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# SOME CHARACTERIZATIONS OF 2-PRIMAL IDEALS OF A Γ-SEMIRING

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**Abstract**

This paper is a continuation of our previous paper entitled "On 2-primal Γ-semirings". In this paper we have introduced the notion of 2-primal ideal in Γ-semiring and studied it.

**Keywords:** Γ-semiring, nilpotent element, 2-primal Γ-semiring, 2-primal ideal, IFP (insertion of factor property), completely prime ideal, completely semiprime ideal.

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## 1. Introduction

The notion of Γ-ring was introduced by N. Nobuswa [6] in 1964. Later W.E. Barnes [19] weakened the defining conditions of a Γ-ring. The notion of Γ-semiring was introduced by M.M.K. Rao in [4, 5]. Now-a-days there has been a remarkable growth of the theory of Γ-ring as well as of Γ-semiring.

Birkenmeier-Heatherly-Lee [3] introduced the notion of 2-primal ring in 1993. A ring $R$ with identity is called 2-primal if $\mathcal{P}(R) = \mathcal{N}(R)$, where $\mathcal{P}(R)$ denotes the intersection of all prime ideals of $R$ and $\mathcal{N}(R)$ denotes the set of all nilpotent elements of $R$. An ideal $I$ of $R$ is called 2-primal if $\mathcal{P}(R/I) = \mathcal{N}(R/I)$. Birkenmeier-Heatherly-Lee obtained some characterizations of 2-primal ideal in ring. They proved that an ideal $I$ is 2-primal if and only if $\mathcal{P}(I)$ is a completely semiprime ideal of $R$.

In this paper we introduce the notion of 2-primal ideal in a $\Gamma$-semiring. We obtain some characterizations of 2-primal ideal in a $\Gamma$-semiring. Also we introduce the notion of $N_I(P)$ and $N_I^P$ etc. in $\Gamma$-semiring and using them we obtain some characterizations of 2-primal ideals.

## 2. Preliminaries

We first give the definition of a $\Gamma$-semiring.

**Definition** (See [12]). Let $S$ and $\Gamma$ be two additive commutative semigroups. Then $S$ is called a $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ (the image to be denoted by $a\alpha b$, for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$

(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Let $S$ be a $\Gamma$-semiring. If there exists an element $0 \in S$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$ and for all $\alpha \in \Gamma$ then '0' is called the zero element or simply the zero of the $\Gamma$-semiring $S$. In this case we say that $S$ is a $\Gamma$-semiring with zero.

**Throughout this paper we assume that a $\Gamma$-semiring always contains a zero element and $S^*$ denotes the set of all nonzero elements of $S$ i.e., $S^* = S \setminus \{0\}$.**

**Definition** (See [12]). Let $S$ be a $\Gamma$-semiring and $F$ be the free additive commutative semigroup generated by $S \times \Gamma$. Then the relation $\rho$ on $F$ defined by $\sum_{i=1}^{m}(x_i, \alpha_i)\rho \sum_{j=1}^{n}(y_j, \beta_j)$ if and only if $\sum_{i=1}^{m} x_i\alpha_i s = \sum_{j=1}^{n} y_j\beta_j s$ for all $s \in S$ ($m, n \in \mathbb{Z}^+$, the set of all positive integers), is a congruence on $F$. We denote the congruence class containing $\sum_{i=1}^{m}(x_i, \alpha_i)$ by $\sum_{i=1}^{m}[x_i, \alpha_i]$. Then $F/\rho$ is an additive commutative semigroup. Now $F/\rho$ forms a semiring with the multiplication defined by $(\sum_{i=1}^{m}[x_i, \alpha_i])(\sum_{j=1}^{n}[y_j, \beta_j]) = \sum_{i,j}[x_i\alpha_i y_j, \beta_j]$. We denotes this semiring by $L$ and call it the left operator semiring of the $\Gamma$-semiring $S$.

Dually, we define the right operator semiring $R$ of the $\Gamma$-semiring $S$ where $R = \{\sum_{i=1}^{m}[\alpha_i, x_i] : \alpha_i \in \Gamma, x_i \in S, i = 1, 2, \ldots, m; m \in \mathbb{Z}^+\}$ and the multiplication on $R$ is defined as $(\sum_{i=1}^{m}[\alpha_i, x_i])(\sum_{j=1}^{n}[\beta_j, y_j]) = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$.

Let $S$ be a $\Gamma$-semiring and $L$ be the left operator semiring and $R$ be the right operator semiring of $S$. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L$ (respectively $\sum_{j=1}^{n}[\nu_j, f_j] \in R$) such that $\sum_{i=1}^{m} e_i\delta_i a = a$ (respectively $\sum_{j=1}^{n} a\nu_j f_j = a$) for all $a \in S$ then $S$ is said to have the *left unity* $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively the *right unity* $\sum_{j=1}^{n}[\nu_j, f_j]$).

**Definition** (See [7])**.** If $R$ is a commutative semiring and $R-\{0\}$ is a multiplicative group then $R$ is called a $\Gamma$*-semifield*.

**Definition** (See [16])**.** Let $A$ be a nonempty subset of a $\Gamma$-semiring $S$. The right annihilator of $A$ with respect to $\Phi \subseteq \Gamma$ in $S$, denoted by $r(A,\ \Phi)$, is defined by $r(A,\ \Phi) = \{s \in S : A\Phi s = \{0\}\}$.

In particular, if $\Phi = \Gamma$ we denote $r(A,\ \Phi)$ by $ann_R(A)$. Again if $A = \{a\}$, then we denote $ann_R(A)$ by $ann_R(a)$.

Analogusly we can define left annihilator $l(\Phi,\ A)$ and for $\Phi = \Gamma$ it is denoted by $ann_L(A)$.

**Proposition 1** (See [16])**.** *The right annihilator $r(A,\ \Phi)$ of $A$ with respect to $\Phi$ in a $\Gamma$-semiring $S$ is a right ideal of $S$.*

**Remark 2.** Similar result holds for left annihilator.

For other preliminaries we refer to [17].

**Throughout this paper we assume that a $\Gamma$-semiring $S$ always contain a unity whose every ideal is a $k$-ideal.**

## 3. 2-primal ideals

We begin with the following examples of $\Gamma$-semiring in which every ideal is a $k$-ideal.

**Example 3.** Let $M$ be a $\Gamma$-ring with unity. Then $M$ is a $\Gamma$-semiring with unity and every ideal of $M$ is a $k$-ideal.

**Example 4.** Let $R$ be a $\Gamma$-ring with unity, $S = \{r\omega : r \in \mathbb{R}_0^+\}$ and $\Gamma_1 = \{r\omega^2 : r \in \mathbb{R}_0^+\}$, where $\omega$ be a cube root of unity and $\mathbb{R}_0^+$ is the set of all non negetive real numbers. Then $S$ is a $\Gamma_1$-semiring with unity with usual addition and multiplication. Also $R \times S$ is a $\Gamma \times \Gamma_1$-semiring with unity which is not a $\Gamma \times \Gamma_1$-ring but every ideal of $R \times S$ is a $k$-ideal.

**Example 5.** Let $L$ be a bounded distributive lattice with maximal element 1. Then $L$ is a $\Gamma$-semiring with unity, where $\Gamma = L$. Now $L$ is not a $\Gamma$-ring. Also every ideal of $L$ is a $k$-ideal.

Now we recall the following definitions:

**Definition** (See [13])**.** An element $a$ of a $\Gamma$-semiring $S$ is said to be *nilpotent* if for any $\gamma \in \Gamma$ there exists a positive integer $n = n(\gamma, a)$ such that $(a\gamma)^{n-1}a = 0$ and an element $a$ of a $\Gamma$-semiring $S$ is said to be *strongly nilpotent* if there exists a positive integer $n$ such that $(a\Gamma)^{n-1}a = 0$.

**Definition** (See [17])**.** A $\Gamma$-semiring $S$ is said to be a 2-*primal* $\Gamma$-*semiring* if $\mathcal{P}(S) = \mathcal{N}(S)$, where $\mathcal{P}(S)$ denotes the intersection of all prime ideals of the $\Gamma$-semiring $S$ i.e., the prime radical of $S$ and $\mathcal{N}(S)$ denotes the set of all nilpotent elements of $S$.

**Definition** (See [17])**.** A one sided ideal $I$ of a $\Gamma$-semiring $S$ is said to have the *insertion of factors property* or simply IFP if for any $a, b \in S$, $a\Gamma b \subseteq I$ implies $a\Gamma S\Gamma b \subseteq I$.

**Definition** (See [17])**.** For a prime ideal $P$ of a $\Gamma$-semiring $S$, we define
$N(P) = \{x \in S : x\Gamma S\Gamma y \subseteq \mathcal{P}(S) \text{ for some } y \in S \setminus P\}$,
$N_P = \{x \in S : x\Gamma y \subseteq \mathcal{P}(S) \text{ for some } y \in S \setminus P\}$,
$\overline{N}_P = \{x \in S : (x\Gamma)^{n-1}x \subseteq N_P, \text{ for some positive integer } n\}$.

**Definition.** Let $S$ be a $\Gamma$-semiring and $I$ be an ideal of $S$. Then $I$ is said to be a 2-*primal ideal* of $S$ if $S/I$ is a 2-primal $\Gamma$-semiring i.e. if $\mathcal{P}(S/I) = \mathcal{N}(S/I)$, where $\mathcal{P}(S/I)$ denotes the intersection of all prime ideals of the factor $\Gamma$-semiring $S/I$ and $\mathcal{N}(S/I)$ denotes the set of all nilpotent elements of $S/I$.

**Definition.** Let $I$ be any ideal of a $\Gamma$-semiring $S$ and $P$ be a prime ideal of $S$. Then we define
$N_I(P) = \{x \in S : x\Gamma S\Gamma y \subseteq \mathcal{P}(I) \text{ for some } y \in S \setminus P\}$,
$N_I^P = \{x \in S : x\Gamma y \subseteq \mathcal{P}(I) \text{ for some } y \in S \setminus P\}$,
$\overline{N_I(P)} = \{x \in S : (x\Gamma)^{n-1}x \subseteq N_I(P), \text{ for some positive integer } n\}$,
$\overline{N_I^P} = \{x \in S : (x\Gamma)^{n-1}x \subseteq N_I^P, \text{ for some positive integer } n\}$.

**Example 6.** Let $F$ be a semifield. Consider the sets:

$$S = \left\{ \begin{pmatrix} d_1 & d_2 \\ 0 & d_3 \\ 0 & 0 \end{pmatrix} : d_1, d_2, d_3 \in F \right\}, \Gamma = \left\{ \begin{pmatrix} d_4 & d_5 & d_6 \\ 0 & d_7 & d_8 \end{pmatrix} : d_4, d_5, d_6, d_7, d_8 \in F \right\}$$

and $I = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : d \in F \right\}$. Then $S$ is a 2-primal Γ-semiring with respect to the usual matrix addition and usual matrix multiplication and $I$ is a 2-primal ideal of $S$.

**Proposition 7.** *Let $S$ be a Γ-semiring and $I$ be an ideal of $S$. Then for any prime ideal $P$ we have, $N(P) \subseteq N_I(P), N_P \subseteq N_I^P, I \subseteq N_I(P) \subseteq \overline{N_I(P)}$ and $I \subseteq N_I^P \subseteq \overline{N_I^P}$.*

**Definition** (See [17])**.** A Γ-semiring $S$ is said to satisfy (SI) if for each $a \in S$, $ann_R(a)$ is an ideal of $S$.

**Definition** (See [17])**.** A Γ-semiring $S$ is said to be *SN Γ-semiring* if $\mathcal{N}(S) = \mathcal{N}_\Gamma(S)$, where $\mathcal{N}_\Gamma(S)$ is the set of all strongly nilpotent elements of $S$.

**Definition** (See [17])**.** A Γ-semiring $S$ is said to be *right symmetric* if for $a, b, c \in S$, $a\Gamma b\Gamma c = 0$ implies $a\Gamma c\Gamma b = 0$. An ideal $I$ of a Γ-semiring $S$ is said to be right symmetric if $a\Gamma b\Gamma c \subseteq I$ implies $a\Gamma c\Gamma b \subseteq I$ for $a, b, c \in S$.

Analogusly we can define *left symmetric Γ-semiring* and *left symmetric ideal.*

**Proposition 8.** *Let $S$ be a Γ-semiring and $I$ be an ideal of $S$. Then $\mathcal{P}(S/I) = \mathcal{P}(I)/I$.*

***Proof.*** Let $s/I \in \mathcal{P}(S/I)$
$\Leftrightarrow s/I \in Q/I$ for all prime ideals $Q$ of $S$ containing $I$
$\Leftrightarrow s \in Q$ for all prime ideals $Q$ of $S$ containing $I$, as $Q$ is a $k$-ideal
$\Leftrightarrow s \in \mathcal{P}(I)$
$\Leftrightarrow s/I \in \mathcal{P}(I)/I$.
Therefore, $\mathcal{P}(S/I) = \mathcal{P}(I)/I$. ∎

**Proposition 9.** *Let $S$ be an SN Γ-semiring and $I$ be an ideal of $S$. If $(x\Gamma)^{n-1}x \subseteq I \Longrightarrow x \in \mathcal{P}(I)$, then $I$ is 2-primal.*

***Proof.*** For any Γ-semiring $S$ and any ideal $I$ of $S$ we have $\mathcal{P}(S/I) \subseteq \mathcal{N}(S/I)$ (Cf. Ref. Proposition 3.10 [17]). On the other hand let, $x/I \in \mathcal{N}(S/I)$. Since $S$ is an SN Γ-semiring, $S/I$ is an SN Γ-semiring. Then there exists a positive ineger say $n$ such that $((x/I)\Gamma)^{n-1}x/I = 0/I$ which implies that $(x\Gamma)^{n-1}x \subseteq I$. By hypothesis $x \in \mathcal{P}(I)$. This implies that $x/I \in \mathcal{P}(I)/I$. Then by Proposition 8, $x/I \in \mathcal{P}(S/I)$. Thus $\mathcal{N}(S/I) \subseteq \mathcal{P}(S/I)$. Therefore, $\mathcal{P}(S/I) = \mathcal{N}(S/I)$. Hence $I$ is 2-primal. ∎

**Proposition 10.** *Let $S$ be an SN Γ-semiring and $I$ be an ideal of $S$. Then the following statements are equivalent:*

(1) *$I$ is a 2-primal ideal of $S$.*

(2) *$\mathcal{P}(I)$ is completely semiprime ideal of $S$.*

(3) *$\mathcal{P}(I)$ is a left and right symmetric ideal of $S$.*

(4) *$\mathcal{P}(I)$ has the IFP.*

***Proof.*** (1) implies (2). Let $I$ be a 2-primal ideal of $S$. Then $S/I$ is a 2-primal $\Gamma$-semiring. So $\mathcal{P}(S/I)$ is completely semiprime (Cf. Ref. Theorem 3.25 [17]). Now by Proposition 8, we have $\mathcal{P}(S/I) = \mathcal{P}(I)/I$. Thus $\mathcal{P}(I)/I$ is completely semiprime, so $\mathcal{P}(I)$ is completely semiprime.

(2) implies (3). Let $a\Gamma b\Gamma c \subseteq \mathcal{P}(I)$, where $a, b, c \in S$. Now $(c\Gamma a\Gamma b)\Gamma(c\Gamma a\Gamma b) = c\Gamma(a\Gamma b\Gamma c)\Gamma a\Gamma b \subseteq \mathcal{P}(I)$. Since $\mathcal{P}(I)$ is completely semiprime, $c\Gamma a\Gamma b \subseteq \mathcal{P}(I)$. Now $(a\Gamma b\Gamma a\Gamma c)\Gamma(a\Gamma b\Gamma a\Gamma c) = a\Gamma b\Gamma a\Gamma(c\Gamma a\Gamma b)\Gamma a\Gamma c \subseteq \mathcal{P}(I)$ as $\mathcal{P}(I)$ is an ideal of $S$. This implies that $a\Gamma b\Gamma a\Gamma c \subseteq \mathcal{P}(I)$. Again by similar argument $(b\Gamma a\Gamma c\Gamma b\Gamma a)$ $\Gamma(b\Gamma a\Gamma c\Gamma b\Gamma a) = b\Gamma a\Gamma c\Gamma b\Gamma(a\Gamma b\Gamma a\Gamma c)\Gamma b\Gamma a \subseteq \mathcal{P}(I) \Rightarrow b\Gamma a\Gamma c\Gamma b\Gamma a \subseteq \mathcal{P}(I) \Rightarrow$ $(a\Gamma c\Gamma b)\Gamma(a\Gamma c\Gamma b)\Gamma(a\Gamma c\Gamma b)\Gamma(a\Gamma c\Gamma b) = a\Gamma c\Gamma(b\Gamma a\Gamma c\Gamma b\Gamma a)\Gamma c\Gamma b\Gamma a\Gamma c\Gamma b \subseteq \mathcal{P}(I) \Rightarrow$ $a\Gamma c\Gamma b \subseteq \mathcal{P}(I)$ as $\mathcal{P}(I)$ is completely semiprime. Hence $\mathcal{P}(I)$ is a right symmetric ideal of $S$. Also $(b\Gamma a\Gamma c)\Gamma(b\Gamma a\Gamma c) = b\Gamma(a\Gamma c\Gamma b)\Gamma a\Gamma c \subseteq \mathcal{P}(I) \Rightarrow b\Gamma a\Gamma c \subseteq \mathcal{P}(I)$. Hence $\mathcal{P}(I)$ is a left symmetric ideal of $S$. Therefore, $\mathcal{P}(I)$ is a left and a right symmetric ideal of $S$.

(3) implies (4). Let $x\Gamma y \subseteq \mathcal{P}(I)$, where $x, y \in S$. Suppose $s \in S$, then $s\Gamma x\Gamma y \subseteq \mathcal{P}(I)$. As $\mathcal{P}(I)$ is left symmetric, $x\Gamma s\Gamma y \subseteq \mathcal{P}(I)$. Therefore $x\Gamma S\Gamma y \subseteq \mathcal{P}(I)$. Hence $\mathcal{P}(I)$ has the IFP.

(4) implies (1). For any $\Gamma$-semiring $S$ and any ideal $I$ of $S$ we have $\mathcal{P}(S/I) \subseteq \mathcal{N}(S/I)$. On the other hand let, $x/I \in \mathcal{N}(S/I)$. Since $S$ is an SN $\Gamma$-semiring, $S/I$ is an SN $\Gamma$-semiring. Then $((x/I)\Gamma)^{n-1}x/I = 0/I$ implies that $(x\Gamma)^{n-1}x \subseteq I$. Now we claim that $x \in \mathcal{P}(I)$. Suppose $x \notin \mathcal{P}(I)$, then there exists a prime ideal $P$ of $S$ containing $I$ such that $x \notin P$, i.e. $x \in S - P$. Since $P$ is a prime ideal of $S$, $S - P$ is an m-system. Then there exist $s_1 \in S, \alpha_1, \beta_1 \in \Gamma$ such that $x\alpha_1 s_1\beta_1 x \in S \setminus P$. Again since $x\alpha_1 s_1 \beta_1 x, x \in S \setminus P$, applying m-system property $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x \in S \setminus P$, for some $\alpha_2, \beta_2 \in \Gamma$ and $s_2 \in S$. Applying m-system property after finite step, we have $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x \dots \alpha_{n-1}s_{n-1}\beta_{n-1}x \in S \setminus P$ for some $s_i \in S, \alpha_i, \beta_i \in \Gamma$, where $i = 1, 2, \dots, (n-1)$. Since $(x\Gamma)^{n-1}x \subseteq \mathcal{P}(I)$ and $\mathcal{P}(I)$ has the IFP, $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x \dots \alpha_{n-1}s_{n-1}\beta_{n-1}x \in \mathcal{P}(I)$ i.e., $x\alpha_1 s_1\beta_1 x\alpha_2 s_2\beta_2 x \dots \alpha_{n-1}s_{n-1}\beta_{n-1}x \in P$, a contradiction. Therefore $x \in \mathcal{P}(I)$. Hence $x/I \in \mathcal{P}(I)/I = \mathcal{P}(S/I)$ by Proposition 8. So $\mathcal{P}(S/I) = \mathcal{N}(S/I)$ i.e., $S/I$ is a 2-primal $\Gamma$-semiring. Hence $I$ is a 2-primal ideal of $S$. ■

**Proposition 11.** *Let $S$ be a $\Gamma$-semiring and $I$ be an ideal of $S$. If $S/I$ satisfies (SI) then $x\Gamma y \subseteq I$ implies that $x\Gamma S\Gamma y \subseteq I$ for all $x, y \in S$ i.e., $I$ has the IFP.*

***Proof.*** Let $S$ be a $\Gamma$-semiring and $I$ be an ideal of $S$ such that $S/I$ satisfies (SI). Let $x\Gamma y \subseteq I$. Then $(x/I)\Gamma(y/I) = 0/I$. So $y/I \in ann_R(x/I)$. Since $S/I$ satisfies (SI), $(S/I)\Gamma ann_R(x/I) \subseteq ann_R(x/I)$ i.e., $(S/I)\Gamma(y/I) \subseteq ann_R(x/I)$ i.e., $(x/I)\Gamma(S/I)\Gamma(y/I) = 0/I$ i.e., $x\Gamma S\Gamma y \subseteq I$. This completes the proof. ∎

**Proposition 12.** *Let $S$ be a $\Gamma$-semiring and $I$ be an ideal of $S$. If $S/\mathcal{P}(I)$ has no nonzero nilpotent elements, then $S/\mathcal{P}(I)$ satisfy (SI).*

***Proof.*** Let $S$ be a $\Gamma$-semiring and $I$ be an ideal of $S$ such that $S/\mathcal{P}(I)$ has no nonzero nilpotent elements. Then $S/\mathcal{P}(I)$ is a 2-primal $\Gamma$-semiring (Cf. Ref. Proposition 3.11, [17]). Hence $\mathcal{P}(I)$ is a 2-primal ideal of $S$. Now by Proposition 10(4), $\mathcal{P}(\mathcal{P}(I))$ has the IFP. Now $\mathcal{P}(\mathcal{P}(I)) = \mathcal{P}(I)$ (Cf. Ref. [10]). So $\mathcal{P}(I)$ has the IFP. Now we show that for any $a/\mathcal{P}(I) \in S/\mathcal{P}(I)$, $ann_R(a/\mathcal{P}(I))$ is an ideal of $S/\mathcal{P}(I)$. Let $b/\mathcal{P}(I) \in ann_R(a/\mathcal{P}(I))$. Then $(a/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) = 0/\mathcal{P}(I)$. Then $a\Gamma b \subseteq \mathcal{P}(I)$, where $a, b \in S$. Since $\mathcal{P}(I)$ has the IFP, $a\Gamma S\Gamma b \subseteq \mathcal{P}(I)$, which implies that $(a/\mathcal{P}(I))\Gamma(S/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) = 0/\mathcal{P}(I)$. Hence $(S/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) \subseteq ann_R(a/\mathcal{P}(I))$. So $ann_R(a/\mathcal{P}(I))$ is a left ideal of $S/\mathcal{P}(I)$. Again we know $ann_R(a/\mathcal{P}(I))$ is a right ideal of $S/\mathcal{P}(I)$. Consequently $ann_R(a/\mathcal{P}(I))$ is an ideal of $S/\mathcal{P}(I)$. Therefore, $S/\mathcal{P}(I)$ satisfy (SI). ∎

**Proposition 13.** *Let $S$ be a $\Gamma$-semiring with unity. Then*

(i) *$I \subseteq P$ if and only if $N_I(P) \subseteq P$ for any ideal $I$ and any prime ideal $P$ of $S$.*

(ii) *$N_I(P) \subseteq N_I^P$ for any prime ideal $P$ and ideal $I$ of $S$.*

(iii) *If $I = P$ then $N_I(P) = P$ for any ideal $I$ and any prime ideal $P$ of $S$.*

(iv) *If $P = Q$ if and only if $N_Q(P) = P$ for any prime ideals $P$ and $Q$ of $S$.*

***Proof.*** (i) Suppose $I \subseteq P$, then $\mathcal{P}(I) \subseteq P$. So, for any element $x \in N_I(P)$, there exists $b \in S - P$ such that $x\Gamma S\Gamma b \subseteq \mathcal{P}(I) \subseteq P$. Since $P$ is a prime ideal of $S$ and $b \in S - P$, we have $x \in P$. Therefore, $N_I(P) \subseteq P$. Conversely, let $N_I(P) \subseteq P$. Let $x \in I$. Now for any $y \in S - P$, $x\Gamma S\Gamma y \subseteq I$. Again $I \subseteq \mathcal{P}(I)$, so we have $x\Gamma S\Gamma y \subseteq \mathcal{P}(I)$. Hence $x \in N_I(P) \subseteq P$. This completes the proof.

(ii) Let $x \in N_I(P)$. Then there exists $b \in S - P$ such that $x\Gamma S\Gamma b \subseteq \mathcal{P}(I)$. Since $P$ is a prime ideal of $S$, there exists $s \in S$ and $\alpha, \beta \in \Gamma$ such that $b\alpha s\beta b \in S - P$. Thus we have $x\Gamma b\alpha s\beta b \subseteq x\Gamma S\Gamma b \subseteq \mathcal{P}(I)$. Now since $b\alpha s\beta b \in S - P$, $x \in N_I^P$. Therefore, $N_I(P) \subseteq N_I^P$.

(iii) Let $P = I$ and $x \in I$. Since $I \subseteq \mathcal{P}(I)$ and $I$ is an ideal, $x\Gamma S\Gamma S \subseteq I \subseteq \mathcal{P}(I)$. So for any $y \in S - P$, $x\Gamma S\Gamma y \subseteq \mathcal{P}(I)$. Hence $x \in N_I(P)$. Therefore, $P \subseteq N_I(P)$. Now by (i) $N_I(P) \subseteq P$. This completes the proof.

(iv) Suppose that $P = Q$, then by (iii), $N_Q(P) = P$. On the other hand, let $N_Q(P) = P$. Then $Q \subseteq N_Q(P) = P$ i.e., $Q \subseteq P$. Let $x \in P$. Then $x \in N_Q(P)$.

Then there exists $b \in S - P$ such that $x\Gamma S\Gamma b \subseteq \mathcal{P}(Q) \subseteq Q$ as $Q$ is prime. Since $Q \subseteq P$, $b \in S - P \subseteq S - Q$. Hence $x \in Q$ as $Q$ is prime. Therefore, $P = Q$. ■

**Lemma 14.** *Let $S$ be a $\Gamma$-semiring and '$a$' be a nonzero strongly nilpotent element of $S$. Then there exists a nonzero element $b$ in $S$ such that $b\Gamma b = 0$.*

***Proof.*** Let '$a$' be a nonzero strongly nilpotent element of $S$. Let $n$ be the smallest positive integer such that $(a\Gamma)^{n-1}a = 0$.

*Case* 1. Suppose that $n$ is odd say $n = 2k + 1$, where $1 \leq k < n$. Then we have $(a\Gamma)^{2k}a = 0$ which implies $(a\Gamma)^{2k}a\Gamma a = 0$. So

$$\underbrace{(a\Gamma a\Gamma a\ldots a\Gamma a)}_{\text{`}a\text{' } appears(k+1)times}\ \Gamma\ \underbrace{(a\Gamma a\Gamma a\ldots a\Gamma a)}_{\text{`}a\text{' } appears(k+1)times} = 0.$$

$$\Rightarrow \underbrace{(a\gamma a\gamma a\ldots a\gamma a)}_{\text{`}a\text{' } appears(k+1)times}\ \Gamma\ \underbrace{(a\gamma a\gamma a\ldots a\gamma a)}_{\text{`}a\text{' } appears(k+1)times} = 0 \text{ for all } \gamma \in \Gamma.$$

Let $b = \underbrace{a\gamma a\gamma a\ldots a\gamma a}_{\text{`}a\text{' } appears\ (k+1)-times}$ for some nonzero $\gamma \in \Gamma$. Then $b \neq 0$ and $b\Gamma b = 0$.

*Case* 2. Suppose that $n$ is even say $n = 2k$, where $1 \leq k < n$. Then we have $(a\Gamma)^{2k-1}a = 0$ which implies

$$\underbrace{(a\Gamma a\Gamma a\ldots a\Gamma a)}_{\text{`}a\text{' } appears\ k-times}\ \Gamma\ \underbrace{(a\Gamma a\Gamma a\ldots a\Gamma a)}_{\text{`}a\text{' } appears\ k-times} = 0.$$

$$\Rightarrow \underbrace{(a\gamma a\gamma a\ldots a\gamma a)}_{\text{`}a\text{' } appears\ k-times}\ \Gamma\ \underbrace{(a\gamma a\gamma a\ldots a\gamma a)}_{\text{`}a\text{' } appears\ k-times} = 0 \text{ for all } \gamma \in \Gamma.$$

Let $b = \underbrace{a\gamma a\gamma a\ldots a\gamma a}_{\text{`}a' \ appears\ k-times}$ for some nonzero $\gamma \in \Gamma$. Then $b \neq 0$ and $b\Gamma b = 0$. ■

**Note 15.** *Spec*$(S)$ denotes the set of all prime ideals of $S$.

**Theorem 16.** *Let $S$ be an SN $\Gamma$-semiring with unity and $I$ be an ideal of $S$. Then the following are equivalent:*

(i) *$I$ is a 2-primal ideal of $S$,*

(ii) *$\mathcal{P}(I)$ has the IFP,*

(iii) *$N_I(P)$ has the IFP for each $P \in Spec(S)$,*

(iv) *$N_I(P) = \overline{N_I^P}$ for each $P \in Spec(S)$,*

(v) *$N_I(P) = N_I^P$ for each $P \in Spec(S)$,*

(vi) *$N_I^P \subseteq P$ for each prime ideal $P$ which contains $I$,*

(vii) *$N_{P/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$ for each prime ideal $P$ which contains $I$,*

(viii) $\overline{N_J^Q} \subseteq N_I(P)$ *for any ideal* $J \subseteq I$ *and prime ideals* $P, Q$ *of* $S$ *such that* $P \subseteq Q$,

(ix) $N_J^Q \subseteq N_I(P)$ *for any ideal* $J \subseteq I$ *and prime ideals* $P, Q$ *of* $S$ *such that* $P \subseteq Q$,

(x) $\overline{N_J^Q} \subseteq P$ *for any ideal* $J$ *and prime ideals* $P, Q$ *of* $S$ *such that* $J \subseteq I \subseteq P \subseteq Q$,

(xi) $N_J^Q \subseteq P$ *for any ideal* $J$ *and prime ideals* $P, Q$ *of* $S$ *such that* $J \subseteq I \subseteq P \subseteq Q$,

(xii) $N_{Q/\mathcal{P}(I)} \subseteq P/P(I)$ *for each prime ideal* $P, Q$ *of* $S$*, such that* $I \subseteq P \subseteq Q$.

***Proof.*** (i) implies (ii). Let $I$ be a 2-primal ideal of $S$. Then $S/I$ is a 2-primal Γ-semiring. Let $x/\mathcal{P}(I) \in \mathcal{N}(S/\mathcal{P}(I))$. Since $S$ is a SN Γ-semiring with unity, $S/\mathcal{P}(I)$ is a SN Γ-semiring with unity. Then there exists a positive integer $n$ such that $(x/\mathcal{P}(I)\Gamma)^{n-1}(x/\mathcal{P}(I)) = \mathcal{P}(I)$, i.e., $((x\Gamma)^{n-1}x)/\mathcal{P}(I)) = \mathcal{P}(I)$, i.e., $(x\Gamma)^{n-1}x \subseteq \mathcal{P}(I)$. Since $I$ is a 2-primal ideal of $S$, then by Proposition 10, $\mathcal{P}(I)$ is a completely semipriime ideal of $S$. Hence $x \in \mathcal{P}(I)$, i.e., $x/\mathcal{P}(I) = 0/\mathcal{P}(I)$. Hence $S/\mathcal{P}(I)$ has no strongly nilpotent elements. Then by Proposition 12, $S/\mathcal{P}(I)$ satisfies (SI). Hence by Proposition 11, $\mathcal{P}(I)$ has the IFP.

(ii) implies (iii). Let $P \in Spec(S)$ and $x\Gamma y \subseteq N_I(P)$ . Then $x\Gamma y\Gamma S\Gamma b \subseteq \mathcal{P}(I)$ for any $b \in S - P$. Now by (ii), $\mathcal{P}(I)$ has the IFP, so $x\Gamma S\Gamma y\Gamma S\Gamma b \subseteq \mathcal{P}(I)$ i.e. $(x\Gamma S\Gamma y)\Gamma S\Gamma b \subseteq \mathcal{P}(I)$, where $b \in S - P$. Therefore, $x\Gamma S\Gamma y \subseteq N_I(P)$ for each $P \in Spec(S)$. Thus $N_I(P)$ has the IFP for each $P \in Spec(S)$.

(iii) implies (i). Let $(a\Gamma)^{n-1}a \in I$, for some positive integer $n$. Claim: $a \in \mathcal{P}(I)$. Suppose $a \notin \mathcal{P}(I)$. Then there exists a prime ideal $P$ which contains $I$, such that $a \notin P$ i.e., $a \in S \setminus P$. Since $P$ is prime, $S \setminus P$ is an m-system. Then there exist $s_1 \in S, \alpha_1, \beta_1 \in \Gamma$ such that $a\alpha_1 s_1\beta_1 a \in S \setminus P$. Again since $a\alpha_1 s_1\beta_1 a, a \in S \setminus P$, applying m-system property $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a \in S \setminus P$, for some $\alpha_2, \beta_2 \in \Gamma$ and $s_2 \in S$. Applying m-system property after finite step, we have $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a \ldots \alpha_{n-1}s_{n-1}\beta_{n-1}a \in S \setminus P$ for some $s_i \in S, \alpha_i, \beta_i \in \Gamma$, where $i = 1, 2, \ldots, (n-1)$. Since $(a\Gamma)^{n-1}a \in I \subseteq N_I(P)$ and $N_I(P)$ has the IFP, $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a \ldots \alpha_{n-1}s_{n-1}\beta_{n-1}a \in N_I(P)$. Again by Proposition 13 (i), $N_I(P) \subseteq P$, then $a\alpha_1 s_1\beta_1 a\alpha_2 s_2\beta_2 a \ldots \alpha_{n-1}s_{n-1}\beta_{n-1}a \in P$, a contradiction. Hence $a \in \mathcal{P}(I)$. Now by Proposition 9, $I$ is a 2-primal ideal of $S$.

(i) implies (iv). Let $a \in \overline{N_I^P}$ for each $P \in Spec(S)$. Then $(a\Gamma)^{n-1}a \subseteq N_I^P$, for some positive integer $n$. Hence there exists $b \in S - P$ such that $(a\Gamma)^{n-1}a\Gamma b \subseteq \mathcal{P}(I)$ i.e., $(a\Gamma)^n b \subseteq \mathcal{P}(I)$. Since $I$ is a 2-primal ideal of $S$, by Proposition 10(3), $\mathcal{P}(I)$ is a left and a right symmetric ideal of $S$. Suppose $n = 1$, $a\Gamma b \subseteq \mathcal{P}(I)$. Let $n = 2$, $a\Gamma a\Gamma b \subseteq \mathcal{P}(I) \Rightarrow a\Gamma b\Gamma a \subseteq \mathcal{P}(I)$ (as $\mathcal{P}(I)$ is a right symmetric ideal of $S$) $\Rightarrow a\Gamma b\Gamma a\Gamma b \subseteq \mathcal{P}(I)$ (as $\mathcal{P}(I)$ is an ideal of $S$). Now by Proposition 10(2), $\mathcal{P}(I)$ is a completely semiprime ideal of $S$, then we have $a\Gamma b \subseteq \mathcal{P}(I)$. Let $n = 3$. Then $a\Gamma a\Gamma a\Gamma b \subseteq \mathcal{P}(I) \Rightarrow b\Gamma a\Gamma a\Gamma a\Gamma b \subseteq \mathcal{P}(I)$ (as $\mathcal{P}(I)$ is an ideal of $S$) $\Rightarrow a\Gamma b\Gamma a\Gamma a\Gamma b \subseteq \mathcal{P}(I)$ (as $\mathcal{P}(I)$ is a left symmetric ideal of $S$) $\Rightarrow a\Gamma b\Gamma a\Gamma b\Gamma a \subseteq \mathcal{P}(I)$ (as $\mathcal{P}(I)$ is a right symmetric ideal of $S$)$\Rightarrow a\Gamma b\Gamma a\Gamma b\Gamma a\Gamma b\Gamma a\Gamma b \subseteq \mathcal{P}(I)$ (as $\mathcal{P}(I)$ is an ideal of $S$). Hence by Proposition 10(2), $\mathcal{P}(I)$ is a completely semiprime ideal of $S$, then we have $a\Gamma b \subseteq \mathcal{P}(I)$. Continueing this process for $n \geq 2$, $(a\Gamma)^n b \subseteq \mathcal{P}(I)$ $\Rightarrow \underbrace{(a\Gamma b)\Gamma(a\Gamma b)\Gamma(a\Gamma b)\Gamma \ldots \Gamma(a\Gamma b)}_{(n-times)} \subseteq \mathcal{P}(I)$. If $n$ is even, then $a\Gamma b \subseteq \mathcal{P}(I)$ (by Proposition 10(2), $\mathcal{P}(I)$ is a completely semiprime ideal of $S$). If $n$ is odd, then multiplying by $a\Gamma b$ and applying Proposition 10(2) we have $a\Gamma b \subseteq \mathcal{P}(I)$. Now by Proposition 10(4), we have $a\Gamma S\Gamma b \subseteq \mathcal{P}(I)$, where $b \in S - P$. Hence $a \in N_I(P)$. Again by Proposition 13, we have, $N_I(P) \subseteq N_I^P \subseteq \overline{N_I^P}$. Therefore, $N_I(P) = \overline{N_I^P}$.

(iv) implies (v). Since $N_I(P) \subseteq N_I^P \subseteq \overline{N_I^P}$, by (iv) we have $N_I(P) = N_I^P$.

(v) implies (vi). Let $P$ be a prime ideal of $S$ which contains $I$. Then by Proposition 13 (i), we have $N_I(P) \subseteq P$. Now by (v) we have, $N_I(P) = N_I^P$. Hence $N_I^P \subseteq P$.

(vi) implies (vii). Let $a/\mathcal{P}(I) \in N_{P/\mathcal{P}(I)}$. Then there exists $b/\mathcal{P}(I) \in S/\mathcal{P}(I) - P/\mathcal{P}(I)$ such that $(a/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) \subseteq \mathcal{P}(S/\mathcal{P}(I)) = \mathcal{P}(\mathcal{P}(I))/\mathcal{P}(I)$ (by Proposition 8). This implies that $a\Gamma b \subseteq \mathcal{P}(I)$ as $\mathcal{P}(\mathcal{P}(I)) = \mathcal{P}(I)$, where $b \in S - P$. So $a \in N_I^P$. Hence by (vi), $a \in P$. This implies that $a/\mathcal{P}(I) \in P/\mathcal{P}(I)$. Therefore, $N_{P/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$.

(vii) implies (i). First we shall show that $S/\mathcal{P}(I)$ is reduced. Suppose, $S/\mathcal{P}(I)$ is not reduced. Then there exists a nonzero nilpotent element say $a/\mathcal{P}(I) \in S/\mathcal{P}(I)$. Since $S$ is an SN $\Gamma$-semiring, $S/\mathcal{P}(I)$ is an SN $\Gamma$-semiring. Then $a/\mathcal{P}(I) \in S/\mathcal{P}(I)$ is a strongly nilpotent element. Hence by Lemma 14, there exists a nonzero element say $b/\mathcal{P}(I) \in S/\mathcal{P}(I)$ such that $(b/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) = \mathcal{P}(I)/\mathcal{P}(I)$ i.e., $(b/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) = \mathcal{P}(S/\mathcal{P}(I))$ (by Proposition 8). Since $b/\mathcal{P}(I)$ is a nonzero element of $S/\mathcal{P}(I)$, $b \notin \mathcal{P}(I)$. So $b/\mathcal{P}(I) \in S/\mathcal{P}(I) - P/\mathcal{P}(I)$ for some prime ideal $P$ of $S$ containing $I$. Hence $b/\mathcal{P}(I) \in N_{P/\mathcal{P}(I)}$. Now by (vii) $N_{P/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$, so $b/\mathcal{P}(I) \in P/\mathcal{P}(I)$ which is a contradiction. Therefore, $S/\mathcal{P}(I)$ has no nonzero strongly nilpotent elements. Thus $S/\mathcal{P}(I)$ is reduced. Then using Proposition 12, we have $S/\mathcal{P}(I)$ satisfy (SI). Then by Proposition 11, $\mathcal{P}(I)$ has the IFP. Then by Proposition 10, we have $I$ is a 2-primal ideal of $S$.

(i) implies (viii). Let $x \in \overline{N_J^Q}$. Then there exists a positive integer $n$ such that $(x\Gamma)^{n-1}x \subseteq N_J^Q$. Then $(x\Gamma)^{n-1}x\Gamma y \subseteq \mathcal{P}(J)$ for some $y \in S-Q$. Since $P \subseteq Q$ and $J \subseteq I$, $(x\Gamma)^{n-1}x\Gamma y \subseteq \mathcal{P}(I)$ for some $y \in S-P$. Since $I$ is 2-primal, by Proposition

10, $\mathcal{P}(I)$ is a left and right symmetric ideal and completely semiprime ideal of $S$. Proceeding as in the proof of (i) implies (iv), we get $x\Gamma y \subseteq \mathcal{P}(I)$, where $y \in S-P$. Again by same proposition $\mathcal{P}(I)$ has the IFP, then $x\Gamma y \subseteq \mathcal{P}(I) \Rightarrow x\Gamma S\Gamma y \subseteq \mathcal{P}(I)$. Hence $x \in N_I(P)$. Therefore, $\overline{N_J^Q} \subseteq N_I(P)$.

(viii) implies (ix). It is obvious as $N_J^Q \subseteq \overline{N_J^Q}$.

(ix) implies (xi). Let $J \subseteq I \subseteq P \subseteq Q$. Then by (ix), $N_J^Q \subseteq N_I(P)$. Again since $I \subseteq P$, by Proposition 13(i), $N_I(P) \subseteq P$. Hence $N_J^Q \subseteq P$ for any ideal $J$ and prime ideals $P, Q$ of $S$ such that $J \subseteq I \subseteq P \subseteq Q$.

(xi) implies (vi). On assuming $I = J$ and $P = Q$ in (xi), we get $N_I^P \subseteq P$.

(vi) implies (xii). Let $a/\mathcal{P}(I) \in N_{Q/\mathcal{P}(I)}$. Then there exists $b/\mathcal{P}(I) \in S/\mathcal{P}(I) - Q/\mathcal{P}(I)$ such that $(a/\mathcal{P}(I))\Gamma(b/\mathcal{P}(I)) \subseteq \mathcal{P}(S/\mathcal{P}(I)) = \mathcal{P}(I)/\mathcal{P}(I)$ (by Proposition 8). This implies that $a\Gamma b \subseteq \mathcal{P}(I)$, where $b \in S-Q$. This implies that $a\Gamma b \subseteq \mathcal{P}(I)$, where $b \in S-P$ as $P \subseteq Q$. Hence $a \in N_I^P$. Since $I \subseteq P$, by (vi) $N_I^P \subseteq P$, so $a \in P$. This implies that $a/\mathcal{P}(I) \in P/\mathcal{P}(I)$. Therefore, $N_{Q/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$.

(xii) implies (vii). On assuming $Q = P$ in (xii), we get $N_{P/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$.

(viii) implies (x). Let $I, J$ be two any ideals of $S$ and $P, Q$ be two prime ideals of $S$ such that $J \subseteq I \subseteq P \subseteq Q$. Now by (viii), we have $\overline{N_J^Q} \subseteq N_I(P)$. Again since $I \subseteq P$, by Proposition 13(i), $N_I(P) \subseteq P$. Therefore, $\overline{N_J^Q} \subseteq P$.

(x) implies (xi). Let $I, J$ be two any ideals of $S$ and $P, Q$ be two prime ideals of $S$ such that $J \subseteq I \subseteq P \subseteq Q$. Then by (x), $\overline{N_J^Q} \subseteq P$. Again by Proposition 7, we have $N_J^Q \subseteq \overline{N_J^Q}$. Therefore, $N_J^Q \subseteq P$. ∎

**Corollary 17.** *Let $S$ be an SN $\Gamma$-semiring with unity and $I$ be a 2-primal ideal of $S$. Then for any prime ideal $P$ of $S$, $I \subseteq P$ if and only if $N_I^P \subseteq P$.*

***Proof.*** Let $I \subseteq P$. Then by Theorem 16 (vi), we have $N_I^P \subseteq P$. Conversely let, $N_I^P \subseteq P$. Let $x \in I$. Then for any $y \in S-P$ we have $x\Gamma y \subseteq I \Rightarrow x\Gamma y \subseteq \mathcal{P}(I)$ as $I \subseteq \mathcal{P}(I)$, where $y \in S-P$. Hence $x \in N_I^P$ and so $x \in P$. Thus $I \subseteq P$. ∎

**Corollary 18.** *Let $S$ be an SN $\Gamma$-semiring with unity. Then $I = P$ if and only if $N_I^P = P$ for any completely prime ideal $I$ and prime ideal $P$ of $S$.*

***Proof.*** Let $I$ be any completely prime ideal and $P$ be any prime ideal of $S$ such that $I = P$. Now we have, $I \subseteq \mathcal{P}(I) \subseteq P$. Since $I = P$, $I = \mathcal{P}(I)$. Hence $\mathcal{P}(I)$ is a completely prime ideal and hence a completely semiprime ideal of $S$. Then by Proposition 10, $I$ is a 2-primal ideal of $S$. By Corollary 17, $N_I^P \subseteq P$. Hence $P = I \subseteq N_I^P \subseteq P$. Therefore, $N_I^P = P$. Conversely let, $N_I^P = P$ for any completely prime ideal $I$ and prime ideal $P$ of $S$. Then by Corollary 17, $I \subseteq P$.

Let $x \in P$. Then $x \in N_I^P$. So $x\Gamma y \subseteq \mathcal{P}(I)$ for some $y \in S - P$. Since $I$ is a completely prime ideal of $S$, $I$ is a prime ideal of $S$. Hence $\mathcal{P}(I) = I$. So $x\Gamma y \subseteq I$. Since $I \subseteq P$ and $y \in S - P$, $y \in S - I$. Again $I$ being a completely prime ideal, then $x \in I$ as $y \in S - P$. Hence $I = P$. ■

**Corollary 19.** *Let $S$ be an SN $\Gamma$-semiring with unity. Then the following are equivalent:*

(i) *$S$ is a 2-primal $\Gamma$-semiring,*

(ii) *$\mathcal{P}(S)$ has the IFP,*

(iii) *$N(P)$ has the IFP for each $P \in Spec(S)$,*

(iv) *$N(P) = \overline{N_P}$ for each $P \in Spec(S)$,*

(v) *$N(P) = N_P$ for each $P \in Spec(S)$,*

(vi) *$N_P \subseteq P$ for each $P \in Spec(S)$,*

(vii) *$N_{P/\mathcal{P}(S)} \subseteq P/\mathcal{P}(S)$ for each $P \in Spec(S)$,*

(viii) *$\overline{N_Q} \subseteq N(P)$ for any prime ideals $P, Q$ of $S$ such that $P \subseteq Q$,*

(ix) *$N_Q \subseteq N(P)$ for any prime ideals $P, Q$ of $S$ such that $P \subseteq Q$,*

(x) *$\overline{N_Q} \subseteq P$ for any prime ideals $P, Q$ of $S$ such that $P \subseteq Q$,*

(xi) *$N_Q \subseteq P$ for any prime ideals $P, Q$ of $S$ such that $P \subseteq Q$,*

(xii) *$N_{Q/\mathcal{P}(S)} \subseteq P/P(S)$ for each prime ideals $P, Q$, such that $P \subseteq Q$.*

***Proof.*** The proof follows from Theorem 16. ■


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