Then $A = sI - D, B = sI - E, C = sI - F$ for some $D \geq F \geq E \geq 0$ and $\rho_{n-1}(D) \leq s < \rho(D), s = \rho(E)$. Since $A \in \mathbb{N}_0$ implies that A is irreducible it follows that C is also irreducible. There fore $\rho(F) > \rho(E) = s$. Since $F \leq D$, we have $\rho_{n-1}(F) \leq \rho_{n-1}(D)$ and $\rho_{n-1}(D) \leq s$, so that $\rho_{n-1}(F) \leq s < \rho(F)$. Thus $C \in \mathbb{N}_0$. ■

Corollary 2.3 (Theorem 3.7, [8])**.** Let $A \in \mathbb{N}_0, B \in \mathbb{M}_{sing}$ and $A \leq B$. Then $aA + bB \in \mathbb{N}_0$ for all $a, b > 0$.

Remark 2.2. The following example shows that the conclusion of Theorem 2.5 is stronger then the conclusion of Corollary 2.3. Let $A = \begin{pmatrix} \frac{1}{2} & -3 \\ -3 & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. By appealing to Theorem 1.4, it follows that $A \in \mathbb{N}_0$. Clearly, $B \in \mathbb{M}_{sing}$. Let $C = \begin{pmatrix} 1 & \frac{-5}{2} \\ \frac{-9}{4} & 1 \end{pmatrix}$. Then $C \in int(J)$. Once again, by Theorem 1.4, it follows that $C \in \mathbb{N}_0$. C is not of the form $aA + bB$ for any $a, b > 0$.

In the following theorem we give a condition on the matrices A and B so that all the matrices in the set J are N_0-matrices. This generalizes Theorem 3.10, [8].

Theorem 2.6. *Let $A, B \in \mathbb{N}_0$ with $A \leq B$. Then $J \subseteq \mathbb{N}_0$.*

Proof. Let $C \in J$ and set $s = max\ b_{ii} + 1$. Then $A = sI - D, B = sI - E, C = sI - F$ for some $D \geq F \geq E \geq 0$ and $\rho_{n-1}(D) \leq s < \rho(D), \rho_{n-1}(E) \leq s < \rho(E)$. Since A and B are irreducible it follows that C is irreducible and $\rho(D) > \rho(F) > \rho(E)$. Hence $s < \rho(F)$ and $\rho_{n-1}(F) \leq \rho_{n-1}(D) \leq s$. Thus $C \in \mathbb{N}_0$. ■

Corollary 2.4 (Theorem 3.10, [8])**.** Let $A, B \in \mathbb{N}_0$ with $A \leq B$. Then $aA + bB \in \mathbb{N}_0$ for all $a, b > 0$.

In the following theorem we give some conditions on the matrices A and B so that the matrices in the set $int(J)$ belong to the class $\mathbb{M}_{inv}, \mathbb{M}_{sing}$ and $\mathbb{N}_0$ provided $det\ C > 0, det\ C = 0$ and $det\ C < 0$ respectively. This generalizes Theorem 3.9, [8].

Theorem 2.7. *Let $A \in \mathbb{N}_0$ and $B \in \mathbb{M}_{inv}$, with $A \leq B, A \neq B$ and $C \in int(J)$. Then*

(a) *$C \in \mathbb{M}_{inv}$ if and only if $det\ C > 0$,*

(b) *$C \in \mathbb{M}_{sing}$ if and only if $det\ C = 0$,*

(c) *$C \in \mathbb{N}_0$ if and only if $det\ C < 0$.*

Proof. By Theorem 1.4, all the principal sub matrices of A belong to $\mathbb{M}$. Also, all the principal submatrices of B belong to $\mathbb{M}_{inv}$ and hence all the principal submatrices of C belong to $\mathbb{M}_{inv}$ (by Theorem 2.3). The result now follows. ■

Remark 2.3. In the above theorem, if we replace the condition $B \in \mathbb{M}_{inv}$ by $B \in \mathbb{M}_{sing}$, then the same conclusions hold.

Corollary 2.5 (Theorem 3.9, [8])**.** Let $A \in \mathbb{N}_0$ and $B \in \mathbb{M}_{inv}$, with $A \leq B$. Then, for all $a, b > 0$

(a) $aA + bB \in \mathbb{M}_{inv}$ if and only if $det\ (aA + bB) > 0$,

(b) $aA + bB \in \mathbb{M}_{sing}$ if and only if $det\ (aA + bB) = 0$,

(c) $aA + bB \in \mathbb{N}_0$ if and only if $det\ (aA + bB) < 0$.

Remark 2.4. Let $A = \begin{pmatrix} \frac{1}{2} & -3 \\ -3 & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Then $A \in \mathbb{N}_0$ and $B \in \mathbb{M}_{inv}$. Now, consider the matrix $C_1 = \begin{pmatrix} \frac{3}{2} & -1 \\ -2 & \frac{3}{2} \end{pmatrix}$. Then $C_1 \in int(J)$ and $det(C_1) > 0$, so that $C_1 \in \mathbb{M}_{inv}$. C_1 is not of the form $aA + bB$ for any $a, b > 0$. If

$C_2 = \begin{pmatrix} \frac{3}{2} & -1 \\ \frac{-3}{2} & 1 \end{pmatrix}$ and $C_3 = \begin{pmatrix} \frac{3}{2} & -2 \\ -3 & \frac{3}{2} \end{pmatrix}$, then $C_2 \in int(J)$, and $det(C_2) = 0$ so that $C_2 \in \mathbb{M}_{sing}$. Also $C_3 \in int(J)$ and $det(C_3) < 0$, so that $C_3 \in \mathbb{N}_0$. Again, both C_2 and C_3 are not of the form $aA + bB$ for any $a, b > 0$.

The following is a special case of Theorem 2.7 since $\mathbb{N} \subseteq \mathbb{N}_0$.

Theorem 2.8. *Let $A \in \mathbb{N}$ and $B \in \mathbb{M}_{inv}$, with $A \leq B$ and $C \in int(J)$. Then*

(a) *$C \in \mathbb{M}_{inv}$ if and only if det $C > 0$,*

(b) *$C \in \mathbb{M}_{sing}$ if and only if det $C = 0$,*

(c) *$C \in \mathbb{N}$ if and only if det $C < 0$.*

Remark 2.5. In the above theorem, if we replace the condition $B \in \mathbb{M}_{inv}$ by $B \in \mathbb{M}_{sing}$, then the same conclusions hold.

Remark 2.6. Suppose $A \in \mathbb{M}_{sing}, B \in \mathbb{Z}$ and $A \leq B$. Set $s = max\ b_{ii} + 1$. Then $A = sI - D, B = sI - E$ for some $D \geq E \geq 0$ and $s = \rho(D)$. Also $\rho(E) \leq \rho(D)$, so that $\rho(E) \leq s$. Thus $B \notin \mathbb{N}_0$.

The following result was proved by Ky Fan [5].

Theorem 2.9 (Lemma 3, [5])**.** *Let $A, B \in \mathbb{N}$ such that $A \leq B$. Then $J \subseteq \mathbb{N}$.*

In the following result, we show that if we replace the condition $A \in \mathbb{N}$ by $A \in \mathbb{N}_0$ then all the matrices in the set $int(J)$ belong to $\mathbb{N}$. Also, we give a counter example to show that the result is not true for the interval J, in general.

Theorem 2.10. *If $A \in \mathbb{N}_0, B \in \mathbb{N}$ and $A \leq B$. Then $int(J) \subseteq \mathbb{N}$.*

Proof. Let $C \in int(J)$ and set $s = max\ b_{ii} + 1$. Then $A = sI - D, B = sI - E, C = sI - F$ for some $D \geq F \geq E \geq 0$ and $\rho_{n-1}(D) \leq s < \rho(D), \rho_{n-1}(E) < s < \rho(E)$. Hence $s < \rho(F)$. Now, by Theorem 1.4, any principal submatrix of A is an M-matrix and any principal submatrix of B is an invertible M-matrix. So $C \in int(J)$ implies that all the principal submatrices of C are invertible M-matrices. Thus $\rho_{n-1}(F) < s < \rho(F)$ and so $C \in \mathbb{N}$. ■

Remark 2.7. The following example shows that the conclusion in Theorem 2.10 need not hold for the interval J. Let $A = \begin{pmatrix} \frac{1}{4} & -4 \\ -4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{2} & -3 \\ -3 & \frac{1}{2} \end{pmatrix}$. Then $A \in \mathbb{N}_0$ and $B \in \mathbb{N}$ and $A \leq B$. Consider $C = \begin{pmatrix} \frac{1}{4} & \frac{-7}{2} \\ -4 & 0 \end{pmatrix}$, then $C \in J$ but $C \notin \mathbb{N}$.

In the concluding result of this article, we show that if $A \in \mathbb{N}$ and $B \in \mathbb{N}_0$, then B must belong to $\mathbb{N}$.

Theorem 2.11. *Let $A \in \mathbb{N}$, $B \in \mathbb{N}_0$ and $A \leq B$. Then $B \in \mathbb{N}$.*

Proof. Let $A \in \mathbb{N}$, $B \in \mathbb{N}_0$ and $A \leq B$. Set $s = max\ |b_{ii}| + 1$. Then $A = sI - D, B = sI - E$ for some $D \geq E \geq 0, \rho_{n-1}(D) < s < \rho(D)$ and $\rho_{n-1}(E) \leq s < \rho(E)$. Also $\rho_{n-1}(E) \leq \rho_{n-1}(D)$, so that $\rho_{n-1}(E) < s$. Thus $B \in \mathbb{N}$. ■

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